CLOSED CONVEX ℓ-SUBGROUPS AND RADICAL CLASSES OF CONVERGENCE ℓ-GROUPS

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Abstract. In this paper we prove that the system of all closed convex ℓ-subgroups of a convergence ℓ-group is a Brouwer lattice and that a similar result is valid for radical classes of convergence ℓ-groups.

Keywords: convergence ℓ-group, closed convex ℓ-subgroup, radical class of convergence ℓ-groups

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All ℓ-groups considered in the present paper are assumed to be abelian. For convergence ℓ-groups (shorter: cl-groups) we apply the same notation and definitions as in [6].

Let (G, α) be a cl-group (where G is an ℓ-group and α is a convergence in G). For the definition of a closed ℓ-subgroup (shorter: cl-subgroup) of (G, α) cf. Section 1 below. The system of all convex cl-subgroups of (G, α) will be denoted by $c(G, \alpha)$; this system is partially ordered by the set-theoretical inclusion.

In the present paper we prove that $c(G, \alpha)$ is a Brouwer lattice. The lattice operations in $c(G, \alpha)$ are constructively described.

For $X \subseteq G$ the meaning of $\lim_\alpha X$ is defined in a natural way. We show that if X is an ℓ-subgroup of G such that X can be represented as a direct product of a finite number of linearly ordered groups, then

$$\lim_\alpha \lim_\alpha X = \lim_\alpha X.$$

A nonempty class A of cl-groups is called a radical class of cl-groups if it is closed with respect to isomorphisms, convex cl-subgroups and joins of convex cl-subgroups. For radical classes A_1 and A_2 we put $A_1 \leqslant A_2$ if A_1 is a subclass of A_2.

We prove that a certain form of distributive law (analogous to the condition applied when defining a Brouwer lattice) is valid for radical classes of cl-groups.

The analogous notion of a radical class of ℓ-groups was introduced in [5] and studied in several papers (cf., e.g., [1], [2], [7], [8]).

1. Preliminaries

For an ℓ-group G we denote by $c(G)$ the system of all convex ℓ-subgroups of G; this system is partially ordered by the set theoretical inclusion. Then $c(G)$ is a complete lattice. The lattice operations in $c(G)$ will be denoted by $\vee$ and $\wedge$.

Let $\mathcal{G}$ be the class of all ℓ-groups. A nonempty subclass X of $\mathcal{G}$ is said to be a radical class of ℓ-groups if it satisfies the following conditions:

(i) X is closed with respect to isomorphisms.
(ii) Whenever $G \in X$ and $G_1 \in c(G)$, then $G_1 \in X$.
(iii) Whenever $G \in \mathcal{G}$ and $\{G_i\}_{i\in I}$ is a nonempty subset of $X \cap c(G)$, then $\bigvee\limits_{i\in I} G_i$ belongs to X.

We suppose that the reader is acquainted with the definitions from Section 1 of [6].

Let (G,α) and (G_1,α_1) be cl-groups.

1.1. Definition. (G_1,α_1) is said to be a cl-subgroup of (G,α) if

(i) G_1 is an ℓ-subgroup of G;
(ii) whenever (x_n) is a sequence in G_1, $x \in G$ and $x_n \to_\alpha x$, then $x \in G_1$ and $x_n \to_{\alpha_1} x$;
(iii) whenever (x_n) is a sequence in G_1, $x \in G_1$ and $x_n \to_{\alpha_1} x$, then $x_n \to_\alpha x$.

If (G_1,α_1) is a cl-subgroup of (G,α), then we often write (G_1,α) instead of (G_1,α_1).

The meaning of a convex cl-subgroup of (G,α) is obvious. The system of all convex cl-subgroups of (G,α) will be denoted by $c(G,\alpha)$. If (G_1,α_1) and (G_2,α_2) belong to $c(G,\alpha)$ and $G_1 \subseteq G_2$, then we put $(G_1,\alpha_1) \leqslant (G_2,\alpha_2)$. It is easy to verify that under the relation $\leqslant$, the system $c(G,\alpha)$ is a complete lattice. The lattice operations in $c(G,\alpha)$ will be denoted by $\vee^c$ and $\wedge^c$.

1.2. Definition. A mapping φ of G into G_1 is called a cl-homomorphism if

(i) φ is a homomorphism of the ℓ-group G into the ℓ-group G_1;
(ii) whenever (x_n) is a sequence in G, $x \in G$ and $x_n \to_\alpha x$, then $\varphi(x_n) \to_{\alpha_1} \varphi(x)$.

If there exists a cl-homomorphism of (G,α) onto (G_1,α_1), then (G_1,α_1) is said to be a homomorphic image of (G,α).

