MAGIC POWERS OF GRAPHS

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Abstract. Necessary and sufficient conditions for a graph G that its power G^i, $i \geqslant 2$, is a magic graph and one consequence are given.

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1. Introduction

In the paper only finite, undirected connected graphs are considered. By a *magic valuation* of a graph $\mathbf{G}$ we mean such an assignment of the edges of $\mathbf{G}$ by pairwise different positive numbers that the sum of assignments of edges meeting the same vertex is constant. A graph is called *magic* if it allows a magic valuation. This notion was introduced by J. Sedláček in [6]. Now, the *i-th power* $\mathbf{G}^i$, $i \geqslant 2$, of a graph $\mathbf{G}$ is the graph with the same vertex set as $\mathbf{G}$ and such that two vertices of $\mathbf{G}^i$ are adjacent if and only if the distance between these vertices in $\mathbf{G}$ is at most i.

Various properties of $\mathbf{G}^i$ have been studied, such as hamiltonicity, existence of some factors, etc. Some results can be found in [1] and [2] and [5].

Two different characterizations of magic graphs were published in [3] and [4]. Since, except of the complete graph $\mathbf{K}_2$ of order 2, no graph with less than 5 vertices is magic we confine ourselves to graphs of order $n \geqslant 5$.

By an *I-graph* we mean a graph $\mathbf{G}$ with a 1-factor $\mathbf{F}$ whose every edge is incident with an *end-vertex* (a vertex of degree 1) of $\mathbf{G}$. The symbol $\mathbf{P}_5$ denotes a path of length 5.

The aim of this paper is the following theorem.

Theorem. *Let a graph* $\mathbf{G}$ *have order* $n \geqslant 5$*. The graph* $\mathbf{G}^2$ *is magic if and only if* $\mathbf{G}$ *is not an* I*-graph and it is different from the path* $\mathbf{P}_5$*. The graph* $\mathbf{G}^i$ *is magic for all* $i \geqslant 3$.

2. Proof of the theorem

First we shall formulate several necessary definitions. We say that a graph $\mathbf{G}$ is of *type* $\mathbb{A}$ if it has two edges e, f such that $\mathbf{G} - e - f$ is a balanced bipartite graph with the partition V_1, V_2, and the edge e joins two vertices of V_1 and f joins two vertices of V_2. A graph $\mathbf{G}$ is of *type* $\mathbb{B}$ if it has two edges e_1, e_2 such that $\mathbf{G} - e_1 - e_2$ is a graph with two components $\mathbf{G}_1$ and $\mathbf{G}_2$ such that $\mathbf{G}_1$ is a balanced bipartite graph with partition V_1, V_2 and $\mathbf{G}_2$ is a non-bipartite graph, and e_i joins a vertex of V_i with a vertex of $V(\mathbf{G}_2)$. As usual, $\Gamma(S)$ denotes the set of vertices adjacent to a vertex in the set S.

The proof of Theorem is an immediate consequence of the following five Lemmas and Theorem 1.

Theorem 1. (Jeurissen [3].) *A non-bipartite graph* $\mathbf{G}$ *is magic if and only if* $\mathbf{G}$ *is neither of type* $\mathbb{A}$ *nor of type* $\mathbb{B}$*, and* $|\Gamma(S)| > |S|$ *for every independent subset* $S \neq \emptyset$ *of* $V(\mathbf{G})$.

Lemma 1. *If* $\mathbf{G}$ *is an* I*-graph or it is a path* $\mathbf{P}_5$*, then* $\mathbf{G}^2$ *is not a magic graph.*

Proof. Every I-graph of order $2n$ has n end-vertices which form an independent subset S such that $|\Gamma(S)| = |S|$. Let $\mathbf{P}_5$ be a path $v_1v_2v_3v_4v_5v_6$. By omitting the edges v_2v_3 and v_4v_5 from $\mathbf{P}_5^2$ we obtain a bipartite graph which is a graph of type $\mathbb{A}$. $\square$

Lemma 2. *If* $\mathbf{G} + e$ *is a graph which arises by adding an arbitrary edge* e *to a non-bipartite magic graph* $\mathbf{G}$*, then* $\mathbf{G} + e$ *is a magic graph.*

The proof follows from Theorem 1 because by omitting an arbitrary edge of a graph of type $\mathbb{A}$ or $\mathbb{B}$ we do not obtain a magic graph.

Let $\mathbf{T}$ be a spanning tree of a graph $\mathbf{G}$. Lemma 2 implies that if the square of $\mathbf{T}$ is magic, then $\mathbf{G}^i$, for all $n \geqslant 2$, is magic. Therefore, in the next part we shall confine ourselves to graphs which are trees.

Lemma 3. *If* $\mathbf{T}$ *is a tree then* $|\Gamma(S)| \geqslant |S|$ *for every non-empty independent subset* S *of* $V(\mathbf{T}^2)$.

