CHARACTERIZING THE INTERVAL FUNCTION OF A CONNECTED GRAPH

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Abstract. As was shown in the book of Mulder [4], the interval function is an important tool for studying metric properties of connected graphs. An axiomatic characterization of the interval function of a connected graph was given by the present author in [5]. (Using the terminology of Bandelt, van de Vel and Verheul [1] and Bandelt and Chepoi [2], we may say that [5] gave a necessary and sufficient condition for a finite geometric interval space to be graphic).

In the present paper, the result given in [5] is extended. The proof is based on new ideas.

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The letters h–n will be reserved for denoting non-negative integers. By a graph we will mean a finite undirected graph without multiple edges and loops (i.e. a graph in the sense of Chartrand and Lesniak [3], for example). If U is a nonempty set, then we denote by $\Omega(U)$ the set of all mappings of U into the set of all subsets of U.

Let G be a connected graph, and let $V(G)$, $E(G)$ and d_G denote its vertex set, its edge set, and its distance function, respectively. Following Mulder [4], we define the interval function I_G of G as follows:

$$I_G(x,z) = \{y \in V(G);\ y \text{ belongs to an } x\text{-}z \text{ path of length } d_G(x,z) \text{ in } G\}$$

for all x, $z \in V(G)$. Obviously, $I_G \in \Omega\bigl(V(G)\bigr)$.

Proposition 1. *Let G be a connected graph, and let J denote the interval function of G. Put $U = V(G)$. Then J fulfils the following Axioms* A–G *(for arbitrary u, v, x, $y \in U$):*

A *if $v \in J(u,x)$, then $J(v,x) \subseteq J(u,x)$;*

B *if* $v \in J(u,x)$ *and* $y \in J(v,x)$, *then* $v \in J(u,y)$;
C $u \in J(u,x)$;
D $|J(u,u)| = 1$;
E $J(u,x) = J(x,u)$;
F *if* $|J(u,v)| = 2 = |J(x,y)|$, $v \in J(u,x)$ *and* $u \in J(v,y)$, *then* $x \in J(v,y)$;
G *if* $|J(u,v)| = 2 = |J(x,y)|$ *and* $v \in J(u,x)$, *then either* $x \in J(v,y)$ *or* $y \in J(u,x)$ *or* $v \in J(u,y)$.

The validity of Axioms A–E follows from 1.1.2 in [4]. The verification of Axiom G was given in [5].

Verification of Axiom F: Let the assumption in F hold. Then $d_G(v,y) \leqslant d_G(v,x) + 1 = d_G(u,x) \leqslant d_G(u,y) + 1 = d_G(v,y)$. Hence $x \in J(v,y)$.

As will be shown in our theorem, Axioms A–G can be used for characterizing the interval function of a connected graph. A similar result was originally proved in [5].

In the theorem of [5], instead of Axiom F the following Axiom F_0 was used (u, v, x, y are arbitrary elements in U):

F_0 if $|J(u,v)| = 2 = |J(x,y)|, v \in J(u,x), u \in J(v,y)$ and $y \in J(u,x)$, then $x \in J(v,y)$.

Because of the proof of our theorem we prefer Axiom F to Axiom F_0.

Proposition 2. *Let* U *be a nonempty set, let* $J \in \Omega(U)$, *and let* J *fulfil Axioms* A–E *and* G. *Then it fulfils Axiom* F *if and only if it fulfils Axiom* F_0.

Proof. Obviously, F implies F_0. Conversely, let J fulfil Axiom F_0. Consider arbitrary $u, v, x, y \in U$ such that $|J(u,v)| = 2 = |J(x,y)|$, $v \in J(u,x)$ and $u \in J(v,y)$. We will show that $x \in J(v,y)$. Suppose, to the contrary, that $x \notin J(v,y)$. Axiom F_0 implies that $y \notin J(u,x)$. By Axiom G, $v \in J(u,y)$. Since $u \in J(v,y)$, we conclude that $u = v$, which is a contradiction. Thus J fulfils Axiom F. □

Proofs of the following two lemmas are not difficult and will be omitted. Note that the proof of Lemma 2 depends on the fact that U is finite.

