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ATOMICITY OF THE BOOLEAN ALGEBRA OF DIRECT FACTORS OF A DIRECTED SET

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Abstract. In the present paper we deal with the relations between direct product decompositions of a directed set L and direct product decompositions of intervals of L.

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1. Introduction

Basic results on direct product decompositions of partially ordered sets were proved in [1].

For a directed set L and an element s^0 of L we apply the notion of the internal direct product decomposition

$$\varphi^0 \colon L \longrightarrow \prod_{i\in I} X_i^0$$

with the central element s^0 in the same sense as in [5]; cf. also Section 2 below. Here, X_i^0 are convex subsets of L containing the element s^0; they are called internal direct factors of L (with the central element s^0).

We denote by $D(L,s^0)$ the system of all direct factors of L with the central element s^0. This system is partially ordered by the set-theoretical inclusion. Then $D(L,s^0)$ is a Boolean algebra.

If s^1 is another element of L, then the Boolean algebras $D(L,s^0)$ and $D(L,s^1)$ are isomorphic. Hence, if we consider the Boolean algebra $D(L,s^0)$ up to isomorphism, then it suffices to write $D(L)$ instead of $D(L,s^0)$.

In the case when L can be represented as a direct product of directly indecomposable direct factors we obtain that the Boolean algebra $D(L)$ is atomic. The converse implication does not hold in general.

Sufficient conditions for $D(L)$ to be atomic were found in [4] in the case when L is a lattice. In [6] sufficient conditions were given under which a complete lattice is a direct product of directly indecomposable direct factors. This result was generalized in [4]. For related results cf. also [2], [3].

We denote by

$\mathcal{L}_a$—the class of all directed sets L such that the Boolean algebra $D(L)$ is atomic;

$\mathcal{L}_b$—the class of all directed sets L such that L is a direct product of directly indecomposable direct factors.

If $L \in \mathcal{L}_a$ and if L_1 is an interval of L then L_1 need not belong to $\mathcal{L}_a$.

In the present paper the following result will be proved:

(A) Let L be a directed set and let $\{L_i\}_{i\in I}$ be a system of intervals of L such that

(i) the system $\{L_i\}_{i\in I}$ is a chain (under the partial order defined by the set-theoretical inclusion) and $\bigcup\limits_{i\in I} L_i = L$;

(ii) all L_i belong to $\mathcal{L}_b$.

Then L belongs to $\mathcal{L}_a$.

2. Internal direct factors

We start by recalling some definitions and results from [5] concerning internal direct product decompositions of directed sets.

In the whole paper L denotes a directed set. For $u, v \in L$ with $u \leqslant v$ we denote by $[u, v]$ the corresponding interval of L. If X is a nonempty subset of L, then we consider X to be partially ordered (with the partial order inherited from L).

Let L_i $(i \in I)$ be directed sets; their direct product will be denoted by $\prod\limits_{i\in I} L_i$. If φ is an isomorphism of L onto $\prod\limits_{i\in I} L_i$, then the relation

$$\varphi\colon L \longrightarrow \prod_{i\in I} L_i \tag{1}$$

is called a *direct product decomposition* of L.

For $i \in I$ and $x \in L$ we denote by $x(L_i, \varphi)$ the component of x in L_i under the morphisms φ. If $X \subseteq L$, then we put

$$X(L_i, \varphi) = \{x(L_i, \varphi)\colon x \in X\}.$$

L is called directly indecomposable if, whenever (1) is valid, then there is $i(1) \in I$ such that $\operatorname{card} L_i = 1$ for each $i \in I \setminus \{i(1)\}$. In such a case L is isomorphic to $L_{i(1)}$.

Suppose that (1) holds. For each $i \in I$ and $x \in L$ we put

$$[x](L_i, \varphi_i) = \{y \in L \colon y(L_j, \varphi) = x(L_j, \varphi) \quad \text{for each } j \in I \setminus \{i\}\}.$$

Let s^0 be a fixed element of L,

$$L_i^0 = [s^0](L_i, \varphi).$$

Given $x \in L$, there exists a uniquely determined element x_i in L_i^0 such that

$$x(L_i, \varphi) = x_i(L_i, \varphi).$$

The mapping

$$\varphi^0 \longrightarrow \prod_{i \in I} L_i^0 \tag{2}$$

defined by

$$\varphi^0(x) = (\ldots, x_i, \ldots)_{i \in I} \tag{2'}$$

is also a direct product decomposition of L. We call (2) an internal direct product decomposition with the central element s^0. The direct factors L_i^0 are called internal. For each $i \in I$, L_i^0 is isomorphic to L_i.

