UPPER AND LOWER SOLUTIONS FOR SINGULARLY PERTURBED SEMILINEAR NEUMANN'S PROBLEM

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Abstract. The paper establishes sufficient conditions for the existence of solutions of Neumann's problem for the differential equation $\mu y'' + ky = f(t,y)$ which tend to the solution of the reduced problem $ky = f(t,y)$ on $[0,1]$ as $\mu \to 0$.

Keywords: singularly perturbed equation, Neumann's problem

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1. Introduction

We will consider the two-point problem

$$\begin{aligned} \mu y'' + ky &= f(t,y), \quad t \in [0,1], \\ y'(0,\mu) = 0, &\quad y'(1,\mu) = 0, \end{aligned} \tag{1}$$

where μ is a small, positive parameter, k a negative constant and $f \in C^1\bigl([0,1]\times\mathbb{R}\bigr)$.

We can view this equation as the mathematical model of the nonlinear dynamical system with a high-speed feedback. We apply the method of upper and lower solutions to prove the existence of a solution for (1).

As usual, we say that $\alpha \in C^2\bigl([0,1]\bigr)$ is a lower solution for (1) if $\alpha'(0,\mu) \geqslant 0$, $\alpha'(1,\mu) \leqslant 0$, and $\mu\alpha''(t,\mu) + k\alpha(t,\mu) \geqslant f\bigl(t,\alpha(t,\mu)\bigr)$ for every $t \in [0,1]$. An upper solution $\beta \in C^2\bigl([0,1]\bigr)$ satisfies $\beta'(0,\mu) \leqslant 0$, $\beta'(1,\mu) \geqslant 0$, and $\mu\beta''(t,\mu) + k\beta(t,\mu) \leqslant f\bigl(t,\beta(t,\mu)\bigr)$ for every $t \in [0,1]$.

Lemma 1. (Cf. [2], pp. 20–30) *If α, β are lower and upper solutions for (1) such that $\alpha \leqslant \beta$ on $[0,1]$, then there exists a solution y of (1) with $\alpha \leqslant y \leqslant \beta$ on $[0,1]$.*

Denote $D(u) = \{(t,y);\, 0 \leqslant t \leqslant 1, |y - u(t)| < \delta\}, \delta > 0$ is a constant and u is a solution of the reduced problem $ky = f(t,y)$ on $[0,1]$.

The main result is the following theorem.

2. Existence and asymptotic behavior of solutions

Theorem 1. *Let f be a function such that $f \in C^1\bigl(D(u)\bigr)$ and*

$$\text{(i)} \qquad \left|\frac{\partial f(t,y)}{\partial y}\right| \leqslant w < -k \text{ for every } (t,y) \in D(u).$$

Then there exits μ_0 such that for each $\mu \in (0,\mu_0]$ the problem (1) has a unique solution satisfying the inequality

$$\bigl|y(t,\mu) - u(t)\bigr| \leqslant v_1(t,\mu) + v_2(t,\mu) + C\mu \text{ on } [0,1],$$

where

$$v_1(t,\mu) = \bigl|u'(0)\bigr| \frac{\exp\bigl[-(m/\mu)^{1/2}(1-t)\bigr] + \exp\bigl[-(m/\mu)^{1/2}(t-1)\bigr]}{(m/\mu)^{1/2}\bigl(\exp\bigl[(m/\mu)^{1/2}\bigr] - \exp\bigl[-(m/\mu)^{1/2}\bigr]\bigr)},$$

$$v_2(t,\mu) = \bigl|u'(1)\bigr| \frac{\exp\bigl[(m/\mu)^{1/2}t\bigr] + \exp\bigl[-(m/\mu)^{1/2}t\bigr]}{(m/\mu)^{1/2}\bigl(\exp\bigl[(m/\mu)^{1/2}\bigr] - \exp\bigl[-(m/\mu)^{1/2}\bigr]\bigr)},$$

$m = -k - w$, C is a positive constant and u is a solution of the reduced problem $ky = f(t,y)$ on $[0,1]$.

P r o o f. We define lower solutions by

$$\alpha(t,\mu) = u(t) - v_1(t,\mu) - v_2(t,\mu) - \Gamma(\mu)$$

and upper solutions by

$$\beta(t,\mu) = u(t) + v_1(t,\mu) + v_2(t,\mu) + \Gamma(\mu);$$

here $\Gamma(\mu) = \mu\tau/m$, where τ is a constant which will be defined below.

