NOETHER THEOREM AND FIRST INTEGRALS OF CONSTRAINED LAGRANGEAN SYSTEMS

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Abstract. The dynamics of singular Lagrangean systems is described by a distribution the rank of which is greater than one and may be non-constant. Consequently, these systems possess two kinds of conserved functions, namely, functions which are constant along extremals (constants of the motion), and functions which are constant on integral manifolds of the corresponding distribution (first integrals). It is known that with the help of the (First) Noether theorem one gets constants of the motion. In this paper it is shown that every constant of the motion obtained from the Noether theorem is a first integral; thus, Noether theorem can be used for an effective integration of the corresponding distribution.

Keywords: Lagrangean system, Lepagean two-form, Euler-Lagrange form, singular Lagrangian, constrained system, Noether theorem, symmetry, constant of the motion, first integral.

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1. Introduction

The theory of symmetries and first integrals becomes more complicated if, instead of classical Lagrangean systems, one considers *constrained systems*, i.e., Lagrangean systems coming from singular Lagrangians (cf. [1], [2], [3], [11], and references therein).

According to [8], every mechanical Lagrangean system can be represented by a certain closed two-form, called the *Lepagean two-form*, and the corresponding dynamics is described by the characteristic distribution of the Lepagean two-form. For regular Lagrangean systems this distribution is of rank one, i.e., locally spanned by *one vector field* (Hamilton vector field), and the integral sections of this vector field

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are in one-to-one correspondence with the extremals. For constrained systems, however, this distribution is of rank greater than one, or even of a non-constant rank. Moreover, the correspondence between extremals and integral sections of the distribution is no more one-to-one. Consequently, constrained systems possess two kinds of conserved functions: functions which are constant along extremals, called *constants of the motion*, and functions which are constant on integral manifolds of the characteristic distribution, called *first integrals* [10]. While for Lagrangean systems of classical mechanics first integrals and constants of the motion coincide, constrained systems may possess constants of the motion which are not first integrals.

The aim of this paper is to complete the results of [10], where symmetries and conserved functions for constrained systems in higher-order mechanics were studied. In particular, it is well-known that from the First Theorem of E. Noether one gets constants of the motion. Here we shall show that *every constant of the motion obtained in this way is a first integral*. In the light of recently developed geometric integration methods for constrained Lagrangean systems based on first integrals [9], this result increases the applicability of the Noether Theorem for practical integration of constrained systems.

2. Lagrangean systems

Throughout this paper, we denote by ∂ the Lie derivative, i the contraction, * the pull-back, and J^r the r-jet prolongation. We consider a fibered manifold $\pi\colon Y \to X$, $\dim X = 1$, $\dim Y = m+1$, and its jet prolongations $\pi_r\colon J^rY \to X$, $r \geqslant 1$. Local fibered coordinates on Y (resp. associated coordinates on J^rY) are denoted by (t, q^σ) (resp. $(t, q^\sigma, q^\sigma_1, \ldots, q^\sigma_r)$). Recall that the projection $\pi_{r,0}\colon J^rY \to Y$ is a fibered manifold. A section δ of the fibered manifold π_r is called *regular* if $\delta = J^r\gamma$ for a section γ of π. A form ϱ on J^rY is called *horizontal* (with respect to the projection π_r) if $i_\xi\varrho = 0$ for every π_r-vertical vector field on J^rY, and is called *contact* if $J^r\gamma^*\varrho = 0$ for every (local) section γ of π. The ideal of contact forms on J^rY is generated by the contact one-forms

$$\omega^\sigma_i = \mathrm{d}q^\sigma_i - q^\sigma_{i+1}\,\mathrm{d}t, \quad 0 \leqslant i \leqslant r-1.$$

A contact form ϱ is called *one-contact* (resp. *two-contact*) if for every vertical vector field ξ the form $i_\xi\varrho$ is horizontal (resp. one-contact). Notice that every one-contact (resp. two-contact) two-form ϱ is expressed in fibered coordinates by $\varrho = \sum\limits_{i=0}^{r-1} \varrho^i_\sigma\,\omega^\sigma_i \wedge \mathrm{d}t$ (resp. $\varrho = \sum\limits_{i,k=0}^{r-1} \varrho^{ik}_{\sigma\nu}\,\omega^\sigma_i \wedge \omega^\nu_k$). We use the notation h, p, p_1, and p_2 for the horizontal, contact, one-contact, and two-contact part of forms, respectively.