In what follows, whenever we consider an internal direct product decomposition of L or of a subset of L, then we always suppose that the corresponding central element is s^0.

From the definition of the internal product decomposition we immediately obtain:

2.1. Lemma. *Let (2) be an internal direct product decomposition and let $i \in I, x \in L$. Then the following conditions are equivalent:*

(i) $x \in L_i^0$;

(ii) $x(L_i^0, \varphi^0) = x$;

(iii) $x(L_j^0, \varphi^0) = s^0$ *for each* $j \in I \setminus \{i\}$.

2.2. Proposition. (Theorem (A) of [5].) *Suppose that two internal direct product decompositions are given,*

$$\psi_1 \colon L \longrightarrow \prod_{i \in I} A_i, \quad \psi_2 \colon L \longrightarrow \prod_{j \in J} B_j$$

such that there exist $i(1) \in I$ and $j(1) \in J$ with $A_{i(1)} = B_{j(1)}$. Then for each $x \in L$ the relation

$$x(A_{i(1)}, \psi_1) = x(B_{j(1)}, \psi_2)$$

is valid.

Hence, if (2) is as above, then instead of $x(L_i^0, \varphi^0)$ it suffices to write $x(L_i^0)$; for $X \subseteq L$, the meaning of $X(L_i^0)$ is analogous. Also, we will write

$$L = \prod_{i \in I} L_i^0 \tag{2'}$$

instead of (2).

2.3. Lemma. *Let (2′) be valid and let $u, v \in L$, $u \leqslant s^0 \leqslant v$, $i \in I$. Then*

$$v(L_i^0) = \max\{t_1 \in L_i^0 \colon s^0 \leqslant t_1 \leqslant v\}, \tag{2.3.1}$$

$$u(L_i^0) = \min\{t_2 \in L_i^0 \colon s^0 \geqslant t_2 \geqslant u\}. \tag{2.3.2}$$

Moreover, $v = \sup\{v(L_i^0)\}_{i \in I}$ and $u = \inf\{(L_i^0)\}_{i \in I}$.

P r o o f. The relation (2.3.1) was proved in [5], Lemma 3.2. The relation (2.3.2) can be proved dually.

Further, in view of (2.3.1) we have $v(L_i^0) \leqslant v$ for each $i \in I$. Let $t \in L$ be such that $t \geqslant v(L_i^0)$ for each $i \in I$. Then for each $i \in I$ we have

$$t(L_i^0) \geqslant (v(L_i^0))(L_i^0) = v(L_i^0),$$

yielding that $t \geqslant v$. Therefore $v = \sup\{v(L_i^0)\}_{i \in I}$. The analogous relation for u can be verified dually. □

For $A \in D(L)$ we denote

$$A^+ = \{a \in A \colon a \geqslant s^0\}, \quad A^- = \{a \in A \colon a \leqslant s^0\}.$$

2.4. Lemma. *Let $A, B \in D(L)$. If $A^+ \subseteq B$ and $A^- \subseteq B$, then $A \subseteq B$.*

P r o o f. Suppose that $A^+ \subseteq B$, $A^- \subseteq B$ and $a \in A$. There exist $u \in A^-$ and $v \in A^+$ such that $u \leqslant a \leqslant v$. Then $u, v \in B$. Since B is convex in L we get $a \in B$. □

2.5. Lemma. *Let* $(2')$ *be valid and let* X *be a convex directed subset of* L, $s^0 \in X$, $i \in I$. *Then* $X(L_i^0) = X \cap L_i^0$.

Proof. In view of 2.1 we have $X \cap L_i^0 \subseteq X(L_i^0)$. Let $y \in X(L_i^0)$. Hence there exists $x \in X$ such that $y = x(L_i^0)$. Since X is directed, there exist $u, v \in X$ such that both x and s^0 belong to the interval $[u, v]$. In view of 2.3 we have $u(L_i^0), v(L_i^0) \in X \cap L_i^0$. Clearly $u(L_i^0) \leqslant y \leqslant v(L_i^0)$. Hence $y \in X \cap L_i^0$. $\square$

If $(2')$ is valid, $I_1 \subseteq I$, and if for each $i \in I_1$ we have $\{s^0\} \in Z_i \subseteq L_i^0$, then $\prod\limits_{i \in I_1} Z_i$ denotes the set

$$\bigl\{x \in L\colon x(L_i^0) \in Z_i \quad \text{for each } i \in I_1 \text{ and } x(L_i^0) = s^0 \text{ for each } i \in I \setminus I_1\bigr\}.$$

