ASYMPTOTIC RELATIONSHIP BETWEEN SOLUTIONS OF TWO LINEAR DIFFERENTIAL SYSTEMS

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Abstract. In this paper new generalized notions are defined: $\mathbf{\Psi}$-boundedness and $\mathbf{\Psi}$-asymptotic equivalence, where $\mathbf{\Psi}$ is a complex continuous nonsingular $n \times n$ matrix. The $\mathbf{\Psi}$-asymptotic equivalence of linear differential systems $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ and $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{x}$ is proved when the fundamental matrix of $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ is $\mathbf{\Psi}$-bounded.

Keywords: $\mathbf{\Psi}$-boundedness, $\mathbf{\Psi}$-asymptotic equivalence

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The problem of the asymptotic relationship between solutions of two systems of differential equations was investigated by several authors, e.g. H. Weyl, N. Levinson, A. Wintner, R. Conti, M. Švec, A. Haščák, T. G. Hallam, etc.

The aim of this paper is to study the so-called $\mathbf{\Psi}$-asymptotic equivalence of two linear differential systems of the form

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y}, \tag{1}$$

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{x} \tag{2}$$

where $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are $n \times n$ matrices whose elements are complex functions defined on the interval $[0, \infty)$ and integrable on compact subsets of $[0, \infty)$; $\mathbf{x}$ and $\mathbf{y}$ are n-dimensional column vectors. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are such that the existence of solutions of (1) and (2) on an interval $[a, \infty)$, $a \geqslant 0$ is guaranteed. So under a solution of a differential system we will understand a solution existing on an infinite interval $[a, \infty)$.

In the papers [4–7] and in the monograph [3] it is assumed that $\mathbf{\Psi}$ is a positive real continuous function. The notions of the $\mathbf{\Psi}$-boundedness of a vector function and of the $\mathbf{\Psi}$-asymptotic equivalence of two systems of differential equations are defined there (see e.g. [3] p. 274–275 or [7] Definitions 1 and 2). In this paper these notions are generalized as follows:

Definition 1. Let $\mathbf{\Psi}(t)$ be a complex continuous nonsingular $n \times n$ matrix. An $n \times m$ complex matrix-function $\mathbf{M}(t)$ will be called $\mathbf{\Psi}$-*bounded* on the interval $[a, \infty)$ iff the matrix $\mathbf{\Psi}^{-1}(t)\mathbf{M}(t)$ is bounded on $[a, \infty)$, i.e.

$$\sup_{t \geqslant a} |\mathbf{\Psi}^{-1}(t)\mathbf{M}(t)| < \infty \tag{3}$$

where $|\cdot|$ denotes a suitable matrix norm and $\mathbf{\Psi}^{-1}$ denotes the inverse matrix to $\mathbf{\Psi}$, i.e. $\mathbf{\Psi}\mathbf{\Psi}^{-1} = \mathbf{\Psi}^{-1}\mathbf{\Psi} = \mathbf{E}$, where $\mathbf{E}$ is the unit matrix.

Definition 2. Let $\mathbf{Y}(t)$ and $\mathbf{X}(t)$ be fundamental matrices of the systems (1) and (2), respectively and let $\mathbf{\Psi}(t)$ be a complex continuous nonsingular matrix. The systems (1) and (2) will be called $\mathbf{\Psi}$-*asymptotically equivalent* iff for every fundamental matrix $\mathbf{Y}(t)$ of (1) there is a fundamental matrix $\mathbf{X}(t)$ of (2) such that

$$|\mathbf{\Psi}^{-1}(t)[\mathbf{X}(t) - \mathbf{Y}(t)]| \to 0 \quad \text{as} \quad t \to \infty \tag{4}$$

and conversely, for each fundamental matrix $\mathbf{X}(t)$ of (2) there is $\mathbf{Y}(t)$ of (1) such that (4) holds.

Since every solution $\mathbf{y}(t)$ of (1) and $\mathbf{x}(t)$ of (2) may be represented in the form $\mathbf{y}(t) = \mathbf{Y}(t)\mathbf{c}$ and $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c}$, where $\mathbf{c}$ is a suitable constant column vector, (4) implies

$$|\mathbf{\Psi}^{-1}(t)[\mathbf{x}(t) - \mathbf{y}(t)]| \to 0 \quad \text{as} \quad t \to \infty. \tag{4'}$$

So the systems (1) and (2) are $\mathbf{\Psi}$-asymptotically equivalent iff for every solution $\mathbf{y}(t)$ of (1) there is a solution $\mathbf{x}(t)$ of (2) and conversely, for each solution $\mathbf{x}(t)$ of (2) there is a solution $\mathbf{y}(t)$ of (1) such that (4′) hods.