1.3. Definition. Let φ be a cl-homomorphism of (G,α) onto (G_1,α_1) such that

(i) φ is a monomorphism;
(ii) the inverse mapping φ^{-1} is a cl-homomorphism of (G_1,α_1) onto (G,α).

Then φ is an isomorphism of (G,α) onto (G_1,α_1); if such φ does exist, then (G_1,α_1) is said to be cl-isomorphic to (G,α).

Let $\mathcal{G}_c$ be the class of all cl-groups.

1.4. Definition. A nonempty subclass Y of $\mathcal{G}_c$ is said to be a radical class of a cl-group if the following conditions are satisfied:

(i) Y is closed with respect to cl-isomorphisms;
(ii) whenever $(G,\alpha)\in Y$ and $(G_1,\alpha_1)\in c(G,\alpha)$, then $(G_1,\alpha_1)\in Y$;
(iii) whenever $(G,\alpha)\in\mathcal{G}_c$ and $\{(G_i,\alpha_i)\}_{i\in I}$ is a nonempty subset of $Y\cap c(G,\alpha)$, then $\bigvee\limits_{i\in I}^{c}(G_i,\alpha_i)\in Y$.

We shall often apply without quotation the following facts:

(a_1) If $a_n\to_\alpha a$ and $a_n\leqslant a$ for each $n\in\mathbb{N}$, then $\bigvee\limits_{n\in\mathbb{N}} a_n=a$ (and dually).
(a_2) If G is linearly ordered, $a_n\to_\alpha a$, $c_1<a<c_2$, then there is $m\in\mathbb{N}$ such that for each $n>m$ the relation $c_1<a_n<c_2$ is valid.

(The assertion (a_1) is easy to verify; (a_2) is a consequence of (a_1).)

2. The system $c(G,\alpha)$

Again, let $(G,\alpha)\in\mathcal{G}_c$.

A subset S of G is said to be closed with respect to (G,α) if, whenever (x_n) is a sequence in S, $x\in G$ and $x_n\to_\alpha x$, then $x\in S$.

2.1. Lemma. *Let H be an ℓ-subgroup of G such that it is closed with respect to (G,α). For a sequence (x_n) in H and $x\in H$ we put $x_n\to_{\alpha(H)} x$ if $x_n\to_\alpha x$. Then*

(i) *$(H,\alpha(H))$ is a cl-group.*
(ii) *$(H,\alpha(H))$ is a cl-subgroup of (G,α).*

P r o o f. The first assertion is an immediate consequence of the definition of the cl-group. Since H is closed with respect to (G,α) and in view of (i), the assertion (ii) holds as well. $\square$

In view of the above remark concerning the notation (cf. Section 1) we will write (H,α) instead of $(H,\alpha(H))$.

Let X be a nonempty subset of G. We denote by $\lim\limits_{\alpha} X$ the set of all $y \in G$ such that there exists a sequence (x_n) in X with $x_n \to_\alpha y$.

2.2. Lemma. *Let H be an ℓ-subgroup of G. Then $\lim\limits_{\alpha} H$ is an ℓ-subgroup of G. If, moreover, H is convex in G, then $\lim\limits_{\alpha} H$ is convex in G as well.*

Proof. Let $y_1, y_2 \in \lim\limits_{\alpha} H$. Hence there are sequences (x_n^i) in H such that $x_n^i \to_\alpha y_i$ $(i=1,2)$. Thus $x_n^1 + x_n^2 \to_\alpha y_1 + y_2$, and analogously for the operations $\wedge$ and $\vee$. Also, $-x_n^1 \to -y_1$. Hence $\lim\limits_{\alpha} H$ is an ℓ-subgroup of G.

Now suppose that H is convex in G and that $z \in G$, $y_1 \leqslant z \leqslant y_2$. Then

$$x_n^1 \wedge x_n^2 \to_\alpha y_1, \quad x_n^1 \vee x_n^2 \to_\alpha y_2.$$

Put

$$z_n = ((x_n^1 \wedge x_n^2) \vee z) \wedge (x_n^1 \vee x_n^2).$$

Hence $z_n \in H$ and $z_n \to_\alpha (y_1 \vee z) \wedge y_2 = z$. Thus $z \in \lim\limits_{\alpha} H$. □

Let H be as in 2.2. We put $H_0 = H$ and for each ordinal $t > 0$ we construct H_t by transfinite induction as follows. Suppose that for $t_1 < t$ all H_{t_1} are already defined and that they are ℓ-subgroups of G such that, whenever $t_1 < t_2 < t$, then $H_{t_1} \subseteq H_{t_2}$. If t is a limit ordinal, then we put

$$H_t = \bigcup_{t_1 < t} H_{t_1}.$$

If t is non-limit, then there exists t_1 with $t = t_1 + 1$. In this case we set

$$H_t = \lim_{\alpha} H_{t_1}.$$

There exists an ordinal t such that $H_t = H_{t_2}$ whenever $t_2 > t$. We denote

$$\overset{\alpha}{\lim} H = H_t.$$

From 2.1, 2.2 and from the construction of $\overset{\alpha}{\lim} H$ we immediately obtain