Obviously, $\alpha \leqslant \beta$ in $[0,1]$ and α, β satisfy the boundary conditions prescribed for the lower and upper solutions of (1).

Now we show that $\mu\alpha''(t,\mu) + k\alpha(t,\mu) \geqslant f\bigl(t,\alpha(t,\mu)\bigr)$ and $\mu\beta''(t,\mu) + k\beta(t,\mu) \leqslant f\bigl(t,\beta(t,\mu)\bigr)$ on $[0,1]$. Denote $h(t,y) = f(t,y) - ky$. By the Taylor theorem we obtain

$$h\bigl(t,\alpha(t,\mu)\bigr) = h\bigl(t,\alpha(t,\mu)\bigr) - h\bigl(t,u(t)\bigr) = \frac{\partial h\bigl(t,\theta(t,\mu)\bigr)}{\partial y}\bigl(v_1(t,\mu) + v_2(t,\mu) + \Gamma(\mu)\bigr),$$

where $(t, \theta(t,\mu))$ is a point between $(t, \alpha(t,\mu))$ and $(t, u(t))$, and $(t, \theta(t,\mu)) \in D(u)$ for sufficiently small μ, for instance if $\mu \in (0, \mu_0]$. Then

$$\mu\alpha''(t,\mu) - h(t, \alpha(t,\mu)) \geqslant \mu u'' - \mu v_1{}'' - \mu v_2{}'' + m(v_1 + v_2 + \Gamma) \geqslant -\mu|u''| + \mu\tau$$

(because $\mu v_1'' = m v_1$ and $\mu v_2''$ on $[0,1]$) for every $t \in [0,1]$. If we choose a constant τ such that $\tau \geqslant |u''(t)|$, $t \in [0,1]$ then $\mu\alpha''(t,\mu) \geqslant h(t, \alpha(t,\mu))$ in $[0,1]$. The inequality for β can be proved similarly. The existence of a solution of (1) satisfying the above inequality follows from Lemma 1. $\square$

R e m a r k 1. Applying the technique of the proof of Theorem 1 we obtain immediately the uniform boundedness of $\{y'(t,\mu), \mu \in (0,\mu_0]\}$ and $\{y''(t,\mu), \mu \in (0,\mu_0]\}$ on every compact set $K \subset (0,1)$. Moreover, if $u'(0) = 0$ $(u'(1) = 0)$ then y' an y'' are uniformly bounded on $K \subset [0,1)$ $(K \subset (0,1])$ and if $u'(0) = u'(1) = 0$ then y' and y'' are uniformly bounded on $[0,1]$ for $\mu \in (0,\mu_0]$.

R e m a r k 2. If a solution of the reduced problem does not satisfy the prescribed boundary conditions, then unlike the Dirichlet problem (see e.g. [1], [3]), in the case of Neumann's problem the initial and/or endpoint nonuniformities do not arise in y', but in y''.

3. Asymptotic behavior of solutions at endpoints

E x a m p l e 1. We consider the linear problem

$$\mu y'' - y = \sin 2\pi t, \quad y'(0,\mu) = y'(1,\mu) = 0.$$

Its unique solution

$$\begin{aligned} y(t,\mu) = -\frac{\sin 2\pi t}{4\pi^2\mu + 1} &+ \frac{2\pi\big(\exp\big[(1/\mu)^{1/2}(1-t)\big] + \exp\big[(1/\mu)^{1/2}(t-1)\big]\big)}{(4\pi^2\mu + 1)(\mu)^{-1/2}\big(\exp\big[-(1/\mu)^{1/2}\big] - \exp\big[(1/\mu)^{1/2}\big]\big)} \\ &- \frac{2\pi\big(\exp\big[(1/\mu)^{1/2}t\big] + \exp\big[-(1/\mu)^{1/2}t\big]\big)}{(4\pi^2\mu + 1)(\mu)^{-1/2}\big(\exp\big[-(1/\mu)^{1/2}\big] - \exp\big[(1/\mu)^{1/2}\big]\big)} \end{aligned}$$

tends (by virtue of Theorem 1) to the solution of the reduced problem as $\mu \to 0^+$ within $[0,1]$. On the other hand, $\lim\limits_{\mu\to 0^+} |y''(0,\mu)| = \lim\limits_{\mu\to 0^+} |y''(1,\mu)| = \infty$.