Remark 1. If in the relation (4′) $\mathbf{\Psi}(t)$ means a positive continuous function on $[a, \infty)$, then the definition of the $\mathbf{\Psi}$-asymptotic equivalence in [7] (or in [3] p. 274) is obtained.

Remark 2. Evidently, if the fundamental matrix of some system is $\mathbf{\Psi}$-bounded, then every solution of the same system is $\mathbf{\Psi}$-bounded.

Remark 3. Let $\mathbf{\Psi}(t)$ be a positive continuous real function on $[a, \infty)$. Then the definitions of $\mathbf{\Psi}$-boundedness for a vector function in this paper and in [7] (Definition 1) or in [3] p. 275 coincide.

Now three new theorems will be presented. The integral will be the Lebesgue integral.

Theorem 1. *Let* $\mathbf{\Psi}(t)$ *and* $\mathbf{\Phi}(t)$ *be complex continuous nonsingular matrices on* $[a,\infty)$ *and let the following hypotheses be satisfied:*
(a) $\int_a^\infty |\mathbf{\Phi}^{-1}(t)\mathbf{B}(t)\mathbf{\Psi}(t)|\,\mathrm{d}t < \infty$,
(b) *the fundamental matrix* $\mathbf{Y}(t)$ *of* (1) *is* $\mathbf{\Psi}$*-bounded on* $[a,\infty)$,
(c) *there is a number* $K_0 > 0$ *such that* $\sup\limits_{a\leqslant s\leqslant t} |\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)| < K_0$.
Then the fundamental matrix $\mathbf{X}(t)$ *of* (2) *is* $\mathbf{\Psi}$*-bounded on* $[a,\infty)$.

P r o o f. Using the method of variation of constants the solution $\mathbf{x}(t)$ and the fundamental matrix $\mathbf{X}(t)$ of (2) will be expressed in the form

$$\mathbf{x}(t) = \mathbf{y}(t) + \int_a^t \mathbf{Y}(t)\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{x}(s)\,\mathrm{d}s, \tag{5}$$

$$\mathbf{X}(t) = \mathbf{Y}(t) + \int_a^t \mathbf{Y}(t)\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}(s)\,\mathrm{d}s. \tag{6}$$

Because according to the assumption the nonsingular matrices $\mathbf{\Psi}(t)$ and $\mathbf{\Phi}(t)$ exist, the relation (6) may be written in the form

$$\begin{aligned} \text{(7)}\quad \mathbf{\Psi}^{-1}(t)\mathbf{X}(t) &= \mathbf{\Psi}^{-1}(t)\mathbf{Y}(t) \\ &\quad + \int_a^t \mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)\mathbf{\Psi}^{-1}(s)\mathbf{X}(s)\,\mathrm{d}s. \end{aligned}$$

By (b) a positive constant K exists such that $|\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)| < K$, hence from (7) by hypothesis (c) the following inequality can be obtained:

$$|\mathbf{\Psi}^{-1}(t)\mathbf{X}(t)| \leqslant K + \int_a^t K_0|\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,|\mathbf{\Psi}^{-1}(s)\mathbf{X}(s)|\,\mathrm{d}s. \tag{8}$$

Using the Gronwall lemma for the inequality (8) we get

$$|\mathbf{\Psi}^{-1}(t)\mathbf{X}(t)| < K\exp\Big(K_0\int_a^t |\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,\mathrm{d}s\Big).$$

From the assumption (a) it follows that there is a number $t^* \geqslant a$ such that

$$K_0\int_{t^*}^\infty |\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,\mathrm{d}s \leqslant \ln 2$$

and therefore

$$|\mathbf{\Psi}^{-1}(t)\mathbf{X}(t)| < K\exp(\ln 2) = 2K,$$

i.e. fundamental matrix $\mathbf{X}(t)$ of (2) is $\mathbf{\Psi}$-bounded.