Hence, if $Z \subseteq L$ with $s^0 \in Z$, then

$$Z \times \{s^0\} = Z.$$

Also, we obviously have

2.6. Lemma. *Let* $(2')$ *be valid and* $i \in I$. *Then*

$$L = L_i^0 \times \prod_{j \in I \setminus \{i\}} L_j^0.$$

Suppose that two internal direct product decompositions are given,

$$L = \prod_{i \in I} A_i, \tag{3}$$

$$L = \prod_{j \in J} B_j. \tag{4}$$

The decomposition (3) is said to be a refinement of (4) if for each $j \in J$ there exists a subset $I(j)$ of I such that

$$B_j = \prod_{i \in I(j)} A_i.$$

2.7. Proposition. *Let* (3) *and* (4) *be valid. Then we have*

$$L = \prod_{i \in I, j \in J} (A_i \cap B_j) \tag{5}$$

and (5) *is a common refinement of both* (3) *and* (4)*. Namely, for each* $i \in I$ *and each* $j \in J$,

$$A_i = \prod_{j \in J} (A_i \cap B_i), \tag{6}$$

$$B_j = \prod_{i \in I} (A_i \cap B_j). \tag{7}$$

P r o o f. In view of Theorem (B) in [1] (cf. the relation (5) in the proof of (B)) we have

$$L = \prod_{i \in I, j \in J} B_j(A_i)$$

and this decomposition is a common refinement of both (3) and (4).

Hence according to 2.5, the relation (5) is valid and it is a common refinement of both (3) and (4).

Let $i(1) \in I$. Since (5) is a refinement of (3), $A_{i(1)}$ is an internal direct product of some $A_i \cap B_j$ $((i,j) \in I \times J)$. Without loss of generality we can assume that $A_{i(1)} \neq \{s^0\}$. Thus it suffices to take into account only those $(i,j) \in I \times J$ for which $A_i \cap B_j \neq \{s^0\}$; the set of these (i,j) will be denoted by M.

Let $i \in I$, $i \neq i(1)$ and $j \in J$. Then $A_{i(1)} \cap A_i = \{s^0\}$, whence according to 2.5,

$$A_{i(1)}(A_i \cap B_j) = A_{i(1)} \cap (A_i \cap B_j) = \{s^0\},$$

yielding that if $(i,j) \in M$, then $i = 1(1)$. Hence

$$A_{i(1)} \subseteq \prod_{j \in J} (A_{i(1)} \cap B_j).$$

The internal direct factors $A_{i(1)} \cap B_j$ which are equal to $\{s^0\}$ can be cancelled in the above relation. Let $j(1) \in J$ and suppose that $A_{i(1)} \cap B_{j(1)} \neq \{s^0\}$. By way of contradiction, assume that

$$A_{i(1)} \subseteq \prod_{j \in J \setminus \{j(1)\}} (A_{i(1)} \cap B_j).$$

There exists $x \in A_{i(1)} \cap B_{j(1)}$ with $x \neq s^0$. If $j \in J$, $j \neq j(1)$, then 2.5 yields that

$$B_{j(1)}(A_{i(1)} \cap B_j) = \{s^0\},$$

whence $x \notin \prod_{j \in J \setminus \{j(1)\}} (A_{i(1)} \cap B_j)$, which is a contradiction. Therefore

$$A_{i(1)} = \prod_{j \in J} (A_{i(1)} \cap B_j).$$

Hence (6) holds. The method of proving (7) is analogous. □

2.8. Lemma. *Let* (2′) *be valid and let* X *be an interval of* L, $s^0 \in X$. *Then*

$$X = \prod_{i \in I} (X \cap L_i^0).$$

If $x \in X$ *and* $i \in I$, *then* $x(L_i^0) = x(L_i^0 \cap X)$.

P r o o f. First, let $i \in I$ be fixed. There are $u, v \in L$ such that $X = [u, v]$. Put $u_i = u(L_i^0)$, $v_i = v(L_i^0)$. Hence $[u_i, v_i]$ is an interval of L_i^0 and $X(L_i^0) \subseteq [u_i, v_i]$. Let $t \in [u_i, v_i]$. There exists $z \in L$ such that $z(L_i^0) = t$ and $z(L_i^0) = s^0$ for each $j \in I \setminus \{i\}$. Since $s^0 \in X$ we obtain that $z \in X$ and then $t \in X(L_i^0)$. Therefore $[u_i, v_i] = X(L_i^0)$.

We clearly have $X \subseteq \prod\limits_{i \in I} X(L_i^0)$. Let $z \in \prod\limits_{i \in I} X(L_i^0)$. Then $z(L_i^0) \in [u_i, v_i]$ for each $i \in I$, whence $z \in [u, v]$. Thus $X = \prod\limits_{i \in I} X(L_i^0)$. Now it suffices to apply 2.5 and we obtain that $X = \prod\limits_{i \in I} (X \cap L_i^0)$.