2.3. Lemma. *Let H be an ℓ-subgroup of G. Put $\overset{\alpha}{\lim} H = H^*$. Then*

(i) *(H^*, α) is a cl-subgroup of (G,α);*
(ii) *if (K,α) is a cl-subgroup of (G,α) and $H \subseteq K$, then $H^* \subseteq K$;*
(iii) *if, moreover, H is convex in G, then H^* is convex in G as well.*

2.4. Lemma. *Let* $\{(H_i,\alpha)\}_{i\in I}$ *be a nonempty subset of* $c(G,\alpha)$. *Put* $H_0 = \bigcap\limits_{i\in I} H_i$, $H^0 = \bigvee\limits_{i\in I} H_i$. *Then*

(i) $\overset{c}{\bigwedge\limits_{i\in I}} (H_i,\alpha) = (H_0,\alpha)$;

(ii) $\overset{c}{\bigvee\limits_{i\in I}} (H_i,\alpha) = (\overset{\alpha}{\lim} H^0,\alpha)$.

Proof. The first assertion is obvious; the second is a consequence of 2.3. $\square$

2.5. Lemma. *Let* H *be an* ℓ*-subgroup of* G. *Then the following conditions are equivalent:*

(i) H *is closed with respect to* (G,α);

(ii) H^+ *is closed with respect to* (G,α).

Proof. Let (i) be valid and let (x_n) be a sequence in H^+, $x\in G$, $x_n \to_\alpha x$. Then $x_n = x_n \vee 0 \to_\alpha x\vee 0$, whence $x\vee 0 = x$ and thus (ii) holds. Conversely, suppose that (ii) is satisfied. Let (x_n) be a sequence in H, $x\in G$, $x_n\to_\alpha x$. Then $x_n^+ \to_\alpha x^+$ and $x_n^- \to_\alpha x^-$. We have $x_n^+, x_n^- \in H^+$ for each $n\in\mathbb{N}$ and thus, in view of (ii), both x^+ and x^- belong to H^+. Hence $x = x^+ - x^-$ is an element of H. $\square$

For subsets X and Y of G we denote

$$X - Y = \{x-y\colon x\in X \text{ and } y\in Y\}.$$

2.6. Lemma. *Let* X *be a subset of* G^+ *such that*

(i) X *is a sublattice and a subsemigroup of* G^+;

(ii) $0\in X$.

Then $X-X$ *is an* ℓ*-subgroup of* G *and* $(X-X)^+ = X$. *If, moreover,* X *is a convex subset of* G^+, *then* $X-X$ *is a convex* ℓ*-subgroup of* G.

The proof is routine, it will be omitted.

For each nonempty subset X of G we can perform an analogous construction as we did above for H; in this way we obtain a subset of G which will be denoted by $\overset{\alpha}{\lim} X$ or by X^*.

From the construction of X^* we immediately obtain

2.7. Lemma. *Let* X *be as in* 2.6. *Then*

(i) X^* *is a subset of* G^+ *and it satisfies the conditions* (i), (ii) *from* 2.6;

(ii) X^* *is closed with respect to* (G,α);

(iii) *if, moreover,* X *is convex in* G, *then* X^* *is convex in* G *as well.*

2.8. Lemma. *Let H be an ℓ-subgroup of G; put $X = H^+$. Then $H^* = X^* - X^*$.*

Proof. In view of the constructions of H^* and X^* we have $X^* \subseteq H^*$. Then according to 2.3 (i), $X^* - X^* \subseteq H^*$. Further, 2.7 and 2.5 yield that $X^* - X^*$ is closed with respect to (G, α). Moreover, $H = H^+ - H^+ \subseteq X^* - X^*$. Hence according to 2.3 (ii) we obtain the relation $H^* \subseteq X^* - X^*$, which completes the proof. $\square$

2.9. Lemma. *Let $\{(H_i, \alpha)\}_{i \in I}$ and H^0 be as in 2.4. Put $(H^0)^+ = X$. Then*

$$\bigvee_{i \in I}^{c} (H_i, \alpha) = X^* - X^*.$$

Proof. This is a consequence of 2.4 and 2.8. $\square$

Now, let (A, α) and (B_i, α) $(i \in I)$ be elements of $c(G, \alpha)$. Put

$$X = X_0 = \Big(\bigvee_{i \in I} B_i\Big)^+,$$

and let X^* be as above. For each ordinal t we define X_t analogously as when defining H_t.