Theorem 2. *Let a function $f \in C^2(D(u))$ satisfy the condition from Theorem 1 and let $\frac{\partial f}{\partial t}(0,y) \neq 0$ $\Big(\frac{\partial f}{\partial t}(1,y) \neq 0\Big)$ for every $y \in D(u)$. Then the set*

$\{y''(t,\mu);\ \mu\in(0,\mu_0], t\in[0,1]\}$ *is unbounded. (More precisely,* $\lim_{\mu\to0^+}|y''(0,\mu)|\big(=\lim_{\mu\to0^+}|y''(1,\mu)|\big)=\infty$*).*

Proof. Assume to the contrary that $\{y''(t,\mu);\ \mu\in(0,\mu_0], t\in[0,1]\}$ is bounded (this implies, on the basis of Remark 1, the uniform boundedness of $y'(t,\mu)$ on $[0,1]$, $\mu\in(0,\mu_0]$), and the existence of a sequence $\mu_n\to0^+$ such that $\lim_{n\to\infty}y''(0,\mu_n)$ $\big(\lim_{n\to\infty}y''(1,\mu_n)\big)$ exists.

The problem (1) is equivalent to the integral equation

$$
\begin{aligned}
y(t,\mu)={}&\int_0^t \frac{\exp\big[-(-k/\mu)^{1/2}(s+t)\big]+\exp\big[-(-k/\mu)^{1/2}(t-s)\big]}{2(-k\mu)^{1/2}\big(\exp\big[-2(-k/\mu)^{1/2}\big]-1\big)}f\big(s,y(s,\mu)\big)\,\mathrm{d}s\\
&+\int_0^t \frac{\exp\big[-(-k/\mu)^{1/2}(2+s-t)\big]+\exp\big[-(-k/\mu)^{1/2}(2-t-s)\big]}{2(-k\mu)^{1/2}\big(\exp\big[-2(-k/\mu)^{1/2}\big]-1\big)}f\big(s,y(s,\mu)\big)\,\mathrm{d}s\\
&+\int_t^1 \frac{\exp\big[-(-k/\mu)^{1/2}(s+t)\big]+\exp\big[-(-k/\mu)^{1/2}(s-t)\big]}{2(-k\mu)^{1/2}\big(\exp\big[-2(-k/\mu)^{1/2}\big]-1\big)}f\big(s,y(s,\mu)\big)\,\mathrm{d}s\\
&+\int_t^1 \frac{\exp\big[-(-k/\mu)^{1/2}(2-s-t)\big]+\exp\big[-(-k/\mu)^{1/2}(2+t-s)\big]}{2(-k\mu)^{1/2}\big(\exp\big[-2(-k/\mu)^{1/2}\big]-1\big)}f\big(s,y(s,\mu)\big)\,\mathrm{d}s
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
\frac{y(0,\mu_n)}{\mu_n}&=\int_0^1 \frac{\exp\big[-(-k/\mu_n)^{1/2}s\big]+\exp\big[-(-k/\mu_n)^{1/2}(2-s)\big]}{(-k)^{1/2}(\mu_n)^{3/2}\big(\exp\big[-2(-k/\mu_n)^{1/2}\big]-1\big)}f\big(s,y(s,\mu_n)\big)\,\mathrm{d}s\\
\bigg(\frac{y(1,\mu_n)}{\mu_n}&=\int_0^1 \frac{\exp\big[-(-k/\mu_n)^{1/2}(1+s)\big]+\exp\big[-(-k/\mu_n)^{1/2}(1-s)\big]}{(-k)^{1/2}(\mu_n)^{3/2}\big(\exp\big[-2(-k/\mu_n)^{1/2}\big]-1\big)}f\big(s,y(s,\mu_n)\big)\,\mathrm{d}s\bigg).
\end{aligned}
$$