From (6) the following expression for the fundamental matrix $\mathbf{Y}(t)$ of (1) can be obtained:

$$\mathbf{Y}(t) = \mathbf{X}(t) - \int_a^t \mathbf{Y}(t)\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}(s)\,\mathrm{d}s,$$

which implies $\square$

Corollary 1. *If a fundamental matrix $\mathbf{X}(t)$ of (2) is $\mathbf{\Psi}$-bounded and the hypotheses (a) and (c) of Theorem 1 are satisfied, then the fundamental matrix $\mathbf{Y}(t)$ of (1) is $\mathbf{\Psi}$-bounded.*

Theorem 2. *Let $\mathbf{Y}(t)$ be a fundamental matrix of (1). Let $\mathbf{\Psi}(t)$ and $\mathbf{\Phi}(t)$ be complex continuous nonsingular matrices for $t \geqslant a \geqslant 0$.*

Suppose

(i) *the assumptions (a) and (b) of Theorem 1 are fulfilled*

(ii) *there exist supplementary projections $\mathbf{P}_1$, $\mathbf{P}_2$ and constants $K_1 > 0$, $K_2 > 0$ such that*

$$\sup_{a\leqslant s\leqslant t} |\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)| < K_1,$$
$$\sup_{t\leqslant s\leqslant \infty} |\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)| < K_2;$$

(iii) $|\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_1| \to 0$ *as* $t \to \infty$.

Then the systems (1) and (2) are $\mathbf{\Psi}$-asymptotically equivalent.

Proof. Let $\mathbf{Y}(t)$ be a fundamental matrix of (1) which is $\mathbf{\Psi}$-bounded by hypothesis, i.e. $|\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)| \leqslant K$, $K > 0$. Consider the integral equation

$$\mathbf{X}(t) = \mathbf{Y}(t) + \int_a^t \mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}(s)\,\mathrm{d}s - \int_t^\infty \mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}(s)\,\mathrm{d}s \tag{9}$$

for the matrix function $\mathbf{X}(t)$, $\mathbf{P}_1$, $\mathbf{P}_2$ being the given supplementary projections, $\mathbf{P}_2 = \mathbf{I} - \mathbf{P}_1$ where $\mathbf{I}$ denotes the $n \times n$ identity matrix and $\mathbf{P}_1\mathbf{P}_1 = \mathbf{P}_1$, $\mathbf{P}_2\mathbf{P}_2 = \mathbf{P}_2$.

If $\mathbf{X}(t)$ satisfies (9) then it is a matter of routine to verify that

$$\mathbf{X}' = \mathbf{A}(t)\mathbf{X} + \mathbf{B}(t)\mathbf{X}. \tag{10}$$

The existence of a solution $\mathbf{X}$ of the equation (9) is not evident; the existence of such a solution will be proved using the method of successive approximations.

The successive approximations for (9) are defined to be the matrix-functions $\mathbf{X}_0, \mathbf{X}_1, \ldots,$ given recursively by the formulae

$$\begin{aligned} &\mathbf{X}_0(t) = \mathbf{Y}(t), \\ (11) \qquad &\mathbf{X}_{m+1}(t) = \mathbf{Y}(t) + \int_b^t \mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}_m(s)\,\mathrm{d}s \\ &\qquad - \int_t^\infty \mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}_m(s)\,\mathrm{d}s, \end{aligned}$$

$m = 0, 1, 2, \ldots$, where $b \geqslant a$ is a number such that

$$(12) \qquad (K_1 + K_2)\int_b^\infty |\mathbf{\Phi}^{-1}(t)\mathbf{B}(t)\mathbf{\Psi}(t)|\,\mathrm{d}t < \tfrac{1}{2}$$

and K_1, K_2 are numbers which fulfil (ii).

It remains to prove the convergence of the sequence $\{\mathbf{X}_m\}$. Let $\mathbf{G}_m$, $m = 0, 1, 2, \ldots$ be defined by

$$\mathbf{G}_m(t) = \mathbf{X}_{m+1}(t) - \mathbf{X}_m(t), \quad t \in [b, \infty).$$

Then (11) by subtraction yields

(13)

$$\begin{aligned} \mathbf{G}_0(t) &= \int_b^t \mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}_0(s)\,\mathrm{d}s - \int_t^\infty \mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}_0(s)\,\mathrm{d}s, \\ \mathbf{G}_m(t) &= \int_b^t \mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{G}_{m-1}(s)\,\mathrm{d}s \\ &\quad - \int_t^\infty \mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{G}_{m-1}(s)\,\mathrm{d}s, \quad m = 1, 2, \ldots. \end{aligned}$$

Since it is assumed that there are nonsingular matrices $\mathbf{\Psi}(t)$ and $\mathbf{\Phi}(t)$, (13) yields