Every one-form (resp. two-form) ϱ on J^rY admits a unique decomposition into a sum of a horizontal and a contact form (resp. a one-contact and a two-contact form), namely

$$\pi^*_{r+1,r}\varrho = h\varrho + p\varrho, \quad \text{resp.} \quad \pi^*_{r+1,r}\varrho = p_1\varrho + p_2\varrho.$$

For more details on the calculus of horizontal and contact forms on fibered manifolds we refer to [7].

Horizontal one-forms on J^rY are called *Lagrangians of order* r. One-contact two-forms on J^rY, horizontal with respect to the projection $\pi_{r,0}$, are called *dynamical forms of order* r. A closed two-form α on J^rY, $r \geqslant 0$, is called a *Lepagean two-form* of order r if $p_1\alpha$ is a dynamical form. If α is a Lepagean two-form then every (local) one-form θ such that $\alpha = \mathrm{d}\theta$ is called a *Cartan form* (or a *Lepagean one-form*).

Lepagean two-forms were introduced in [8] as forms generalizing symplectic, presymplectic, cosymplectic and precosymplectic forms to any Lagrangian, and to any finite order. Lepagean two-forms are global closed counterparts of variational equations, and they are *equivalent with Euler-Lagrange equations* in the following sense: If α is a Lepagean two-form (of order r) then the form $p_1\alpha$ is *locally variational*, i.e., its fiber-chart components are Euler-Lagrange expressions (of order $r+1$); conversely, to any Euler-Lagrange expressions E_σ, $1 \leqslant \sigma \leqslant m$, of order r one can find a unique Lepagean two-form α of order $r-1$ such that the E_σ's are the components of $p_1\alpha$. Moreover, regular integral sections of the characteristic distribution of a Lepagean two-form coincide with the solutions of the corresponding Euler-Lagrange equations (extremals).

The above properties of Lepagean two-forms enable us to introduce the concept of a *Lagrangean system of order* r as a *Lepagean two-form on* J^rY; the manifold Y is then called the *configuration space*, and J^rY is called the *phase space*. Notice that in this setting, a Lagrangean system is a *family of Lagrangians* which generally are local and of different orders.

Let $s \geqslant 1$, and let α be a Lagrangean system of order $s-1$. Then on the phase space one has two distributions, the *characteristic distribution* $\mathcal{D}$ of α, spanned by the system of one-forms $i_\xi\alpha$ where ξ runs over the set of *all* vector fields on $J^{s-1}Y$, and the *Euler-Lagrange distribution* Δ, spanned by the system of one-forms $i_\xi\alpha$ where ξ runs over the set of π_{s-1}-*vertical* vector fields on $J^{s-1}Y$. These distributions are generally different, and of a non-constant rank. Integral sections of the Euler-Lagrange distribution are called Hamilton extremals. Regular Hamilton extremals are in one-to-one correspondence with extremals, and coincide with regular integral sections of the characteristic distribution.

3. Point symmetries of Lagrangean systems

Let α be a Lepagean two-form (Lagrangean system) of order $s-1$. A π-projectable vector field ξ on an open subset of Y will be called *a point symmetry of the Lagrangean system* α if one of the following conditions is satisfied:

(1) $\partial_{J^{s-1}\xi}\alpha = 0$,
(2) $\partial_{J^s\xi}E = 0$, where E is the locally variational form defined by $E = p_1\alpha$,
(3) there exists an integer r and a (local) Lagrangian of order r for α such that $\partial_{J^r\xi}\lambda = 0$,
(4) there exists an integer r and a (local) Lepagean one-form θ of order r for α such that $\partial_{J^r\xi}\theta = 0$.

Recall that the equation $\partial_{J^r\xi}\lambda = 0$ (resp. $\partial_{J^s\xi}E_\lambda = 0$) is called the *Noether equation* (resp. the *Noether-Bessel-Hagen equation*).

If λ is a (possibly local) Lagrangian, denote by θ_λ its Cartan form, and by E_λ its Euler-Lagrange form. Let us recall a theorem relating various point symmetries of a Lagrangean system represented by a two-form α [10].

