ON r-EXTENDABILITY OF THE HYPERCUBE Q_n

Nirmala B. Limaye, Mumbai, Dinesh G. Sarvate, Charleston

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Abstract. A graph having a perfect matching is called r-extendable if every matching of size r can be extended to a perfect matching. It is proved that in the hypercube Q_n, a matching S with $|S| \leqslant n$ can be extended to a perfect matching if and only if it does not saturate the neighbourhood of any unsaturated vertex. In particular, Q_n is r-extendable for every r with $1 \leqslant r \leqslant n-1$.

Keywords: 1-factor, r-extendability, hypercube

MSC 1991: 05C70

1. Introduction

We consider only finite, simple graphs. A set S of edges in a graph G is called a *matching* if no two edges of S have a common vertex. A matching S is called a *perfect matching* if every vertex of G is an end vertex of some edge in S. Let r and p be positive integers and let G be a graph on $2p$ vertices having a perfect matching, that is having a 1-factor. Then G is said to be *r-extendable* if every matching of size r in G can be extended to a perfect matching of G. The r-extendable graphs were studied in [2] and [3]. Plummer proved [3] that for $p \geqslant 2$ and $p + r \leqslant k \leqslant 2p-1$ any graph G on $2p$ vertices with the minimum degree $\delta(G) \geqslant k$ is r-extendable. Moreover, if $r \leqslant p-1$, then any r-extendable graph is $(r-1)$-extendable and $(r+1)$-connected.

The tetrahedron, the hypercube Q_n, the dodecahedron, the icosahedron, the complete bipartite graphs $K_{n,n}$ with $n \geqslant 2$ are all 2-extendable, but the octahedron and the Petersen graph are not. The extendability of generlized Petersen graphs was studied in [1] and [4]. In this note we study r-extendability of the hypercube Q_n and prove that Q_n is r-extendable for every r with $1 \leqslant r \leqslant n-1$.

2. The hypercube Q_n

For a positive integer n with $n \geqslant 2$, the hypercube Q_n is the graph whose vertex set $V(Q_n)$ is given by $\{\bar{a} = (a_1, \ldots, a_n) \mid a_i = 0 \text{ or } 1 \text{ for each } i\}$ and whose edge set $E(Q_n)$ is given by $\{\bar{a}\,\bar{b} \mid a_i \neq b_i \text{ for exactly one } i\}$. Clearly Q_n is a graph on 2^n vertices and is regular with the degree of regularity equal to n. The following properties of Q_n are useful.

(i) Any two adjacent edges of Q_n belong to a unique 4-cycle.

(ii) For a fixed vertex $\bar{a}$, let L_i be the set of all vertices at a distance i from $\bar{a}$. This set is called the ith level of the vertex $\bar{a}$. Clearly $L_i = \emptyset$ for all $i > n$. Moreover, every vertex $\bar{b}$ in L_i has precisely i neighbours in L_{i-1} and $n-i$ neighbours in L_{i+1}.

By $\bar{0}$ we denote the vertex having all coordinates equal to 0 and by $\overline{e_i}$ we denote the vertex having the ith coordinate equal to 1 and all the other coordinates equal to 0.