(14,1)

$$\begin{aligned} \mathbf{\Psi}^{-1}(t)\mathbf{G}_0(t) &= \int_b^t \mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)\mathbf{\Psi}^{-1}(s)\mathbf{X}_0(s)\,\mathrm{d}s \\ &\quad - \int_t^\infty \mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)\mathbf{\Psi}^{-1}(s)\mathbf{X}_0(s)\,\mathrm{d}s, \end{aligned}$$

(14,2)

$$\begin{aligned} &\mathbf{\Psi}^{-1}(t)\mathbf{G}_m(t) \\ &\quad = \int_b^t \mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)\mathbf{\Psi}^{-1}(s)\mathbf{G}_{m-1}(s)\,\mathrm{d}s \\ &\quad - \int_t^\infty \mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)\mathbf{\Psi}^{-1}(s)\mathbf{G}_{m-1}(s)\,\mathrm{d}s \end{aligned}$$

for $m = 1, 2, \ldots$

Next, the validity of the estimate

$$|\mathbf{\Psi}^{-1}(t)\mathbf{G}_m(t)| \leqslant \frac{K}{2^{m+1}},\ m = 0, 1, 2, \ldots \tag{15}$$

for some $K > 0$ will be proved by induction.

The equation (14,1) gives

(16)

$$\begin{aligned} &|\mathbf{\Psi}^{-1}(t)\mathbf{G}_0(t)| \\ &\quad \leqslant \int_b^t |\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)|\,|\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,|\mathbf{\Psi}^{-1}(s)\mathbf{X}_0(s)|\,\mathrm{d}s \\ &\qquad + \int_t^\infty |\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)||\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,|\mathbf{\Psi}^{-1}(s)\mathbf{X}_0(s)|\,\mathrm{d}s. \end{aligned}$$

Using (ii) and the fact that $\mathbf{Y}$ is $\mathbf{\Psi}$-bounded, we have

$$|\mathbf{\Psi}^{-1}(t)\mathbf{X}_0(t)| = |\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)| \leqslant K,\ K > 0.$$

From (16) it follows that

$$\begin{aligned} |\mathbf{\Psi}^{-1}(t)\mathbf{G}_0(t)| \leqslant K_1K \int_b^t |\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,\mathrm{d}s \\ + K_2K \int_t^\infty |\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,\mathrm{d}s. \end{aligned} \tag{17}$$

Further it is evident that from (17) we can obtain

$$|\mathbf{\Psi}^{-1}(t)\mathbf{G}_0(t)| \leqslant K(K_1 + K_2) \int_b^\infty |\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,\mathrm{d}s$$

and finally by (12)

$$|\mathbf{\Psi}^{-1}(t)\mathbf{G}_0(t)| \leqslant \tfrac{1}{2}K.$$

Consequently, (15) holds for $m = 0$.

Assume that

$$|\mathbf{\Psi}^{-1}(t)\mathbf{G}_{m-1}(t)| \leqslant \frac{K}{2^m}, \quad m = 1, 2, \ldots \tag{18}$$

From (14,2) it follows that

(19)

$$\begin{aligned} &|\mathbf{\Psi}^{-1}(t)\mathbf{G}_m(t)| \\ &\quad < \int_b^t |\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)||\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)||\mathbf{\Psi}^{-1}(s)\mathbf{G}_{m-1}(s)|\,\mathrm{d}s \\ &\qquad + \int_t^\infty |\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)||\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,|\mathbf{\Psi}^{-1}(s)\mathbf{G}_{m-1}(s)|\,\mathrm{d}s. \end{aligned}$$

Using (ii) and (18) we obtain from (19) that

$$
\begin{aligned}
&|\boldsymbol{\Psi}^{-1}(t)\mathbf{G}_m(t)| \\
&\qquad \leqslant K_1 \frac{K}{2^m} \int_b^t |\boldsymbol{\Phi}^{-1}(s)\mathbf{B}(s)\boldsymbol{\Psi}(s)| \,\mathrm{d}s + K_2 \frac{K}{2^m} \int_t^\infty |\boldsymbol{\Phi}^{-1}(s)\mathbf{B}(s)\boldsymbol{\Psi}(s)| \,\mathrm{d}s
\end{aligned}
$$

and evidently

$$
|\boldsymbol{\Psi}^{-1}(t)\mathbf{G}_m(t)| \leqslant \frac{K}{2^m}(K_1 + K_2) \int_b^\infty |\boldsymbol{\Phi}^{-1}(s)\mathbf{B}(s)\boldsymbol{\Psi}(s)| \,\mathrm{d}s \leqslant \frac{K}{2^{m+1}}
$$

if finally (12) is used. So (15) is proved. From (15) it follows that

$$
|\mathbf{G}_m(t)| \leqslant |\boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t)\mathbf{G}_m(t)| < |\boldsymbol{\Psi}(t)|\ |\boldsymbol{\Psi}^{-1}(t)\mathbf{G}_m(t)| < |\boldsymbol{\Psi}(t)| \frac{K}{2^{m+1}}. \tag{20}
$$