Theorem 3.1. *Let ξ be a π-projectable vector field on Y, let λ be a Lagrangian for α.*

(1) *If ξ is a point symmetry of θ_λ then it is a point symmetry of λ, and of α.*
(2) *If ξ is a point symmetry of α then it is a point symmetry of E_λ.*
(3) *If ξ is a point symmetry of λ then it is a point symmetry of E_λ.*

It is known that if ξ is a point symmetry of an Euler-Lagrange form E, and if λ is a Lagrangian of E then ξ need not be a point symmetry of λ. Similarly, one can find examples showing that if ξ is a point symmetry of a Lepagean two-form α, and if θ_λ is a corresponding Lepagean one-form then ξ need not be a point symmetry of θ_λ. However, let us show that *mechanical systems* possess the following properties:

Theorem 3.2. *Let ξ be a π-projectable vector field on Y. Let α be a Lagrangean system, λ a Lagrangian for α.*

(1) *If ξ is a point symmetry of λ then it is a point symmetry of θ_λ.*
(2) *If ξ is a point symmetry of E_λ then it is a point symmetry of α.*

The proof is based on the following assertions.

Proposition 3.1. [6] *Let ξ be a π-projectable vector field on Y. If θ is a Lepagean one-form on J^rY then $\partial_{J^r\xi}\theta$ is a Lepagean one-form and*

$$h\partial_{J^r\xi}\theta = \partial_{J^{r+1}\xi}h\theta.$$

Proposition 3.2. *Let ξ be a π-projectable vector field on Y. If α is a Lepagean two-form on $J^{s-1}Y$ then $\partial_{J^{s-1}\xi}\alpha$ is a Lepagean two-form and*

$$p_1 \partial_{J^{s-1}\xi}\alpha = \partial_{J^s\xi} p_1 \alpha.$$

Proof of the propositions. Obviously, to prove that $\partial_{J^r\xi}\theta$ is a Lepagean one-form, and $\partial_{J^r\xi}\alpha$ is a Lepagean two-form, it is sufficient to show that for every π-projectable vector field ξ on Y, the form $p_1\, \mathrm{d}(i_{J^{s-1}\xi}\alpha)$ is dynamical. Expressing α in the form

$$\pi^*_{s,s-1}\alpha = E_\sigma\, \omega^\sigma \wedge \mathrm{d}t + \sum_{j,k=0}^{s-1} F^{jk}_{\sigma\nu}\, \omega^\sigma_j \wedge \omega^\nu_k,$$

computing $p_1\, \mathrm{d}(i_{J^{s-1}\xi}\alpha)$, and using the identities

$$\begin{aligned}
\frac{\partial E_\sigma}{\partial q^\nu_k} - 2\frac{\mathrm{d}}{\mathrm{d}t}F^{0k}_{\sigma\nu} - F^{0,k-1}_{\sigma\nu} &= 0, \quad 1 \leqslant k \leqslant s-1,\\
\frac{\partial E_\sigma}{\partial q^\nu_s} - 2F^{0,s-1}_{\sigma\nu} &= 0,\\
\frac{\mathrm{d}}{\mathrm{d}t}F^{jk}_{\sigma\nu} + F^{j-1,k}_{\sigma\nu} + F^{j,k-1}_{\sigma\nu} &= 0, \quad 1 \leqslant j,k \leqslant s-1,\\
F^{s-1,k}_{\sigma\nu} &= 0, \quad 1 \leqslant k \leqslant s-1,
\end{aligned}$$

[8], and the prolongation formula

$$\xi^\sigma_i = \frac{\mathrm{d}\xi^\sigma_{i-1}}{\mathrm{d}t} - q^\sigma_i\, \frac{\mathrm{d}\xi^0}{\mathrm{d}t}, \quad 1 \leqslant i \leqslant s-1$$

for the components of ξ [5], we get the result. These computations also yield

$$p_1\, \mathrm{d}(i_{J^{s-1}\xi}\alpha) = i_{J^s\xi}\, \mathrm{d}E + \mathrm{d}i_{J^s\xi}E,$$

where $E = p_1\alpha$, proving that $p_1\partial_{J^{s-1}\xi}\alpha = \partial_{J^s\xi}p_1\alpha$.

The relation $h\partial_{J^r\xi}\theta = \partial_{J^{r+1}\xi}h\theta$ is the infinitesimal first variation formula [5]. □

Using the above propositions and the relation

$$\partial_{J^{2r}\xi}E_\lambda = E_{\partial_{J^r\xi}\lambda}$$

[6], we immediately get

Corollary. *Let ξ be a π-projectable vector field on Y, λ a Lagrangian on J^rY. Then*

$$\partial_{J^{2r-1}\xi}\theta_\lambda = \theta_{\partial_{J^r\xi}\lambda}, \quad \partial_{J^{2r-1}\xi}\alpha_{E_\lambda} = \alpha_{\partial_{J^{2r}\xi}E_\lambda} = \alpha_{E_{\partial_{J^r\xi}\lambda}}.$$

Now, we can easily prove Theorem 3.2.