For a positive integer i, $1 \leqslant i \leqslant n$, by the ith *decomposition* of the hypercube Q_n we mean the partition $\{V_1, V_2\}$ of the vertex set $V(Q_n)$, where $V_1 = \{\bar{a} \mid a_i = 0\}$ and $V_2 = \{\bar{a} \mid a_i = 1\}$. Clearly, the induced subgraphs on V_1 as well as on V_2 are isomorphic to the cube Q_{n-1}. We denote these smaller hypercubes by G_1 and G_2. The edge set $E(Q_n)$ also gets partitioned into three subsets: $E(G_1), E(G_2)$ and a perfect matching $\{\bar{x}\,\bar{y} \mid x_j = y_j, 1 \leqslant j \leqslant n, j \neq i\}$. The edges of this perfect matching are called the *cross edges* in the ith decomposition. Every vertex $\bar{x}$ in G_1 (or G_2), is adjacent to a unique vertex in G_2 (G_1, respectively). This vertex is called the *mirror image* of $\bar{x}$ and is denoted by $m(\bar{x})$. By taking mirror images of vertices as well as edges, one can see that for a subgraph H of G_1 (or G_2), there is an isomorphic copy of it in G_2 (G_1, respectively). It is denoted by $m(H)$. For a set S of edges in Q_n, by the set $A(S)$ of *associated integers* of S we mean the set $\{j \mid \text{the end vertices of some edge } e \in S \text{ differ in the } j\text{th coordinate}\}$. If $S = \{e\}$ and $A(S) = \{i\}$, then we say that the integer i is the *associated integer* of the edge e. If S is a set of edges in Q_n, we say that S *saturates* a vertex $\bar{x}$ if some edge e of S is incident with the vertex $\bar{x}$, otherwise $\bar{x}$ is said to be *unsaturated*.

For a vertex $\bar{x}$ in G_1, by $L_1^{'}, L_2^{'}, \ldots$ we mean the levels of $\bar{x}$ in G_1. Similarly, the levels of $m(\bar{x})$ in G_2 will be denoted by $L_1^{''}, L_2^{''}, \ldots$. Clearly, $L_i = L_i^{'} \cup L_{i-1}^{''}$ for all i.

Theorem. *Let S be a matching in Q_n such that $|S| \leqslant n$. Then S can be extended to a perfect matching of Q_n if and only if S does not saturate the neighbourhood of any unsaturated vertex.*

In particular, Q_n is r-extendable for each r with $1 \leqslant r \leqslant n-1$.

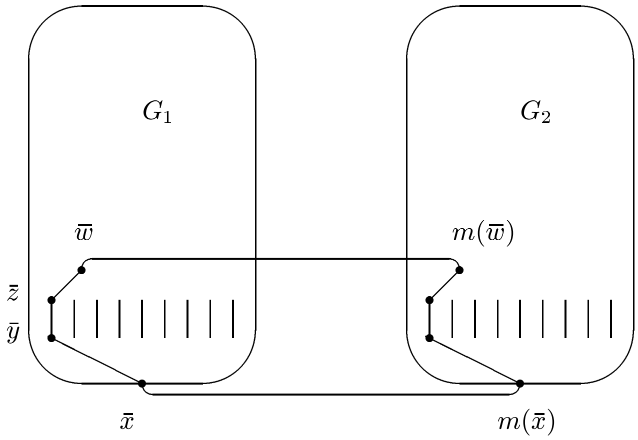
Proof. It is easy to see that if S can be extended to a perfect matching, then it does not saturate the neighbourhood of any unsaturated vertex. For the converse, we use induction on n. One can easily see that the theorem is true for $n = 2, 3$ and 4. Let $n \geqslant 5$.

Case 1: $|A(S)| < n$.

Subcase 1(a): $|S| \leqslant n-1$. Choose an integer $i \notin A(S)$ and consider the ith decomposition of Q_n. Let $S_t = S \cap E(G_t)$, $t = 1, 2$. Clearly, $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. If $|S_1| < n-1$ and $|S_2| < n-1$, then by induction we can extend each S_t to a perfect matching F_t in G_t, $t = 1, 2$. Let $F = F_1 \cup F_2$.

If S_1 is of size $n-1$ and $S_2 = \emptyset$, we proceed as follows. If S_1 does not saturate the neighbourhood in G_1 of any unsaturated vertex, then by induction we extend S_1 to a perfect matching F_1 of G_1. Choose any perfect matching F_2 of G_2 and let $F = F_1 \cup F_2$.

If S_1 saturates the neighbourhood in G_1 of an unsaturated vertex $\bar{x}$, remove any edge $e = \bar{y}\,\bar{z}$ in S_1, where $\bar{y}$ is a neighbour of $\bar{x}$.