Consider the series

$$
\sum_{m=0}^{\infty} |\mathbf{G}_m(t)| = \sum_{m=0}^{\infty} |\mathbf{X}_{m+1}(t) - \mathbf{X}_m(t)|. \tag{21}
$$

(20) shows that the series (21) is majorized by the convergent number series

$$
K|\boldsymbol{\Psi}(t)| \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} = |\boldsymbol{\Psi}(t)|K \tag{22}
$$

and therefore the series (21) is uniformly convergent on every compact subset of the interval $[b, \infty)$. Thus the series

$$
\mathbf{X}_0(t) + \sum_{m=0}^{\infty} [\mathbf{X}_{m+1}(t) - \mathbf{X}_m(t)]
$$

is absolute and uniformly convergent on every compact subset of the interval $[b, \infty)$; consequently the partial sum

$$
\mathbf{X}_0(t) + \sum_{m=0}^{k-1} [\mathbf{X}_{m+1}(t) - \mathbf{X}_m(t)] = \mathbf{X}_k(t) \tag{23}
$$

tends uniformly on every compact subset of interval $[b, \infty)$ to a continuous limit function $\mathbf{X}$ satisfying

$$
|\mathbf{X}(t)| \leqslant 2K|\boldsymbol{\Psi}(t)| \quad \text{and} \quad |\boldsymbol{\Psi}^{-1}(t)\mathbf{X}(t)| \leqslant 2K \tag{24}
$$

[(22) and (23) imply (24)]. This means that the limit $\mathbf{X}(t)$ is $\mathbf{\Psi}$-bounded.

Clearly, $|\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{X}_m)| = |\mathbf{\Psi}^{-1}\mathbf{X}-\mathbf{\Psi}^{-1}\mathbf{X}_m| \leqslant |\mathbf{\Psi}^{-1}|\,|\mathbf{X}-\mathbf{X}_m|$ and also $\mathbf{\Psi}^{-1}\mathbf{X}_m$ converges uniformly to a continuous function $\mathbf{\Psi}^{-1}\mathbf{X}$ on every compact subset of the interval $[b,\infty)$.

It will be shown that $\mathbf{X}$ satisfies the integral equation (9) and therefore it is a solution of (10), i.e. $\mathbf{X}$ is the fundamental matrix of (2).

It is necessary to prove that

$$\begin{aligned}\text{(25)}\quad &\lim_{m\to\infty}\Big[\int_b^t \mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}_m(s)\,\mathrm{d}s - \int_t^\infty \mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}_m(s)\,\mathrm{d}s\Big]\\ &\quad= \int_b^t \mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}(s)\,\mathrm{d}s - \int_t^\infty \mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}(s)\,\mathrm{d}s.\end{aligned}$$

Clearly,

$$\begin{aligned}|\mathbf{U}| = \Big|&\int_b^t \mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}_m(s)\,\mathrm{d}s - \int_t^\infty \mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}_m(s)\,\mathrm{d}s\\ &- \int_b^t \mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}(s)\,\mathrm{d}s + \int_t^\infty \mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}(s)\,\mathrm{d}s\Big|\\ \leqslant{}& |\mathbf{\Psi}(t)|\\ &\times\Big[\int_b^t |\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)\|\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)\|\mathbf{\Psi}^{-1}(s)[\mathbf{X}(s)-\mathbf{X}_m(s)]|\,\mathrm{d}s\\ &+ \int_t^\infty |\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)\|\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)\|\mathbf{\Psi}^{-1}(s)[\mathbf{X}(s)-\mathbf{X}_m(s)]|\,\mathrm{d}s\Big].\end{aligned}$$

From (24) it follows that

$$\text{(25,1)}\qquad |\mathbf{\Psi}^{-1}[\mathbf{X}-\mathbf{X}_m]| \leqslant 4K.$$