Further, we put

$$Y = Y_0 = \Big(\bigvee_{i \in I} (A \wedge B_i)\Big)^+;$$

the symbols Y^* and Y_t are defined analogously as X^* and X_t.

It is well-known that the relation

$$A \wedge \Big(\bigvee_{i \in I} B_i\Big) = \bigvee_{i \in I} (A \wedge B_i)$$

is valid (cf., e.g., [5]). From this relation we immediately obtain that

$$A \wedge X_0 = Y_0$$

holds. Let t be an ordinal with $t > 0$ and assume that for each ordinal $t_1 < t$ the relation

$$A \wedge X_{t_1} = Y_{t_1}$$

is valid.

a) Suppose that t is a limit ordinal. Then we have

$$Y_t = \bigcup_{t_1<t} Y_{t_1} = \bigcup_{t_1<t} (A \wedge X_{t_1}) = \bigcup_{t_1<t} (A \cap X_{t_1})$$
$$= A \cap \left(\bigcup_{t_1<t} X_{t_1} \right) = A \cap X_t = A \wedge X_t.$$

b) Further, suppose that t is a non-limit ordinal. Hence there is an ordinal t_1 with $t = t_1 + 1$. Then

$$X_t = \lim_{\alpha} X_{t_1}, \quad Y_t = \lim_{\alpha} Y_{t_1} = \lim_{\alpha} (A \cap X_{t_1}).$$

Let $z \in A \wedge X_t$. Hence $z \in A$ and $z \in X_t$. Also, $z \geqslant 0$. There exists a sequence (z_n) in X_{t_1} such that $z_n \to_\alpha z$. Clearly $z_n \geqslant 0$. Then $0 \leqslant z_n \wedge z \leqslant z$, whence $z_n \wedge z \in A \cap X_{t_1} = Y_{t_1}$ and $z_n \wedge z \to_\alpha z$. Thus $z \in Y_t$ and therefore $A \wedge X_t \subseteq Y_t$.

Assume that $v \in Y_t$. There exists a sequence (v_n) in Y_{t_1} with $v_n \to_\alpha v$. We have $v_n \in A$ for each $n \in \mathbb{N}$. Since A is closed with respect to (G, α) we obtain that $v \in A$. Further, $v_n \in X_{t_1}$ for each $n \in \mathbb{N}$ and thus $v \in X_t$. Therefore $v \in A \wedge X_t$.

By summarizing, we obtain the relation

$$A \wedge X_t = Y_t$$

for each ordinal t. Thus

$$(*) \qquad A \wedge X^* = Y^*.$$

2.10. Theorem. *Let (A, α) and (B_i, α), $i \in I$, be elements of (G, α). Then*

$$(A, \alpha) \wedge^c \left(\bigvee_{i \in I}^{c} (B_i, \alpha) \right) = \bigvee_{i \in I}^{c} ((A, \alpha) \wedge^c (B_i, \alpha)).$$

Proof. This is a consequence of 2.8, 2.9 and of the relation $(*)$. $\square$

2.11. Corollary. *The system $c(G, \alpha)$ is a Brouwer lattice.*

Let the symbol ω_1 have the usual meaning. It is easy to verify that if X is a nonempty subset of G and if t is an ordinal with $X^* = X_t$, then $t \leqslant \omega_1$.

If t is the first ordinal with $X^* = X_t$, then t will be said to be the degree of X in (G, α).

Further, let t' be the first ordinal such that, whenever X is a nonempty subset of G, then the degree of X in (G, α) is less or equal to t'. We denote $d(G, \alpha) = t'$.

The following questions remain open:

a) For which ordinals t there exist $(G,\alpha) \in \mathcal{G}_c$ and $X \subseteq G$ such that t is the degree of X in (G,α)?

b) For which ordinals t there exists $(G,\alpha) \in \mathcal{G}_c$ such that $d(G,\alpha) = t$?

For a related open question concerning convergence groups cf. [3].

3. The condition $\overset{2}{\lim\limits_{\alpha}} X = \lim\limits_{\alpha} X$

Let (G,α) be as above. For $X \subseteq G$ we denote $\lim\limits_{\alpha}\lim\limits_{\alpha} X = \overset{2}{\lim\limits_{\alpha}} X$. In this section we prove that if X is an ℓ-subgroup of G such that X is a direct product of a finite number of linearly ordered groups, then the relation

$$\overset{2}{\lim_{\alpha}} X = \lim_{\alpha} X \tag{1}$$

is valid. In other words, the degree of X is either 0 or 1.