The last statement of the lemma is an immediate consequence of the above construction. (Namely, for each $x \in X$, $\varphi^0(u)$ is as in (2′) and then $\varphi^0(x) \in \prod\limits_{i \in I} X(L_i^0)$.)

□

3. Auxiliary results

In this section we deal with the partially ordered set $D(L)$ consisting of all internal direct factors of L. Then $\{s^0\}$ and L are the least element and the greatest element of $D(L)$, respectively.

We call $D(L)$ *atomic* if for each $A \in D(L)$ with $A \neq \{s^0\}$ there exists an atom A_1 of $D(L)$ such that $A_1 \subseteq A$.

If $A, B \in D(L)$ and if $\inf\{A, B\}$ or $\sup\{A, B\}$ does exist in $D(L)$, then we denote these elements by $A \wedge B$ or by $A \vee B$, respectively.

3.1. Lemma. *Let* $L = A_1 \times B_1$, $L = A_2 \times B_2$, $A_1 = A_2$. *Then* $B_1 = B_2$.

P r o o f. We have $A_1 \cap B_1 = \{s^0\} = A_2 \cap B_2$. Hence from 2.7 we obtain

$$B_1 = (B_1 \cap A_2) \times (B_1 \cap B_2) = \{s^0\} \times (B_1 \cap B_2) = B_1 \cap B_2,$$

thus $B_1 \subseteq B_2$. Analogously we get $B_2 \subseteq B_1$. □

3.2. Lemma. *Let* $A \in D(L)$. *Then there exists a unique* $A' \in D(L)$ *such that* $L = A \times A'$.

P r o o f. In view of 2.6, such A' does exist. Then 3.1 implies that A' is uniquely determined. □

3.3. Lemma. *Let $A, B, C, A_1, B_1 \in D(L)$. Suppose that $A_1 = A \times C$, $B_1 = B \times C$, $A_1 \leqslant B_1$. Then $A \leqslant B$.*

P r o o f. Let $a \in A^+$. Hence $a \in A_1$ and $a(C) = \{s^0\}$. At the same time, $a \in B_1$ and thus in view of 2.3 we have

$$a = \sup\{a(B), a(C)\} = \sup\{a(B), s^0\}.$$

From $a \geqslant s^0$ we get $a(B) \geqslant s^0(B) = s^0$. Thus $a = a(B)$ and hence $a \in B$. We have shown that $A^+ \subseteq B$. Analogously we can verify that $A^- \subseteq B$. Then according to 2.4 we have $A \subseteq B$. □

3.4. Lemma. *Let $A, B \in D(L)$. Then $A \wedge B = A \cap B$.*

P r o o f. According to 3.2 we have

$$L = A \times A', \quad L = B \times B'.$$

Thus in view of 2.7,

$$L = (A \cap B) \times (A \cap B') \times (A' \cap B) \times (A' \cap B'). \tag{8}$$

Hence by applying 2.6 we obtain that $A \cap B$ belongs to $D(L)$. If $C \in D(L)$ and $C \leqslant A$, $C \leqslant B$, then $C \leqslant A \cap B$, whence $A \wedge B = A \cap B$. □

3.5. Lemma. *Let $A, B \in D(L)$. Then*

$$A \vee B = (A \cap B) \times (A \cap B') \times (A' \cap B).$$

P r o o f. In view of (8) and 2.6,

$$(A \cap B) \times (A \cap B') \times (A' \cap B) \in L(D);$$

denote this element of $L(D)$ by P. We have

$$A = (A \cap B) \times (A \cap B'), \quad B = (B \cap A) \times (B \cap A'),$$

whence $A \leqslant P$ and $B \leqslant P$. Let $Q \in D(L)$, $Q \geqslant A$ and $Q \geqslant B$. Then from (8) and 2.7 we obtain

$$\begin{aligned} Q &= (Q \cap A \cap B) \times (Q \cap A \cap B') \times (Q \cap A' \cap B) \times (Q \cap A' \cap B') \\ &= (A \cap B) \times (A \cap B') \times (A' \cap B) \times (Q \cap A' \cap B') = P \times (Q \cap A' \cap B'), \end{aligned}$$

thus $Q \geqslant P$. Therefore $A \vee B = P$. □

3.6. Corollary. *The partially ordered set $L(D)$ is a lattice with the least element $\{s^0\}$ and the greatest element L.*

3.7. Lemma. *For each $A \in L(D)$, A' is a complement of A in $L(D)$.*

Proof. From $L = A \times A'$ we obtain $A \cap A' = \{s^0\}$, hence in view of 3.4, $A \wedge A' = \{s^0\}$. Further, in view of 3.5,

$$A \vee A' = (A \cap A') \times (A \cap A'') \times (A' \cap A') = \{s^0\} \times A \times A' = L.$$

$\square$

Consider the mapping $\varphi\colon D(L) \longrightarrow D(L)$ defined by $\varphi(A) = A'$ for each $A \in D(L)$.