Using twice integration by parts we obtain by the mean value theorem for integrals the following relations:

$$
\begin{aligned}
-y''(0,\mu_n)={}&-\frac{2\big(\exp\big[(-k/\mu_n)^{1/2}\big]\big)\frac{\partial}{\partial t}f\big(1,y(1,\mu_n)\big)}{(-k\mu_n)^{1/2}\big(\exp\big[-2(-k/\mu_n)^{1/2}\big]-1\big)}\\
&+\frac{\big(\exp\big[-2(-k/\mu_n)^{1/2}\big]+1\big)\frac{\partial}{\partial t}f\big(0,y(0,\mu_n)\big)}{(-k\mu_n)^{1/2}\big(\exp\big[-2(-k/\mu_n)^{1/2}\big]-1\big)}\\
&+(-k)^{-1}\Big(\frac{\mathrm{d}^2}{\mathrm{d}t^2}f\big(\theta_1(\mu_n),y(\theta_1(\mu_n),\mu_n)\big)\Big)
\end{aligned}
$$

$$\Big(-y''(1,\mu_n) = \frac{2\big(\exp\big[(-k/\mu_n)^{1/2}\big]\big)\frac{\partial}{\partial t} f\big(0, y(0,\mu_n)\big)}{(-k\mu_n)^{1/2}\big(\exp\big[-2(-k/\mu_n)^{1/2}\big]-1\big)} - \frac{\big(\exp\big[-2(-k/\mu_n)^{1/2}\big]+1\big)\frac{\partial}{\partial t} f\big(1, y(1,\mu_n)\big)}{(-k\mu_n)^{1/2}\big(\exp\big[-2(-k/\mu_n)^{1/2}\big]-1\big)} + (-k)^{-1}\Big(\frac{\mathrm{d}^2}{\mathrm{d}t^2} f\big(\tilde{\theta}_1(\mu_n), y(\tilde{\theta}_1(\mu_n),\mu_n)\big)\Big)\Big),$$

where $0 \leqslant \theta_1(\mu_n)\big(\tilde{\theta}_1(\mu_n)\big) \leqslant 1$. Hence we have

$$|y''(0,\mu_n)| \geqslant \frac{\big(\exp\big[-2(-k/\mu_n)^{1/2}\big]+1\big)\big|\frac{\partial}{\partial t} f\big(0, y(0,\mu_n)\big)\big|}{(-k\mu_n)^{1/2}\big(1-\exp\big[-2(-k/\mu_n)^{1/2}\big]\big)} - \frac{2\big(\exp\big[(-k/\mu_n)^{1/2}\big]\big)\big|\frac{\partial}{\partial t} f\big(1, y(1,\mu_n)\big)\big|}{(-k\mu_n)^{1/2}\big(1-\exp\big[-2(-k/\mu_n)^{1/2}\big]\big)} + (k)^{-1}\Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2} f\big(\theta_1(\mu_n), y(\theta_1(\mu_n),\mu_n)\big)\Big| \tag{2}$$

$$\Big(|y''(1,\mu_n)| \geqslant \frac{\big(\exp\big[-2(-k/\mu_n)^{1/2}\big]+1\big)\big|\frac{\partial}{\partial t} f\big(1, y(1,\mu_n)\big)\big|}{(-k\mu_n)^{1/2}\big(1-\exp\big[-2(-k/\mu_n)^{1/2}\big]\big)} - \frac{2\big(\exp\big[(-k/\mu_n)^{1/2}\big]\big)\big|\frac{\partial}{\partial t} f\big(0, y(0,\mu_n)\big)\big|}{(-k\mu_n)^{1/2}\big(1-\exp\big[-2(-k/\mu_n)^{1/2}\big]\big)} + (k)^{-1}\Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2} f\big(\tilde{\theta}_1(\mu_n), y(\tilde{\theta}_1(\mu_n),\mu_n)\big)\Big|\Big). \tag{2'}$$

From the above assumptions it follows that

$$\Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2} f\big(\theta_1(\mu_n), y(\theta_1(\mu_n),\mu_n)\big)\Big| \leqslant c_1,$$
$$\Big(\Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2} f\big(\tilde{\theta}_1(\mu_n), y(\tilde{\theta}_1(\mu_n),\mu_n)\big)\Big| \leqslant \tilde{c_1}\Big).$$

Taking limits on both sides of the inequality $(2)\big((2')\big)$ we come to a contradiction.

□

R e m a r k 3. It is well known that conditions (i) guarantees uniqueness of the solution for the boundary problem (1) in the set $D(u)$, but between different solutions u_1, u_2 of the reduced problem satisfying condition (i) in $D(u_1)$, $D(u_2)$, respectively, there may be such solutions which switch n-times between u_1 and u_2 for any nonnegative integer n. For an autonomous equation, the exact formulation is a straighforward adaptation of the results and conclusions of O'Malley in [1], therefore being omitted. In general, the problem of existence of such solutions for a nonautonomous equation remains open.

References

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