3.1. Lemma. *Let X be a linearly ordered ℓ-subgroup of G and $g \in \lim\limits_{\alpha} X$. Then the set $X \cup \{g\}$ is linearly ordered and there are $x^1, x^2 \in X$ such that $x^1 \leqslant g \leqslant x^2$.*

Proof. In the case $X = \{0\}$ we have $g = 0$. Assume that $X \neq \{0\}$. Then there exists $x_0^1 \in X$ with $x_0^1 > 0$. First we prove that the element g cannot be an upper bound of the set X. By way of contradiction, suppose that $g > x$ for each $x \in X$.

Since $g \in \lim\limits_{\alpha} X$, there is a sequence (x_n) in X such that $x_n \to_\alpha g$. Because $x_n \leqslant g$ for each $n \in \mathbb{N}$, we obtain that

$$\sup\{x_n\}_{n\in\mathbb{N}} = g$$

and this yields that $\sup X = g$. For each $x \in X$ we have $x + x_0^1 \in X$, thus $x + x_0^1 \leqslant g$, hence $x \leqslant g - x_0^1 < g$. This is a contradiction with the relation $\sup X = g$. Hence there is $x^2 \in X$ such that $x^2 \nleqq g$.

If x^2 is any element of X with this property, then there is a positive integer $m(x^2)$ such that for each $n \in \mathbb{N}$ with $n \geqslant m(x^2)$ we have $x_n \leqslant x^2$ (otherwise the relation $g \geqslant x^2$ would be valid). Then $g \leqslant x^2$. By a dual argument we prove that there is $x^1 \in X$ with $x^1 \leqslant g$. Moreover, if x^3 is any element of X with $x^3 \ngeqq g$, then $g \geqslant x^3$. $\square$

3.2. Lemma. *Let X be a linearly ordered ℓ-subgroup of G. Then $\lim\limits_{\alpha} X$ is also a linearly ordered ℓ-subgroup of G.*

P r o o f. In view of 2.2, $\lim_\alpha X$ is an ℓ-subgroup of G. Hence it suffices to verify that whenever g_1 and g_2 are distinct elements of $\lim_\alpha X$, then g_1 and g_2 are comparable. In view of 3.1 there are ideals X_1 and X_2 of the linearly ordered set X such that

(i) $X_1 \neq X \neq X_2$;
(ii) $x \leqslant g_1$ if $x \in X_1$, and $x > g_1$ if $x \in X \setminus X_1$;
(iii) $x \leqslant g_2$ if $x \in X_2$, and $x > g_2$ if $x \in X \setminus X_2$.

The ideals X_1 and X_2 are comparable. Since $g_1 \neq g_2$, we obtain that $X_1 \neq X_2$. Thus without loss of generality we can suppose that $X_1 \subset X_2$. Hence there is $z \in X_2 \setminus X_1$. Then in view of (ii), $z > g_1$. Further, according to (iii), $z \leqslant g_2$. Therefore $g_1 \leqslant g_2$. □

3.3. Lemma. *Let X be a linearly ordered ℓ-subgroup of G. Then (1) holds.*

P r o o f. Let (y_n) be a sequence in $\lim_\alpha X$, $g \in G$ and $y_n \rightarrow_\alpha g$. Then in view of 3.1 and 3.2, g is comparable with all elements of $\lim_\alpha X$. Hence there exists a subsequence (y_n^1) of (y_n) such that either (i) $y_n^1 \geqslant g$ for each $n \in \mathbb{N}$, or (ii) $y_n^1 \leqslant g$ for each $n \in \mathbb{N}$. Suppose that (i) holds (in the case of (ii) the method is similar). If $y_n^1 = g$ for some $n \in \mathbb{N}$, then $g \in \lim_\alpha X$. Thus it suffices to suppose that $y_n^1 < g$ for each $n \in \mathbb{N}$, and in this case we can assume without loss of generality that $y_n^1 < y_{n+1}^1$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. There exists a sequence $(x_k^n)_{k\in\mathbb{N}}$ in X such that $x_k^n \rightarrow_\alpha y_n^1$ (as $k \rightarrow \infty$). Hence there is $m(n) \in \mathbb{N}$ such that

$$y_{n-1}^1 < x_k^n < y_{n+1}^1$$

whenever $k \geqslant m(n)$. Since $y_{n-1}^1 \rightarrow_\alpha g$ and $y_{n+1}^1 \rightarrow_\alpha g$ we obtain that

$$x_{m(n)}^n \rightarrow_\alpha g$$

and thus $g \in \lim_\alpha X$. Hence (1) is valid. □

3.4. Lemma. *Let L be a distributive lattice with the least element 0. Let A and B be sublattices of L such that*

(i) *$0 \in A \cap B$;*
(ii) *$a \wedge b = 0$ for each $a \in A$ and each $b \in B$;*
(iii) *for each $g \in L$ there are $a \in A$ and $b \in B$ with $g = a \vee b$.*

Then the elements a, b from (iii) *are uniquely determined and the mapping $g \to (a, b)$ gives an isomorphism of L onto the direct product $A \times B$.*

The proof is routine, it will be omitted.