P r o o f. Let S be an independent subset of $V(\mathbf{T}^2)$. Then the distance $d(u,v) \geqslant 3$ in $\mathbf{T}$ for vertices $u, v \in S$, i.e. no vertex of the induced subgraph $\mathbf{H}$ on $V(\mathbf{T}) - S$ is joined with two vertices of S. We choose one internalvertex $w \in V(\mathbf{H})$ and define a mapping f from the set S into $\Gamma(S)$ in the following way: The image $f(v)$ of a vertex v is the vertex such that $d(v,w) - 1 = d(f(v), w)$. The proof follows from the fact that f is an injective mapping. $\square$

Lemma 4. *If $V(\mathbf{T}^2)$ contains an independent subset S such that $|\Gamma(S)| = |S| = n > 0$, then $\mathbf{T}$ is an I-graph of order 2n.*

P r o o f. If v is an internalvertex of $\mathbf{T}$ and $v \in S$ then there exists a vertex $z \in \Gamma(v)$ with $d(z,w) = d(v,w) + 1$ (the internalvertex w is chosen in the same way as in the proof of Lemma 3). The vertex z is not an image of any vertex $u \in S$ in the mapping f because in $\mathbf{T}^2$ the vertex v is joined by an edge with all vertices which in $\mathbf{T}$ have the distance 2. This fact together with the proof of Lemma 3 yield that then $|\Gamma(S)| > |S|$.

If an arbitrary end-vertex $t \notin S$, then t is not the image of any vertex of S and so $|\Gamma(S)| > |S|$.

Every vertex of S is joined to at least two vertices of $\mathbf{T} - S$ and so it follows from the assumption $|\Gamma(S)| = |S|$ that every internal vertex is uniquely assigned to a vertex of S. $\square$

Lemma 5. *No graph $\mathbf{T}^2$, different from $\mathbf{P}_5^2$, is a graph of type $\mathbb{A}$ or $\mathbb{B}$.*

P r o o f. First we show that $\mathbf{T}^2$ different from $\mathbf{P}_5^2$ cannot be a graph of type $\mathbb{A}$. We suppose that the order of $\mathbf{T}$ is at least 6, because any graph of type $\mathbb{A}$ has an even order. If $\mathbf{T}$ has a vertex of degree at least 4, then $\mathbf{T}^2$ has, as a subgraph, the complete graph $\mathbf{K}_5$. By omitting an arbitrary pair of edges from $\mathbf{K}_5$ we obtain a graph with chromatic number 3, i.e. a non-bipartite graph. If $\mathbf{T}$ has a vertex of degree 3, then $\mathbf{T}$ contains a subgraph isomorphic to one of the graphs which are depicted in Fig. 1. In both cases, by omitting two edges we obtain a subgraph with at least one triangle. If $\mathbf{T}$ is a path $\mathbf{P}_n$, $n \geqslant 6$, then $\mathbf{T}^2$ has at least 3 edge-disjoint triangles.

Fig. 1

Every graph of type $\mathbb{B}$ is 2-edge-connected. From $\mathbf{T}^2$ we obtain a disconnected graph only if we omit a pair of edges incident with an end-vertex of $\mathbf{T}$ and so the non-bipartite part consists of one vertex while the other part is not a bipartite graph. □

3. A consequence of the theorem

A spanning subgraph $\mathbf{F}$ of the graph $\mathbf{G}$ is called a (1-2)-*factor* of $\mathbf{G}$ if each of its components is an isolated edge or a circuit. We say that a (1-2)-factor *separates edges* e and f, if at least one of them belongs to $\mathbf{F}$ and neither the edge part nor the circuit part contains both of them. In [4] the following theorem is proved.

Theorem 2. (Jezný, Trenkler) *A graph* $\mathbf{G}$ *is magic if and only if every edge belongs to a* (1-2)*-factor, and every pair of edges* e, f *is separated by a* (1-2)*-factor.*

From Lemma 3 and Theorem 2 we get the following

Corollary. *Let* $\mathbf{G}$ *be a graph of order* $\geqslant 5$ *and* e *its arbitrary edge. The graph* $\mathbf{G}^i$, $i \geqslant 2$, *has a* (1-2)*-factor which contains the edge* e *if and only if* e *is not an internaledge of an* I*-graph and* $i = 2$.

Proof. No pair of end-vertices of an I-graph $\mathbf{G}$ is joined by an edge in $\mathbf{G}^2$ because every (1-2)-factor of $\mathbf{G}^2$ is a 1-factor. The internaledge of an I-graph does not belong to the same (1-2)-factor. Evidently, every edge of $\mathbf{P}_5^2$ belongs to a 2-factor. The sufficient condition follows from Theorem 2. □

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