3.8. Lemma. *The mapping φ is a dual isomorphism of $D(L)$ onto $D(L)$.*

Proof. If $A \in D(L)$, then $\varphi(\varphi(A)) = A$, hence φ is a bijection. Let $A, B \in D(L)$, $A \leqslant B$. In view of 2.7,

$$B' = (B' \cap A) \times (B' \cap A').$$

Since

$$\{s^0\} \leqslant B' \cap A \leqslant B' \cap B = \{s^0\}$$

we get $B' \cap A = \{s^0\}$ and thus $B' = B' \cap A'$ yielding that $B' \leqslant A'$. Conversely, from $B' \leqslant A'$ we obtain that $B = B'' \geqslant A'' = A$. $\square$

3.9. Lemma. *Let $A, B \in L(D)$ be such that B is a complement of A in $L(D)$. Then $B = A'$.*

Proof. According to the assumption we have

$$A \wedge B = \{s^0\}, \quad A \vee B = L.$$

Hence in view of 3.8 we obtain

$$A' \vee B' = L, \quad A' \wedge B' = \{s^0\}.$$

Thus

$$A \cap B = A' \cap B' = \{s^0\}.$$

The relation (8) is valid and hence

(9) $$L = (A \cap B') \times (A' \cap B).$$

Let $a \in A^+$. Then in view of 2.3 we have

$$a(A' \cap B) = s^0.$$

Put $a(A \cap B') = x$. According to (9) and 2.3,

$$a = \sup\{x, s^0\}.$$

Clearly $x \geqslant s^0$, whence $a = x$. Thus $A^+ \subseteq A \cap B'$. Dually we obtain that $A^- \subseteq A \cap B'$. Thus according to 2.4, $A \subseteq A \cap B'$ yielding that $A \subseteq B'$. Analogously we establish the validity of the relation $B' \subseteq A$. Hence $A = B'$ and thus $A' = B$. $\square$

From 3.9 and 3.2 we infer

3.10. Lemma. *Each element of $D(L)$ has a unique complement.*

Now let A, B be elements of $D(L)$, $A \wedge B = P$, $A \vee B = Q$. From $L = P' \times P$ and from 2.7 we obtain

$$Q = (Q \cap P') \times P.$$

Put $Q \cap P' = Q_1$. Hence $Q = Q_1 \times P$. Analogously we have

$$A = A_1 \times P, \quad B = B_1 \times P,$$

where $A_1 = A \cap P'$ and $B_1 = B \cap P'$. Thus in view of 3.3 we get $A_1 \leqslant Q_1$, $B_1 \leqslant Q_1$; also

$$A_1 \wedge B_1 = A_1 \cap B_1 = (A \cap P') \cap (B \cap P') = (A \cap B) \cap P' = P \cap P' = \{s^0\}.$$

Further we have

$$Q = A \vee B = (A \cap B) \times (A \cap B') \times (A' \cap B) = P \times (A \cap B') \times (A' \cap B)$$

and $Q_1 \subseteq Q$, $Q_1 \cap P = \{s^0\}$. Therefore

$$\begin{aligned} Q_1 &= (P \cap Q_1) \times (A \cap B' \cap Q_1) \times (A' \cap B \cap Q_1) \\ &= (A \cap B' \cap Q_1) \times (A' \cap B \cap Q_1). \end{aligned}$$

Let us consider the elements $A' \cap B \cap Q_1$ and $A'_1 \cap B_1$ of $D(L)$.

Let $x \in A'_1 \cap B_1$. Then $x \in B_1$, whence $x \in Q_1$ and $x \in B$. Therefore $x(P) = s^0$. From $L = A \times A' = A_1 \times P \times A'$ we obtain that $A'_1 = P \times A'$. Thus $x \in A'$ and so $A'_1 \cap B_1 \subseteq A' \cap B \cap Q_1$.

Further, let $y \in A' \cap B \cap Q_1$. Thus $y \in B \subseteq Q = Q_1 \times P$ and so in view of $y \in Q_1$ we get $y(P) = \{s^0\}$ yielding that $y \in B_1$. Next we have $y \in A' \subseteq A_1'$. Therefore $A' \cap B \cap Q_1 \subseteq A_1' \cap B_1$.