P r o o f of Theorem 3.2. (1) If ξ is a point symmetry of λ then, by the above corollary, $\partial_{J^{2r-1}\xi}\theta_\lambda = \theta_{\lambda_0}$, where $\lambda_0 = 0$. However, $\theta_{\lambda_0} = 0$, i.e., ξ is a point symmetry of θ_λ.

(2) If ξ is a point symmetry of E_λ then, by the above corollary, $\partial_{J^{s-1}\xi}\alpha$ is the Lepagean equivalent of the identically zero locally variational form, hence, $\partial_{J^{s-1}\xi}\alpha = 0$. □

4. Point symmetries and constants of the motion

Let α be a Lagrangean system of order $s-1$, $s \geqslant 1$. A function f is called a *constant of the motion of the Lagrangean system* α if for every extremal γ

$$f \circ J^{s-1}\gamma = \text{const.} \tag{4.1}$$

Notice that in the above definition, one requires f to be of order $\leqslant s-1$. Only in this case the conservation law (4.1) is an ODE of order *lower* than the order of the Euler-Lagrange equations, and thus can have a practical significance for simplifying the original integration problem.

Let us recall the famous First Noether Theorem and its generalization [12], [5].

Noether Theorem. *Let λ be a Lagrangian of order r, defined on an open subset $W \subset J^rY$, let θ_λ be its Lepagean equivalent. If a π-projectable vector field ξ on Y is a point symmetry of λ, then for every extremal γ of λ defined on $\pi_r(W) \subset X$ we have*

$$i_{J^{2r-1}\xi}\theta_\lambda \circ J^{2r-1}\gamma = \text{const.}$$

It is easy to see that for Lagrangians of order $r \leqslant s$ the function $i_{J^{2r-1}\xi}\theta_\lambda$ is defined *on the phase space*; in other words, for such Lagrangians, it is a *constant of the motion*. Notice that knowing a symmetry of a Lagrangian of order greater than the order of the given variational equations, results in a higher-order conservation law which does not simplify the integration problem.

Generalized Noether Theorem. *Let E be a locally variational form of order s, let a π-projectable vector field ξ on Y be a point symmetry of E. If λ is a (local) Lagrangian of order r for E on an open set W, and ϱ is a closed one-form of order $r-1$ such that $\partial_{J^r\xi}\lambda = h\varrho$, and if γ is an extremal of E defined on $\pi_r(W) \subset X$, then*

$$J^{2r-1}\gamma^*(\,\mathrm{d}i_{J^{2r-1}\xi}\theta_\lambda - \varrho) = 0. \tag{4.2}$$

This means that a point symmetry of a locally variational form E gives rise to conservation laws

$$(i_{J^{2r-1}\xi}\theta_\lambda - g) \circ J^{2r-1}\gamma = \text{const.},$$

where g is a function (defined on an appropriate open set) such that $\mathrm{d}g = \varrho$. Since $\partial_{J^r\xi}\lambda = h\varrho$, we have $\partial_{J^{2r-1}\xi}\theta_\lambda = \varrho$, and we get

$$\mathrm{d}i_{J^{2r-1}\xi}\theta_\lambda - \varrho = -i_{J^{2r-1}\xi}\,\mathrm{d}\theta_\lambda.$$

Independently of the order of the Lagrangian, this closed one-form is *projectable onto an open subset of* $J^{s-1}Y$. Consequently, knowing a point symmetry of E, the Generalized Noether Theorem provides us, for *every Lagrangian*, with *a constant of the motion.*

5. Point symmetries and first integrals

The dynamics of a Lagrangean system α is geometrically described by means of its Euler-Lagrange distribution and the characteristic distribution of the closed two-form α. This means that one can make use of *first integrals* of these distributions for getting an explicit solution of the Euler-Lagrange equations.