The assumption (ii), (12) and (25,1) give

$$\begin{aligned}|\mathbf{U}| \leqslant |\mathbf{\Psi}(t)|\Big[&\frac{K_1}{2(K_1+K_2)}\int_0^t |\mathbf{\Psi}^{-1}(s)[\mathbf{X}(s)-\mathbf{X}_m(s)]|\,\mathrm{d}s\\ &+ 4KK_2\int_t^\infty |\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,\mathrm{d}s\Big].\end{aligned}$$

Let $\varepsilon > 0$ be chosen arbitrarily. Let $t_1 > b$ be such that

$$\int_{t_1}^\infty |\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,\mathrm{d}s < \frac{\varepsilon}{8KK_2}.$$

Such a t_1 exists by the assumption (b) the Theorem 1.

On the interval $[b,t_1]$ $\mathbf{\Psi}^{-1}\mathbf{X}_m$ converges to $\mathbf{\Psi}^{-1}\mathbf{X}$ uniformly. Hence for $\varepsilon>0$ there is $m_0(\varepsilon)$ such that for each $m>m_0(\varepsilon)$

$$|\mathbf{\Psi}^{-1}(s)[\mathbf{X}(s)-\mathbf{X}_m(s)]|<\frac{\varepsilon(K_1+K_2)}{K_1(t_1-b)},\quad s\in[b,t_1].$$

But then $|\mathbf{U}|\leqslant|\mathbf{\Psi}(t)|\varepsilon$ for $m>m_0(\varepsilon)$. The validity (25) is proved. Also $\mathbf{X}$ is the solution of (9).

It may be verified that if $\mathbf{X}(t)$ is a $\mathbf{\Psi}$-bounded fundamental matrix of (2), then there exists a $\mathbf{\Psi}$-bounded fundamental matrix $\mathbf{Y}(t)$ of (1), because by virtue of (9) $\mathbf{Y}(t)$ may be expressed in the form

$$\mathbf{Y}(t)=\mathbf{X}(t)-\int_a^t\mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}(s)\,\mathrm{d}s+\int_t^\infty\mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{X}(s)\,\mathrm{d}s.$$

To establish the $\mathbf{\Psi}$-asymptotic equivalence of (1) and (2) it remains to prove that

$$|\mathbf{\Psi}^{-1}(t)[\mathbf{X}(t)-\mathbf{Y}(t)]|\to 0\quad\text{as}\quad t\to\infty.$$

From (9) it may be obtained that

(26)
$$\begin{aligned}&\mathbf{\Psi}^{-1}(t)[\mathbf{X}(t)-\mathbf{Y}(t)]\\&\quad=\int_a^t\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)\mathbf{\Psi}^{-1}(s)\mathbf{X}(s)\,\mathrm{d}s\\&\qquad-\int_t^\infty\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)\mathbf{\Psi}^{-1}(s)\mathbf{X}(s)\,\mathrm{d}s.\end{aligned}$$

Denoting by $\mathbf{I}_1$ and $\mathbf{I}_2$ the first and the second integral in (26), the following inequality is valid:

$$|\mathbf{\Psi}^{-1}(t)[\mathbf{X}(t)-\mathbf{Y}(t)]|\leqslant|\mathbf{I}_1|+|\mathbf{I}_2|. \tag{27}$$

Assume that an arbitrary $\varepsilon>0$ is given. By the hypothesis (i) part (a) a number $t^*>a$ may be chosen such that

$$\int_{t^*}^\infty|\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,\mathrm{d}s<\frac{\varepsilon}{2KK_1}. \tag{27,1}$$

The assumption (ii) and (24) give

$$
\begin{aligned}
(28)\quad |\mathbf{I}_1| &\leqslant \int_a^t |\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)|\,|\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,|\mathbf{\Psi}^{-1}(s)\mathbf{X}(s)|\,\mathrm{d}s\\
&\leqslant 2K|\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_1|\int_a^{t^*} |\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)|\,|\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,\mathrm{d}s\\
&\quad + 2KK_1\int_{t^*}^{t} |\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,\mathrm{d}s\\
&\leqslant |\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_1|2K\int_a^{t^*} |\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)|\,|\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,\mathrm{d}s\\
&\quad + 2KK_1\int_{t^*}^{\infty} |\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,\mathrm{d}s.
\end{aligned}
$$

The hypothesis (iii) implies that the limit of the first term on the right hand side of the inequality (28) for $t \to \infty$ equals 0. From this and from (27,1) it follows that $\lim\limits_{t\to\infty} |\mathbf{I}_1| = 0$.