Summarizing, we obtained the relation

$$A' \cap B \cap Q_1 = A_1' \cap B_1.$$

Analogously we can prove

$$A \cap B' \cap Q_1 = A_1 \cap B_1'.$$

Hence

$$Q_1 = (A_1 \cap B_1) \times (A_1 \cap B_1') \times (A_1' \cap B_1) = A_1 \vee B_1.$$

Thus we have verified the following result.

3.11. Lemma. *Let A, B, P, Q, A_1 and B_1 be as above. Then A_1 is a complement of B_1 in the lattice $D(P_1)$.*

3.12. Lemma. *Let A, P, Q be as above, $C \in D(L)$, $P \leqslant C \leqslant Q$, $A \neq C$. If $C = C_1 \times P$, then $A_1 \neq C_1$.*

P r o o f. If $C_1 = A_1$, then $A = A_1 \times P$ implies that $C = A$, which is a contradiction. □

3.13. Lemma. *Let $A, P, Q \in D(L)$, $P \leqslant A \leqslant Q$. Then A has exactly one complement in the interval $[P, Q]$ of $D(L)$.*

P r o o f. This is a consequence of 3.10, 3.11 and 3.12. □

3.14. Proposition. *The partially ordered set $D(L)$ is a Boolean algebra.*

P r o o f. It is well-known that 3.13 implies the distributivity of $D(L)$. Hence 3.6 and 3.13 suffice to complete the proof. □

4. Construction of partially ordered sets C_k

In this section we suppose that the assumptions of (A) are satisfied. The case $L = \{s^0\}$ being trivial we can assume without loss of generality that $\operatorname{card} L > 1$ and $\operatorname{card} L_i > 1$ for each $i \in I$.

For each $i(1) \in I$ there exists an internal direct product decomposition

$$L_{i(1)} = \prod_{j \in J(i(1))} A_{i(1)j} \tag{10}$$

such that each $A_{i(1)j}$ is directly indecomposable and $\operatorname{card} A_{i(1)j} > 1$. From 2.7 it follows that such an internal direct product decomposition is uniquely determined.

In view of condition (i) in (A) we can suppose that the set I is linearly ordered and that whenever $i(1), i(2) \in I$, $i(1) < i(2)$, then $L_{i(1)} \subset L_{i(2)}$.

4.1. Lemma. *Let $i(1), i(2) \in I$, $i(1) < i(2)$, $j(1) \in J(i(1))$. Then there exists a uniquely determined $j(2) \in J(i(2))$ such that*

$$A_{i(1)j(1)} \subseteq A_{i(2)j(2)}.$$

Proof. We have

$$L_{i(2)} = \prod_{j \in J(i(2))} A_{i(2)j}, \tag{10'}$$
$$L_{i(1)} \subseteq L_{i(2)}.$$

Hence $L_{i(1)}$ is an interval of $L_{i(2)}$ and thus according to 2.8,

$$L_{i(1)} = \prod_{j \in J(i(2))} (L_{i(1)} \cap A_{i(2)j}).$$

Then, since $A_{i(1)j}$ is a directly indecomposable internal direct factor of $L_{i(1)}$ we infer that there exists $j(2) \in J(i(2))$ such that

$$A_{i(1)j(1)} \subseteq L_{i(1)} \cap A_{i(2)j(2)}.$$

This yields that whenever $j \in J(i(2))$ and $j \neq j(2)$, then

$$A_{i(1)j(1)} \cap A_{i(2)j} = \{s^0\}.$$

Hence the index $j(2)$ is uniquely determined. $\square$

If $i(1) < i(2)$ and if $j(1), j(2)$ are as above, then we denote

$$\varphi_{i(1)i(2)}(j_1) = j(2).$$

For $i(1) = i(2)$ we put

$$\varphi_{i(1)i(2)}(j_1) = j(1).$$

4.2. Lemma. *Let $i(1), i(2), i(3) \in I$, $i(1) \leqslant i(2) \leqslant i(3)$, $j(1) \in J(i(1))$ and $j(2) = \varphi_{i(1)i(2)}(j(1))$. Then*

$$\varphi_{i(1)i(3)}(j(1)) = \varphi_{i(2)i(3)}(j(2)).$$

P r o o f. Denote $\varphi_{i(2)i(3)}(j(2)) = j(3)$. Then

$$A_{i(1)j(1)} \subseteq A_{i(2)j(2)} \subseteq A_{i(3)j(3)},$$

whence $\varphi_{i(1)i(3)}(j(1)) = j(3)$. □

Let $i(1) \in I$ and $j(1) \in J(i(1))$. We put

$$B_{i(1)j(1)} = \bigcup_{i(2),j(2)} A_{i(2)j(2)},$$

where $i(2)$ runs over the set $\{i(2) \in I \colon i(2) \geqslant i(1)\}$ and for each such $i(2)$ we have $j(2) = \varphi_{i(1)i(2)}(j(1))$.