3.5. Lemma. *Let X be an ℓ-subgroup of G such that X is a direct product of linearly ordered groups $X_1, X_2, \ldots, X_k$. Then the ℓ-group $\lim\limits_{\alpha} X$ is a direct product of linearly ordered groups $\lim\limits_{\alpha} X_1, \lim\limits_{\alpha} X_2, \ldots, \lim\limits_{\alpha} X_k$.*

P r o o f. We proceed by induction with respect to k. The case $k = 1$ is trivial. Suppose that $k > 1$ and that the assertion is valid for $k - 1$.

Without loss of generality we can assume that $X_i \neq \{0\}$ for $i = 1, 2, \ldots, k$. Put $Y_i = \lim\limits_{\alpha} X_i$ $(i = 1, 2, \ldots, k)$. According to 3.2, all Y_i are linearly ordered ℓ-subgroups of G. Also, $\lim\limits_{\alpha} X = Y$ is an ℓ-subgroup of G. In view of Theorem 2.3, [4] it suffices to verify that the lattice Y^+ is a direct product of lattices $Y_1^+, \ldots, Y_k^+$.

Let $g \in Y^+$. In the same way as in the proof of 3.1 we can verify that g fails to be an upper bound of the set X^+. For each $x \in X^+$ we have

$$x = x(X_1) \vee \ldots \vee x(X_k), \quad x(X_i) \geqslant 0 \quad (i = 1, 2, \ldots, k),$$

where $x(X_i)$ is the component of x in X_i. Hence g fails to be an upper bound of the set $X_1^+ \cup X_2^+ \cup \ldots \cup X_k^+$. Thus we can suppose that g is not an upper bound of the set X_k^+. Therefore there is $x_0 \in X_k^+$ such that $x_0 \nleqq g$.

There is a sequence (z_n) in X such that $z_n \to_\alpha g$. Put $z_n' = z_n \vee 0$. Then we have $z_n' \to_\alpha g$ as well. Further,

$$z_n' \wedge x_0 = (z_n'(X_1) \vee z_n'(X_2) \vee \ldots \vee z_n'(X_k)) \wedge x_0 =$$

$$= z_n'(X_k) \wedge x_0 \in X_k$$

and $z_n'(X_k) \wedge x_0 \to_\alpha g \wedge x_0$, whence $g \wedge x_0 \in \lim\limits_{\alpha} X_k \subseteq \lim\limits_{\alpha} X$.

Put $N_1 = \{n \in \mathbb{N}\colon z_n'(X_k) \geqslant x_0\}$. If the set N_1 is infinite, then there exists a subsequence (z_n'') of (z_n') such that $z_n'' \geqslant x_0$ for each $n \in \mathbb{N}$ and then we would have $g \geqslant x_0$, which is a contradiction. Hence the set N_1 is finite; thus there is a subsequence (z_n'') of (z_n') such that $z_n''(X_k) < x_0$ for each $n \in \mathbb{N}$, whence

$$z_n''(X_k) \wedge x_0 = z_n''(X_k)$$

and then $z_n''(X_k) \to_\alpha g \wedge x_0$. Therefore

$$z_n'' - z_n''(X_k) = z_n''(X_1) + z_n''(X_2) + \ldots + z_n''(X_{k-1}) \to_\alpha g - (g \wedge x_0).$$

Therefore by the induction hypothesis (since $z''_n(X_1) + \ldots + z''_n(X_{k-1})$ belongs to $X_1 \times \ldots \times X_{k-1}$) the element $g - (g \wedge x_0)$ belongs to the direct product $Y_1 \times Y_2 \times \ldots \times Y_{k-1}$. Since $g - (g \wedge x_0) \geqslant 0$ we obtain, moreover, that this element belongs to the direct product of lattices $Y_1^+, \ldots, Y_{k-1}^+$.

Let $t \in Y_1^+ \times Y_2^+ \times \ldots \times Y_{k-1}^+$. Then by the induction hypothesis, there is a sequence (t_n) in $X_1^+ \times \ldots \times X_{k-1}^+$ such that $t_n \to_\alpha t$. We have

$$t_n \wedge z''_n(X_k) = 0 \quad \text{for each } n \in \mathbb{N},$$

thus

$$t \wedge (g \wedge x_0) = 0,$$
$$t + (g \wedge x_0) = t \vee (g \wedge x_0).$$

In particular,

$$g = (g - (g \wedge x_0)) + (g \wedge x_0) = (g - (g \wedge x_0)) \vee (g \wedge x_0)$$

with $g - (g \wedge x_0) \in Y_1^+ \times \ldots \times Y_{k-1}^+$ and $g \wedge x_0 \in Y_k^+$.