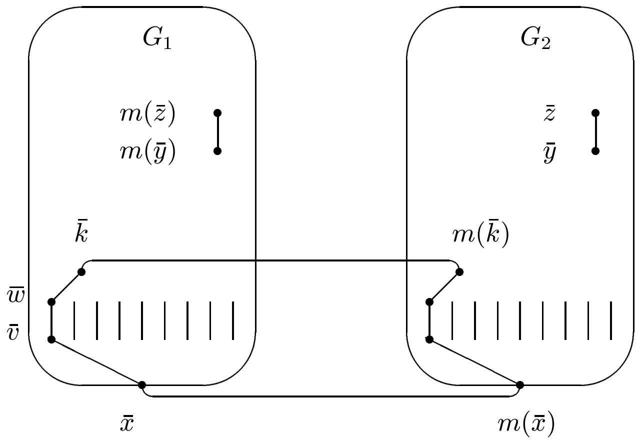
By induction, $S - \{\bar{y}\,\bar{z}\}$ can be extended to a perfect matching F_1 of G_1. Clearly, the edge $\bar{x}\,\bar{y}$ must belong to F_1. Let the edge of F_1 saturating $\bar{z}$ be $\bar{z}\,\overline{w}$. One can now let $F = F_1 \cup m(F_1) \cup \{e, m(e), \bar{x}m(\bar{x}), \overline{w}m(\overline{w})\} - \{\bar{x}\,\bar{y}, m(\bar{x}\,\bar{y}), \bar{z}\,\overline{w}, m(\bar{z}\,\overline{w})\}$. Clearly F is a perfect matching of Q_n containing S.

Subcase 1(b): $|S| = n$. As before, let $S_t = S \cap E(G_t), t = 1, 2$. If $|S_1| = n$ and S_1 does not saturate the neighbourhood of any unsaturated vertex, then choose any edge $e = \bar{y}\,\bar{z}$ from S. Otherwise for $n > 5$, the set S can saturate the neighbourhood of only one unsaturated vertex $\bar{x}$. So choose the edge e such that $\bar{y}$ is a neighbour of $\bar{x}$. If $n = 5$, then the set S can possibly saturate the neighbourhoods of two unsaturated vertices $\bar{x}, \overline{w}$. In this case, choose the edge e in S such that $\bar{y}$ is a neighbour of $\bar{x}$ and $\bar{z}$ is a neighbour of $\overline{w}$. By induction, extend $S - \{e\}$ to a 1-factor F_1 of G_1. One can now see that $F = F_1 \cup m(F_1) \cup \{e, m(e)\} - \{\bar{x}\,m(\bar{x}), \overline{w}\,m(\overline{w})\}$ is the required 1-factor. Here $\bar{x}\,\bar{y}, \bar{z}\,\overline{w}$ are the edges of F_1 saturating $\bar{y}$ and $\bar{z}$, respectively.

If $|S_1| \leqslant n-2$ and $|S_2| \leqslant n-2$, or if $|S_1| = n-1, |S_2| = 1$ but S_1 does not saturate the neighbourhood in G_1 of any unsaturated vertex, then we can extend each S_t to a perfect matching F_t of $G_t, t = 1, 2$. Let $F = F_1 \cup F_2$.

Now suppose that $|S_1| = n-1, |S_2| = 1$ and that S_1 saturates the neighbourhood of an unsaturated vertex $\bar{x}$ in G_1. Let $S_2 = \{\bar{y}\,\bar{z}\}$. By hypothesis, the neighbourhood of $\bar{x}$ in Q_n is not saturated. Hence both $\bar{y}$ and $\bar{z}$ are different from $m(\bar{x})$. Since Q_n is bipartite, distances of $\bar{y}$ and $\bar{z}$ from $m(\bar{x})$ are not the same. Without loss of generality, suppose that $d(m(\bar{x}), \bar{y}) = d$ and $d(m(\bar{x}), \bar{z}) = d+1$.