The second integral satisfies

$$|\mathbf{I}_2| \leqslant \int_t^\infty |\mathbf{\Psi}^{-1}(t)\mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{\Phi}(s)|\,|\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,|\mathbf{\Psi}^{-1}(s)\mathbf{X}(s)|\,\mathrm{d}s.$$

Using (i), (ii) and (24) we obtain

$$|\mathbf{I}_2| \leqslant 2KK_2\int_t^\infty |\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s)|\,\mathrm{d}s \to 0 \quad \text{as} \quad t\to\infty$$

and so (27) implies that $|\mathbf{\Psi}^{-1}(t)[\mathbf{X}(t)-\mathbf{Y}(t)]| \to 0$ as $t\to\infty$.

Thus the $\mathbf{\Psi}$-asymptotic equivalence of the systems (1) and (2) is proved. $\square$

R e m a r k 4. Assume that a positive continuous function $\mathbf{\Psi}_0(t)$ is such that the elements of the matrix $\mathbf{\Psi}(t)$ are majorized by $\mathbf{\Psi}_0(t)$. Then the matrix $\mathbf{\Psi}$-asymptotic equivalence which is defined in this paper implies the $\mathbf{\Psi}_0$-asymptotic equivalence of (1) and (2), i.e. the "scalar" asymptotic equivalence in the sense of the definition in [3] or [7].

Theorem 3. *Let $\mathbf{Y}(t)$ be a fundamental matrix of* (1)*. Let $\mathbf{M}_j$ and $\mathbf{N}_j$ be the sets of all $\mathbf{\Psi}_j$-bounded solutions of the systems* (1) *and* (2)*, respectively. Let there be supplementary projections $\mathbf{P}_1$ and $\mathbf{P}_2$, positive numbers K_1 and K_2 and nonsingular continuous complex matrices $\mathbf{\Psi}_j(t)$ and $\mathbf{\Phi}_j(t)$ on the interval $[a,\infty)$ such that the following conditions are fulfilled.*

(i) $\int_a^\infty |\mathbf{\Phi}_j^{-1}(t)\mathbf{B}(t)\mathbf{\Psi}_j(t)|\,\mathrm{d}t < \infty$,

(ii) $\sup\limits_{a\leqslant s\leqslant t} |\boldsymbol{\Psi}_j^{-1}(t)\mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\boldsymbol{\Phi}_j(s)| < K_1$,
$\sup\limits_{t\leqslant s\leqslant \infty} |\boldsymbol{\Psi}_j^{-1}(t)\mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\boldsymbol{\Phi}_j(s)| < K_2$,

(iii) $|\boldsymbol{\Psi}_j^{-1}(t)\mathbf{Y}(t)\mathbf{P}_1| \to 0$ *as* $t\to\infty$.

Then for every solution $\mathbf{y}(t)\in\mathbf{M}_j$ *there is a solution* $\mathbf{x}(t)\in\mathbf{N}_j$ *and the sets* $\mathbf{M}_j$ *and* $\mathbf{N}_j$ *are* $\boldsymbol{\Psi}_j$*-asymptotic equivalent, i.e.*

$$|\boldsymbol{\Psi}_j^{-1}(t)[\mathbf{x}(t)-\mathbf{y}(t)]| \to 0 \quad \textit{as} \quad t\to\infty.$$

P r o o f. If the solution $\mathbf{x}(t)$ of (2) is considered in the form

$$\mathbf{x}(t) = \mathbf{y}(t) + \int_a^t \mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{x}(s)\,\mathrm{d}s - \int_t^\infty \mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{x}(s)\,\mathrm{d}s \tag{29}$$

and $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ are replaced by $\boldsymbol{\Psi}_j$ and $\boldsymbol{\Phi}_j$ and successive approximations for (29) are given recursively by the formulae

$$\begin{aligned} \mathbf{g}_0(t) &= \mathbf{y}(t),\\ \mathbf{g}_{m+1}(t) &= \mathbf{y}(t) + \int_b^t \mathbf{Y}(t)\mathbf{P}_1\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{g}_m(s)\,\mathrm{d}s\\ &\quad - \int_t^\infty \mathbf{Y}(t)\mathbf{P}_2\mathbf{Y}^{-1}(s)\mathbf{B}(s)\mathbf{g}_m(s)\,\mathrm{d}s, \end{aligned} \tag{30}$$

and if the limit of (30) is denoted by $\mathbf{x}(t)$, then Theorem 3 can be proved in the same way as Theorem 2 was. □

R e m a r k 5. In the concrete cases the new generalized notion of $\boldsymbol{\Psi}$-asymptotic equivalence enables us to determine weaker sufficient conditions.