Hence in view of 3.4 we obtain that for the lattice Y^+ there exists a direct product decomposition

$$Y^+ = Y_1^+ \times Y_2^+ \times \ldots \times Y_k^+.$$

Now we apply again Theorem 2.3 of [4] concluding that the ℓ-group Y has a direct product decomposition

$$Y = Y_1 \times Y_2 \times \ldots \times Y_k. \tag{2}$$

□

3.6. Theorem. *Let X be an ℓ-subgroup of G such that X is a direct product of a finite number of linearly ordered groups. Then* (1) *holds.*

Proof. We apply the notation as in the proof of 3.5 and similarly as in 3.5 we proceed by induction with respect to k. The case $k = 1$ was dealt with in 3.3; let $k > 1$.

Since all Y_i are linearly ordered we can apply 3.5 to the relation (2) obtaining

$$\lim_\alpha Y = \lim_\alpha Y_1 \times \lim_\alpha Y_2 \times \ldots \times \lim_\alpha Y_k.$$

Since $\lim_\alpha Y = \overset{2}{\lim_\alpha} X$ and $\lim_\alpha Y_i = \overset{2}{\lim_\alpha} X_i$ $(i = 1, 2, \ldots, k)$, by applying 3.3 we infer

$$\overset{2}{\lim_\alpha} X = Y_1 \times \ldots \times Y_k = \lim_\alpha X.$$

□

For a class X of cl-groups we denote by

$\operatorname{Sub}_0 X$—the class of all cl-groups (G,α) having the property that there exist (H,β) in X and $(H_1,\beta)\in c(H,\beta)$ such that (G,α) and (H_1,β) are cl-isomorphic;

$\operatorname{Join} X$—the class of all cl-groups (G,α) having the property that there exist (H_i,β_i) in X and $(G_i,\alpha)\in c(G,\alpha)$ $(i\in I)$ such that

a) for each $i\in I$, (H_i,β_i) and (G_i,α) are cl-isomorphic, and

b) $(G,\alpha)=\bigvee\limits_{i\in I}^{c}(G_i,\alpha)$.

4.1. Proposition. *Let X be a nonempty class of cl-groups. Then*

a) $\operatorname{Join}\operatorname{Sub}_c X$ *is a radical class of cl-groups.*

b) *If Y is a radical class of cl-groups and $X\subseteq Y$, then* $\operatorname{Join}\operatorname{Sub}_c X\subseteq Y$.

Proof. Put $\operatorname{Join}\operatorname{Sub}_c X=Z$. We have to verify that Z satisfies the conditions (i), (ii) and (iii) from 1.4. It is obvious that Z is closed with respect to cl-isomorphisms. For each nonempty class Z_1 of cl-groups we have $\operatorname{Join}\operatorname{Join} Z_1=\operatorname{Join} Z_1$, whence Z satisfies the condition (iii) from 1.4.

Let $(G,\alpha)\in Z$ and $(G_1,\alpha)\in c(G,\alpha)$. Hence there exist (H_i,α_i) $(i\in I)$ belonging to $\operatorname{Sub}_c X\cap c(G,\alpha)$ such that

$$(G,\alpha)=\bigvee_{i\in I}^{c}(H_i,\alpha).$$

Then by applying 2.10

$$\begin{aligned}(G_1,\alpha)&=(G_1,\alpha)\wedge^c(G,\alpha)=(G_1,\alpha)\wedge^c\Big(\bigvee_{i\in I}^{c}(H_i,\alpha)\Big)\\&=\bigvee_{i\in I}^{c}((G_1,\alpha)\wedge^c(H_i,\alpha)).\end{aligned}$$

For each $i\in I$, the cl-group $(G_1,\alpha)\wedge^c(H_i,\alpha)$ belongs to $\operatorname{Sub}_c\operatorname{Sub}_c X=\operatorname{Sub}_c X$ and therefore (G_1,α) belongs to Z. Hence the condition (ii) from 1.3 is valid, which completes the proof of a).

Let Y be a radical class of cl-groups and $X\subseteq Y$. Then $\operatorname{Sub}_c X\subseteq\operatorname{Sub}_c Y=Y$ and $\operatorname{Join}\operatorname{Sub}_c X\subseteq\operatorname{Join} Y=Y$. Thus b) is valid. $\square$

Let Y_1 and Y_2 be radical classes of cl-groups. We put $Y_1\leqslant Y_2$ if Y_1 is a subclass of Y_2.