If $d \geqslant 3$, choose a neighbour $\bar{v}$ of $\bar{x}$ in G_1 and an edge $e = \bar{v}\,\bar{w} \in S_1$. If $d = 1$ but $m(\bar{y}\,\bar{z}) \in S_1$, then choose an adge $e = \bar{v}\,\bar{w} \in S_1$ such that $\bar{v} \neq \bar{y}$. By induction, we can extend $S_1 \cup \{m(\bar{y}\,\bar{z})\} - \{e\}$ to a perfect matching F_1 of G_1. Let $\bar{w}\,\bar{k} = f$ be the edge of F_1 saturating $\bar{w}$. The only edgeincidence with the vertex $\bar{x}$ that can belong to F_1 is $\bar{x}\,\bar{v}$.



It is clear that $F = F_1 \cup m(F_1) \cup \{e, m(e), \bar{x}\,m(\bar{x}), \bar{k}\,m(\bar{k})\} - \{f, m(f), \bar{x}\,\bar{v}, m(\bar{x}\,\bar{v})\}$ is a perfect matching of Q_n containing S.

Now suppose $d = 1$ and $m(\bar{y}\,\bar{z}) \notin S_1$. By assumption, $\bar{y}$ is saturated by some edge in S_1. Choose an edge $e = \bar{v}\,\bar{w}$ in S_1 such that $\bar{v} \neq m(\bar{y})$ and $\bar{v}$ is a neighbour of $\bar{x}$. Extend $S_1 - \{e\}$ to a 1-factor F_1 of G_1. Clearly, the edge $\bar{x}\,\bar{v}$ belongs to F_1. If $\bar{w}\,\bar{k}$ is the edge in F_1 saturating $\bar{w}$, then $\bar{k}$ cannot be $m(\bar{y})$ since $m(\bar{y})$ is saturated in S_1, and it cannot be $m(\bar{z})$ since both $\bar{w}$ and $m(\bar{z})$ belong to the level $L_2^{'}$ of $\bar{x}$. This means that the edges $\bar{y}\,\bar{z}, m(\bar{x}\,\bar{v}), m(\bar{w}\,\bar{k})$ are parallel in G_2. By induction, extend this set to a 1-factor F_2 of G_2. As before, we can now let $F = F_1 \cup F_2 \cup \{e, m(e), \bar{x}m(\bar{x}), \bar{k}m(\bar{k})\} - \{\bar{x}\,\bar{v}, m(\bar{x}\,\bar{v}), \bar{w}\,\bar{k}, m(\bar{w}\,\bar{k})\}$.

If $d = 2$ then the distance of $m(\bar{z})$ from $\bar{x}$ is 3. But then there are exactly 3 neighbours of $m(\bar{z})$ on any shortest path from $\bar{x}$ to $m(\bar{z})$. Since $n - 1 \geqslant 4$, we can choose an edge $f \in S_1$ such that $\bar{v}$ is not on a shortest $\bar{x}$-$m(\bar{z})$ path. The rest of the construction is the same as when $d \geqslant 3$.

Case 2: $|A(S)| = n$. If $|A(S)| = n$ then in any ith decomposition of Q_n, there is precisely one edge having one end vertex in G_1 and the other in G_2. Consider the first decomposition of Q_n. Let $\bar{x}\, m(\bar{x})$ be the unique cross edge. As before, let $S_i = S \cap E(G_i), i = 1, 2$ and suppose that $|S_2| \leqslant |S_1|$.

Subcase 2(a): $S_1 \cup m(S_2)$ is a matching in G_1. Let $F_1 = S_1 \cup m(S_1) \cup S_2 \cup m(S_2)$ and $F = F_1 \cup \{$all the cross edges with vertices unsaturated by $F_1\}$.

Subcase 2(b): $S_1 \cup m(S_2)$ is not a matching, but there is a neighbour $\bar{y}$ of $\bar{x}$ in G_1 such that both $\bar{y}, m(\bar{y})$ are unsaturated by S.

Subcase 2(b-I): $|S_1| \leqslant n - 3$, or $|S_1| = n - 2$ but $S_1 \cup \{\bar{x}\,\bar{y}\}$ does not saturate the neighbourhood in G_1 of any unsaturated vertex. Then by induction we extend $S_1 \cup \{\bar{x}\,\bar{y}\}$ to a 1-factor F_1 of G_1, extend $S_2 \cup \{m(\bar{x}\,\bar{y})\}$ to a 1-factor F_2 of G_2 and let $F = F_1 \cup F_2 \cup \{\bar{x}\, m(\bar{x}), \bar{y}\, m(\bar{y})\} - \{\bar{x}\,\bar{y}, m(\bar{x}\,\bar{y})\}$.