Let us remark that if $i(1) \in I$ and $j(1), j'(1)$ are distinct elements of $J(i(1))$, then $B_{i(1)j(1)}$ and $B_{i(1)j'(1)}$ can be equal. Further, if $i(1) < i(2)$ and $j(2) = \varphi_{i(1)i(2)}(j(1))$, then according to 4.2 we have

$$B_{i(1)j(1)} = B_{i(2)j(2)}.$$

Let C_k be the system of all directed sets $B_{i(1)j(1)}$, where $i(1)$ runs over the set I, and for each $i(1) \in I$, $j(1)$ runs over the set $J_{i(1)}$.

Let $i(1) \in I$ and $k \in K$. Consider the relation (10) and denote

$$\begin{aligned}
J^a_{i(1)} &= \{j \in J(i(1)) \colon A_{i(1)j} \subseteq C_k\},\\
J^b_{i(1)} &= J(i(1)) \setminus J^a_{i(1)},\\
L^a_{i(1)} &= \prod_{j \in J^a_{i(1)}} A_{i(1)j},\\
L^b_{i(1)} &= \prod_{j \in J^b_{i(1)}} A_{i(1)j}.
\end{aligned}$$

Then

$$(10'') \qquad L_{i(1)} = L^a_{i(1)} \times L^b_{i(1)}.$$

Also, from the definition of C_k we obtain

4.3. Lemma. *Let $i(1), i(2) \in I$, $i(1) < i(2)$. Then $L^a_{i(1)}$ is an interval of $L^a_{i(2)}$ and $L^b_{i(1)}$ is an interval of $L^b_{i(2)}$. Moreover,*

$$C_k = \bigcup_{i(1)\in I} L^a_{i(1)}.$$

We put

$$C^*_k = \bigcup_{i(1)\in I} L^b_{i(1)}.$$

4.4. Lemma. *Let $i(1), i(2) \in I$, $i(1) < i(2)$, $x \in L_{i(1)}$. Then*

$$x(L^a_{i(1)}) = x(L^a_{i(2)}),$$

$$x(L^b_{i(1)}) = x(L^b_{i(2)}).$$

Proof. This is a consequence of $(10'')$, 4.3 and 2.8. □

Let $x \in L$. There exists $i(1) \in I$ such that $x \in L_{i(1)}$. Denote

$$x^a = x(L^a_{i(1)}), \quad x^b = x(L^b_{i(1)}).$$

In view of 4.4, the mapping $\psi\colon L \longrightarrow L \times L$ defined by

$$\psi(x) = (x^a, x^b)$$

is correctly defined.

Clearly $x^a \in C_k$ and $x^b \in C^*_k$.

4.5. Lemma. *Let $x, y \in L$. Then $x \leqslant y$ if and only if $x^a \leqslant y^a$ and $x^b \leqslant y^b$.*

Proof. There exists $i(1) \in I$ such that both x and y belong to $L_{i(1)}$. Let $x \leqslant y$. Then in view of the definition of the mapping ψ we have $x^a \leqslant y^a$ and $x^b \leqslant y^b$. Conversely, suppose that $x^a \leqslant y^a$ and $x^b \leqslant y^b$. Thus $(10'')$ yields that $x \leqslant y$. □

4.6. Lemma. *Let $z_1 \in C_k$, $z_2 \in C_k^*$. There exists $x \in L$ such that $\psi(x) = (z_1, z_2)$.*

P r o o f. There is $i(1) \in I$ with $z_1, z_2 \in L_{i(1)}$. Then $z_1 \in L^a_{i(1)}$ and $z_2 \in L^b_{i(1)}$. Now it suffices to apply $(10'')$. $\square$

Also, from the definition of ψ we immediately obtain

4.7. Lemma. *Let $x \in L$. Then*

(i) $x \in C_k \Leftrightarrow \psi(x) = (x, s^0)$,
(ii) $x \in C_k^* \Leftrightarrow \psi(x) = (s^0, x)$.

From 4.5, 4.6 and 4.7 we infer

4.8. Lemma. *The mapping ψ defines an internal direct product decomposition*

$$L = C_k \times C_k^*.$$

4.9. Lemma. *Let $A \in D(L)$, $i(1) \in I$, $j(1) \in J(i(1))$, $A \cap A_{i(1)j(1)} \neq \{s^0\}$. Then $B_{i(1)j(1)} \subseteq A$.*

P r o o f. Since $A_{i(1)j(1)}$ is directly indecomposable, from $A \cap A_{i(1)j(1)} \neq \{s^0\}$ we obtain $A \cap A_{i(1)j(1)} = A_{i(1)j(1)}$, thus $A_{i(1)j(1)} \subseteq A$.