We denote by Y_0 the class of all cl-groups (G,α) such that G is a one-element set. Then Y_0 is a radical class of cl-groups and for each radical class Y of cl-groups we have $Y_0 \leqslant Y \leqslant \mathcal{G}_c$.

Let G be an ℓ-group. For a sequence (x_n) in G and for $x \in G$ we put $x_n \to_{\alpha(G)} x$ if there exists $m \in \mathbb{N}$ such that $x_n = x$ for each positive integer n with $n \geqslant m$. Then $(G,\alpha(G))$ is a cl-group; $\alpha(G)$ is the discrete convergence on G.

If X is a class of ℓ-groups, then we put

$$\varphi(X) = \{(G;\alpha(G)) \colon G \in X\}.$$

Then we obviously have

4.2. Lemma. *If X is a radical class of ℓ-groups, then $\varphi(X)$ is a radical class of cl-groups. Moreover, if X_1 and X_2 are distinct radical classes of ℓ-groups, then $\varphi(X_1) \neq \varphi(X_2)$.*

Let $\mathcal{R}_a$ and $\mathcal{R}_c$ be the collection of all radical classes of ℓ-groups or the collection of all radical classes of cl-groups, respectively. (Let us remark that in [5] the symbol $\mathcal{R}$ was used, but in [5] it was not assumed that the ℓ-groups under consideration were abelian.)

There exists an injective mapping of the class of all infinite cardinals into $\mathcal{R}_a$ (this follows from the construction in [5], Section 3). Hence in view of 4.2, there exists an injective mapping of the class of all infinite cardinals into $\mathcal{R}_c$.

Suppose that I is a nonempty class and that for each $i \in I$, Y_i is a radical class of cl-groups. Put

$$Z_1 = \bigcap_{i\in I} Y_i.$$

Then in view of 1.4, Z_1 is a radical class of cl-groups. We obviously have

$$Z_1 = \inf\{Y_i\}_{i\in I}.$$

We express this fact by writing

$$Z_1 = \bigwedge_{i\in I} Y_i.$$

Further, we put

$$Z_2 = \operatorname{Join} \operatorname*{Sub}_c \bigcup_{i\in I} Y_i.$$

Then 4.1 yields that the relation

$$Z_2 = \sup\{Y_i\}_{i\in I}$$

is valid in $\mathcal{R}_c$. We express this fact by writing

$$Z_2 = \bigvee_{i\in I} Y_i.$$

We clearly have

$$\operatorname*{Sub}_c \bigcup_{i\in I} Y_i = \bigcup_{i\in I} \operatorname*{Sub}_c Y_i.$$

Since each Y_i is a radical class of cl-groups we obtain $\mathrm{Sub}_c\, Y_i = Y_i$. Hence

$$\bigvee_{i\in I} Y_i = \mathrm{Join} \bigcup_{i\in I} Y_i.$$

4.3. Theorem. *Let $\{Y_i\}_{i\in I}$ be as above and let Y be a radical class of cl-groups. Then*

$$Y \wedge \Big(\bigvee_{i\in I} Y_i\Big) = \bigvee_{i\in I} (Y \wedge Y_i).$$

Proof. We have

$$\bigvee_{i\in I} (Y \wedge Y_i) \leqslant Y \wedge \Big(\bigvee_{i\in I} Y_i\Big).$$

Let $(G,\alpha) \in Y \wedge \big(\bigvee_{i\in I} Y_i\big)$. Thus $(G,\alpha) \in Y$ and

$$(G,\alpha) \in \mathrm{Join} \bigcup_{i\in I} Y_i.$$

Then there exist cl-groups (G_k,α) $(k \in K)$ such that, for each $k \in K$,

$$(G_k,\alpha) \in c(G,\alpha) \cap \Big(\bigcup_{i\in I} Y_i\Big)$$

and

$$(G,\alpha) = \bigvee_{k\in K}^{c} (G_k,\alpha).$$

Hence for each $k \in K$ there exists $i(k) \in I$ with $(G_k,\alpha) \in Y_{i(k)}$. Denote

$$I_1 = \{i(k) \colon k \in K\}.$$

Thus $(G_k,\alpha) \in Y \wedge Y_{i(k)}$ and

$$(G,\alpha) \in \mathrm{Join} \bigcup_{i\in I_1} (Y \wedge Y_{i(k)}) \leqslant \mathrm{Join} \bigcup_{i\in I} (Y \wedge Y_i) = \bigvee_{i\in I} (Y \wedge Y_i).$$

$\square$

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