Subcase 2(b-II): $|S_1| = n - 2, |S_2| = 1$ and $S_1^{'} = S_1 \cup \{\bar{x}\,\bar{y}\}$ saturates the neighbourhood in G_1 of some unsaturated vertex $\bar{w}$. Clearly, $\bar{w}$ is different from $\bar{x}$ as well as $\bar{y}$, but it is a neighbour of precisely one of them.

Suppose $\bar{w}$ is adjacent to $\bar{x}$. Since S does not saturate the neighbourhood of $\bar{w}$ in Q_n, the vertex $m(\bar{w})$ is unsaturated. Hence we replace the edge $\bar{x}\,\bar{y}$ by the edge $\bar{x}\,\bar{w}$ in the above argument. One can easily check that $S_1 \cup \{\bar{x}\,\bar{w}\}$ does not saturate the neighbourhood in G_1 of any unsaturated vertex. Now we proceed as in Subcase 2(b-I).

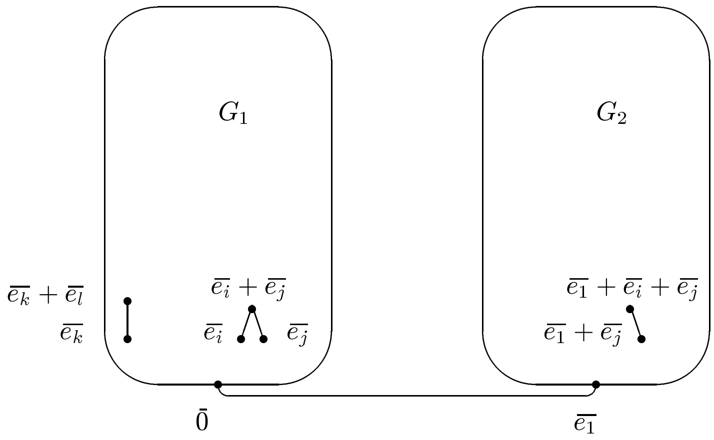
If $\bar{w}$ is a neighbour of $\bar{y}$, then S_1 saturates only one neighbour of $\bar{x}$ in G_1. Since $n - 1 \geqslant 4$ and $|S_2| = 1$, one can choose one more vertex $\bar{u}$ adjacent to the vertex $\bar{x}$ such that $\bar{u}$ and $m(\bar{u})$ are both unsaturated. It is easy to see that $S_1 \cup \{\bar{x}\,\bar{u}\}$ does not saturate the neighbourhood in G_1 of any unsaturated vertex. Now we proceed as in Subcase 2(b-I).

Subcase 2(c): $|A(S)| = n, m(S_2) \cup S_1$ is not a matching in G_1 and every neighbour of $\bar{x}$ in G_1 is saturated by $m(S_2) \cup S_1$.

The graph Q_n is bipartite and hence no edge joins two neighbours of $\bar{x}$. This means $n-1$ edges of $S_1 \cup m(S_2)$ saturate $n-1$ distinct neighbours of $\bar{x}$. Since $S_1 \cup m(S_2)$ is not a matching, the subgraph H induced by this set in G_1 is the union of paths, each having alternating edges in S_1 and $m(S_2)$.