Let $i(2) > i(1)$. Denote $\varphi_{i(1)i(2)}(j(1)) = j(2)$. Hence by the same reasoning as we have applied to $A_{i(1)j(1)}$ we get $A_{i(2)j(2)} \subseteq A$. Therefore $B_{i(1)j(1)} \subseteq A$. $\square$

4.10. Lemma. *Let $k \in K$. Then C_k is directly indecomposable.*

P r o o f. By way of contradiction, suppose that C_k is directly decomposable. Hence it can be represented in the form

$$C_k = A \times B, \quad A \neq \{s^0\} \neq B.$$

There is $i(1) \in I$ and $j(1) \in J(i(1))$ such that $C_k = B_{i(1)j(1)}$. Hence $A_{i(1)j(1)}$ is an interval of C_k. This yields

$$A_{i(1)j(1)} = (A_{i(1)j(1)} \cap A) \times (A_{i(1)j(1)} \cap B).$$

Since $A_{i(1)j(1)}$ is directly indecomposable, without loss of generality we can suppose that

$$A_{i(1)j(1)} = A_{i(1)j(1)} \cap A.$$

Thus in view of 4.9 we obtain the relation $C_k = B_{i(1)j(1)} \subseteq A$, whence $B = \{s^0\}$, which is a contradiction. $\square$

4.11. Lemma. *Let $\{s^0\} \neq A \in D(L)$. Then the following conditions are equivalent:*

(i) *A is an atom of $D(L)$.*
(ii) *A is directly indecomposable.*

The proof is the same as in [4], Lemma 2.1.

4.12. Lemma. *Let $A \in D(L)$, $A \neq \{s^0\}$. Then there exist $i(1) \in I$ and $j(1) \in J(i(1))$ such that $A \cap A_{i(1)j(1)} \neq \{s^0\}$.*

Proof. There exists $x \in A$ with $x \neq s^0$. Further, there exists $i(1) \in I$ such that $x \in L_{i(1)}$. Consider the relation (10). There is $j(1) \in J(i(1))$ such that

$$x(A_{i(1)j(1)}) \neq s^0.$$

Hence $A \cap A_{i(1)j(1)} = A(A_{i(1)j(1)}) \neq \{s^0\}$. □

Proof of (A). It suffices to apply 4.8–4.12. □

5. Examples

Let $\mathcal{L}_a$ and $\mathcal{L}_b$ be as in Section 1.

From 4.11 and 2.7 it easily follows that $\mathcal{L}_b$ is a subclass of $\mathcal{L}_a$.

5.1. Example. Let L be the system of all finite subsets of an infinite set M; this system is partially ordered by the set-theoretical inclusion. Then L belongs to $\mathcal{L}_a$, but it does not belong to $\mathcal{L}_b$.

In particular, let M be the set of all positive integers, $s^0 = \emptyset$. For each $n \in M$ put $v_n = \{1, 2, \ldots, n\}$, $L_n = [s^0, v_n]$. Then $L_n \in \mathcal{L}_b$ for each $n \in M$, $\bigcup_{n \in M} L_n = L$, hence L satisfies the assumptions of (A). Nevertheless, $L \notin \mathcal{L}_b$.

5.2. Example. There exists an infinite Boolean algebra X such that X has no atom. Let $L = X \cup \{y\}$ be such that $y \notin X$ and y is the greatest element of L. Further let s^0 be the least element of X. Then $D(L) = \{\{s^0\}, L\}$, whence $L \in \mathcal{L}_b$. On the other hand, X is an interval of L and for each $x \in X$, the interval $[s^0, x]$ belongs to $D(X)$, hence the partially ordered set $D(X)$ is isomorphic to X. Therefore $D(L)$ fails to be atomic, i.e., X does not belong to $\mathcal{L}_a$.

5.3. Example. The assertion of Lemma 2.8 cannot be extended to the case when X is a convex subset of L with $s^0 \in X$. Indeed, let M be an infinite set and let L be the Boolean algebra of all subsets of M; put $s^0 = \emptyset$. For each $m \in M$ let $L_m = \{\emptyset, \{m\}\}$. Then $L = \prod\limits_{m \in M} L_m$. Let X be the system consisting of all finite subsets of M. This system is directed, convex in L and for each $m \in M$ we have $X \cap L_m = L_m$. However, $X \neq \prod\limits_{m \in M} L_m$.

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