Another advantage of this generalized notion is that every component of the vector-solution of some system may be particularly estimated, while in the case of the scalar function $\boldsymbol{\Psi}$ the estimate of the solution of the system is obtained in the whole, i.e. all components of the vector-solution have the same estimate.

The assertions of Remark 5 will be shown on a simple example.

E x a m p l e. Consider the systems

$$\mathbf{y}' = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\mathbf{y}, \tag{31}$$

$$\mathbf{x}' = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\mathbf{x} + \mathbf{R}(t)\mathbf{x}. \tag{32}$$

The fundamental matrix of (31) has the form

$$\mathbf{Y} = \mathrm{e}^{\lambda t}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}.$$

Choosing the matrix functions

$$\mathbf{\Psi} = \mathrm{e}^{\lambda t}\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}, \qquad \mathbf{\Phi} = \mathrm{e}^{\lambda t}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and the scalar functions $\mathbf{\Psi}_0 = t\mathrm{e}^{\lambda t}$, $\mathbf{\Phi}_0 = \mathrm{e}^{\lambda t}$ and the projections

$$\mathbf{P}_1 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \qquad \mathbf{P}_2 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

we may easily verify that if the matrix $\mathbf{R}(t)$ satisfies the assumption (a) of Theorem 1 the other hypotheses of Theorem 2 are fulfilled as well.

Consequently, the systems (31) and (32) are $\mathbf{\Psi}$-asymptotically equivalent and also $\mathbf{\Psi}_0$-asymptotically equivalent, i.e. every solution of (32) satisfies

$$\left|\mathrm{e}^{-\lambda t}\begin{pmatrix}t^{-1} 0\\ 0 \quad 1\end{pmatrix}[\mathbf{x}(t)-\mathbf{y}(t)]\right| \to 0 \quad \text{as} \quad t\to\infty$$

and

$$\left|t^{-1}\mathrm{e}^{-\lambda t}[\mathbf{x}(t)-\mathbf{y}(t)]\right| \to 0 \quad \text{as} \quad t\to\infty$$

or

$$\mathbf{x}(t) = \mathbf{y}(t) + \mathrm{e}^{\lambda t}\begin{pmatrix}0(t)\\ 0(1)\end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \mathbf{y}(t) + \mathrm{e}^{\lambda t}\begin{pmatrix}0(t)\\ 0(t)\end{pmatrix}.$$

From this it is seen that the assertions about the estimates of the solutions are true.

Further, in the case of the matrix-functions $\mathbf{\Psi}$ and $\mathbf{\Phi}$ the condition

$$\int^{\infty} |\mathbf{\Phi}^{-1}(t)\mathbf{R}(t)\mathbf{\Psi}(t)|\,\mathrm{d}t < \infty \tag{33}$$

(i.e. (a) of Theorem 1) applies if

$$\int^{\infty} t|\mathbf{R}_1(t)|\,\mathrm{d}t < \infty \quad \text{and} \quad \int^{\infty} |\mathbf{R}_2(t)|\,\mathrm{d}t < \infty \tag{34}$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the columns of the matrix $\mathbf{R}$ because

$$\mathbf{\Phi}^{-1}\mathbf{R}\mathbf{\Psi} = \mathrm{e}^{-\lambda t}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}(\mathbf{R}_1,\mathbf{R}_2)\mathrm{e}^{\lambda t}\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix} = (t\mathbf{R}_1,\mathbf{R}_2)$$

while in the case of scalar functions $\mathbf{\Psi}_0$ and $\mathbf{\Phi}_0$, (33) applies if

$$\int^{\infty} t|\mathbf{R}_1(t)|\,\mathrm{d}t < \infty \quad \text{and} \quad \int^{\infty} t|\mathbf{R}_2(t)|\,\mathrm{d}t < \infty. \tag{35}$$

(35) follows from the equalities

$$\mathbf{\Phi}_0^{-1}\mathbf{R}\mathbf{\Psi}_0 = \mathrm{e}^{-\lambda t}\mathbf{R}(t)\mathrm{e}^{\lambda t} = t(\mathbf{R}_1, \mathbf{R}_2) = (t\mathbf{R}_1, t\mathbf{R}_2).$$

Consequently, the conditions (34) are weaker than (35).

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