Let α be a Lagrangean system of order $s-1$, $s \geqslant 1$. Let f be a function defined on an open subset U of the phase space $J^{s-1}Y$. Recall that f is said to be *a first integral of the Lagrangean system* α if f is a first integral of the characteristic distribution $\mathcal{D}$ of α, i.e., if

$$\mathrm{d}f \in \mathcal{D}.$$

Remark. The characteristic distribution $\mathcal{D}$ is a subdistribution of the Euler-Lagrange distribution Δ. Hence, the set of first integrals of the Euler-Lagrange distribution is a subset of the set of first integrals of the characteristic distribution. To see the motivation for the above definition, let us look at *time-independent* Lagrangean systems. It is easy to see that in this case, the Hamiltonian H is a first integral of the characteristic distribution. On the other hand, $\mathrm{d}H \in \Delta$ if and only if $\Delta = \mathcal{D}$, i.e., there exist time-independent constrained systems for which H is *not* a first integral of the Euler-Lagrange distribution; as an example of such a Lagrangean system one can take Cawley's Lagrangian

$$L = \dot{q}^1\dot{q}^3 + \tfrac{1}{2}(q^2)^2q^3.$$

A natural requirement that (similarly to the regular case) for time-independent constrained systems H should be a first integral, leads us to the definition of a first integral of a Lagrangean system as a first integral of its *characteristic* distribution.

Proposition. *Every first integral of a Lagrangean system is a constant of the motion.*

If the Lagrangean system is regular then the set of first integrals coincides with the set of constants of the motion.

Proof. Let α be a Lagrangean system of order $s-1$, let f be a first integral of α. Since for every extremal γ the $J^{s-1}\gamma$ is a Hamilton extremal, and it is an integral section of the characteristic distribution $\mathcal{D}$ of α, we have $\mathrm{d}(J^{s-1}\gamma \circ f) = J^{s-1}\gamma^* \,\mathrm{d}f = 0$.

Suppose that α is regular, and let f be a constant of the motion. Then $\mathcal{D}$ is locally spanned by one nowhere zero vector field ζ such that the prolonged extremals coincide with the integral sections of ζ. Hence, $i_\zeta \,\mathrm{d}f = 0$, proving that $f \in \mathcal{D}$. $\square$

Since generally the set of integral mappings of the characteristic distribution does not coincide with the set of prolonged extremals, it is not surprising that one can find examples of constrained systems with constants of the motion which are not first integrals [4].

The following proposition follows immediately from the definition of the characteristic distribution.

Proposition. *Let α be a Lagrangean system of order $s-1$, $s \geqslant 1$. Let ξ be a point symmetry of the Lepagean equivalent θ_λ of a (local) Lagrangian λ of α. If θ_λ is of order $s-1$ then $i_{J^{s-1}\xi}\theta_\lambda$ is a first integral of the Lagrangean system α.*

Now, using the above proposition and Theorem 3.2 we obtain a *stronger form of the Noether Theorem*:

Theorem 5.1. *Let α be a Lagrangean system on $J^{s-1}Y$, let λ be a (local) Lagrangian of order $r \leqslant s$. If a π-projectable vector field ξ on Y is a point symmetry of λ then $i_{J^{2r-1}\xi}\theta_\lambda$ is a first integral of the Lagrangean system α.*

Let α be a Lagrangean system of order $s-1$, $s \geqslant 1$. Let ξ be a π-projectable vector field on Y. If ξ is a point symmetry of α then, since α is closed, one has in a neighborhood of every point in $J^{s-1}Y$ the relation $i_{J^{s-1}\xi}\alpha = \mathrm{d}f$, where f is a first integral of the characteristic distribution of α. This, together with Theorem 3.2, leads to a *stronger form of the Generalized Noether Theorem*:

Theorem 5.2. *Let α be a Lagrangean system on $J^{s-1}Y$. Let $E = p_1\alpha$, and let a π-projectable vector field ξ on Y be a point symmetry of E. Then for every Lagrangian λ of E, the function $i_{J^{2r-1}\xi}\theta_\lambda - g$, where r is the order of λ and g is given by $\mathrm{d}g = \partial_{J^{2r-1}\xi}\theta_\lambda$, is a first integral of the Lagrangean system α.*

R e m a r k. Let us mention a practical significance of the above theorems. Recently, methods have been developed for integration of (generally higher-order) constrained Lagrangean systems, based on a generalization of the classical Liouville theorem and the Hamilton-Jacobi integration method [9]. To apply them, one needs to know a suitable set of first integrals. However, according to Theorems 5.1 and 5.2, one can easily get first integrals with the help of symmetries of Lagrangians and Euler-Lagrange equations.

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