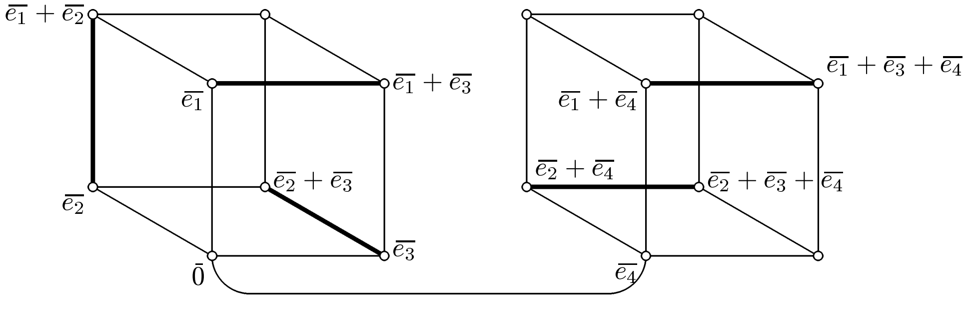
If possible, let there be a path of length at least 3. Then there is a vertex $\bar{z}$ of degree 2 on this path which is on the first level $L_1^{'}$ of $\bar{x}$. But then there is one edge in S_1 and one in $m(S_2)$ saturating this vertex. This contradicts the fact that $n-1$ edges of $S_1 \cup m(S_2)$ saturate $n-1$ distinct neighbours of $\bar{x}$. Hence the subgraph H of G_1 induced by $S_1 \cup m(S_2)$ is the union of paths of length 1 or 2 and there is at least one path of length 2. Moreover, end vertices of every path of length 2 are neighbours of $\bar{x}$.

Without loss of generality, let $\bar{x} = (0, \dots, 0) = \bar{0}$ and consider a path $\{\overline{e_i}, \overline{e_i} + \overline{e_j}, \overline{e_j}\}$, of length 2, where the edge $\overline{e_i}(\overline{e_i} + \overline{e_j})$ is in S_1 and the edge $m(\overline{e_j}(\overline{e_i} + \overline{e_j}))$ is in S_2. The associated integers of these edges are j and i, respectively. All edges in $S_1 \cup m(S_2)$ have one end vertex in $L_1^{'}$ and the other in $L_2^{'}$. If $\overline{e_l}(\overline{e_l} + \overline{e_k})$ is a path of length one in $S_1 \cup m(S_2)$, then the associated integer of this edge is k.



Since $|A(S)| = n$, the integer k is different from i and j. This means that neither of these vertices is a neighbour of $\overline{e_i}$ or of $\overline{e_i} + \overline{e_j}$. Suppose $\{\overline{e_k}, \overline{e_k} + \overline{e_l}, \overline{e_l}\}$ is a path of length two in $S_1 \cup m(S_2)$. Then by the same argument, both k, l are different from i, j. Hence the only neighbours of $\overline{e_i}$ saturated by S are $\bar{0}$ and $(\overline{e_i} + \overline{e_j})$. Similarly, the only neighbour of $\overline{e_i} + \overline{e_j}$ saturated by S is $\overline{e_i} + \overline{e_j} + \overline{e_1}$. Now we can consider the jth decomposition and complete the required 1-factor as in Subcase 2(b). $\square$

E x a m p l e. The condition $|S| \leqslant n$ on the size of the matching S is optimal. We give an example of a set of 5 parallel edges in Q_4, which does not saturate the neighbourhood of any unsaturated vertex but cannot be extended to a 1-factor.



Let $S = \{\overline{e_1}\,(\overline{e_1}+\overline{e_3}), \overline{e_3}\,(\overline{e_2}+\overline{e_3}), \overline{e_2}\,(\overline{e_1}+\overline{e_2}), m(\overline{e_1}\,(\overline{e_1}+\overline{e_3})), m(\overline{e_2}\,(\overline{e_2}+\overline{e_3}))\}$. If this is to be extended to a 1-factor, one is forced to include the edge $\bar{0}\,\overline{e_4}$. But then one is left with no choice of an edge to saturate the vertex $\overline{e_3}+\overline{e_4}$.

We conjecture that a set of $n+1$ parallel edges in Q_n which does not saturate the neighbourhood of any unsaturated vertex can be extended to a 1-factor if $n \geqslant 5$.

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Authors' addresses: *Nirmala B. Limaye*, Department of Mathematics, University of Mumbai, India; *Dinesh G. Sarvate*, Department of Mathematics, University of Charleston, S. C., U.S.A.