Lemma 1. *Let* U *be a nonempty set, let* $J \in \Omega(U)$, *and let* J *fulfil Axioms* A, B *and* E. *Let* $x_0, \ldots, x_{n+m} \in U$, *let*

$$x_{n+1} \in J(x_n, x_0), \ldots, x_{n+m} \in J(x_{n+m-1}, x_0) \tag{1}$$

and

$$x_0 \in J(x_{n+1}, x_1), \ldots, x_{k-2} \in J(x_{n+k-1}, x_{k-1}), \tag{2}$$

where $2 \leqslant k \leqslant m$ *and* $n \geqslant 1$. *Then*

$$\begin{aligned} &x_{n+i+1} \in J(x_{n+i}, x_i), \ldots, x_{n+k} \in J(x_{n+k-1}, x_i) \text{ and} \\ (3) \qquad &x_{i-1} \in J(x_{n+i+1}, x_i), \ldots, x_{i-1} \in J(x_{n+k}, x_i) \text{ for each } i,\ 1 \leqslant i \leqslant k-1. \end{aligned}$$

Lemma 2. *Let* U *be a nonempty finite set, let* $J \in \Omega(U)$, *and let* J *fulfil Axioms* A–E. *If* u, $x \in U$ *and* $u \neq x$, *then*

$$J(u,x) - \{u\} = \bigcup_{\substack{v \in J(u,x) \\ |J(u,v)|=2}} J(v,x).$$

Remark 1. Let U be a finite nonempty set, let $J \in \Omega(U)$, let J fulfil Axioms A–E, let u, $x \in U$ and $u \neq x$. By Axiom C, $u \in J(u,x)$. Lemma 2 implies that there exists $w \in U$ such that $w \in J(u,x)$ and $\bigl|J(u,w)\bigr| = 2$. Consider an arbitrary $v \in J(u,x)$ such that $\bigl|J(u,v)\bigr| = 2$. Recall that U is finite. Lemma 2 implies that there exist w_0, w_1, …, $w_j \in U$, where $j \geqslant 1$, such that $w_0 = u$, $w_1 = v$, $w_j = x$,

$$\bigl|J(w_0, w_1)\bigr| = \ldots = \bigl|J(w_{j-1}, w_j)\bigr| = 2$$

and

$$w_1 \in J(w_0, x), \ldots, w_j \in J(w_{j-1}, x).$$

Let U be a finite nonempty set, and let $J \in \Omega(U)$. We will say that a graph G is the graph of J if $V(G) = U$ and t and z are adjacent in G if and only if $\bigl|J(t,z)\bigr| = 2$ for all distinct t, $z \in U$. Obviously, there exists exactly one graph of J. As follows from Remark 1, if J fulfils Axioms A–E, then the graph of J is connected.

It is clear that if G is a connected graph, then G is the graph of I_G.

The following theorem extends (and partially modifies) the result of [5]:

Theorem. *Let* U *be a finite nonempty set, let* $J \in \Omega(U)$, *and let* G *denote the graph of* J. *Then the following three statements are equivalent:*

(I) G *is connected and* $J = I_G$;

(II) J *fulfils Axioms* A–G *(for arbitrary* $u, v, x, y \in U$*)*;

(III) J *fulfils Axioms* A–F *(for arbitrary* $u, v, x, y \in U$*) and* $I_G(t,z) \subseteq J(t,z)$ *for all* t, $z \in U$.

Remark 2. Let U be a nonempty set, and let $J \in \Omega(U)$. It is not difficult to show that J is a geometric interval space in the sense of Bandelt, van de Vel and Verheul [1], Verheul [7] and Bandelt and Chepoi [2] if and only if J fulfils Axioms A–E. By our theorem, a finite geometric interval space is graphic in the sense of [1] and [2] if and only if it fulfils Axioms F and G.

In proving our theorem, we will need one more lemma. Let U be a finite nonempty set, and let J_1, $J_2 \in \Omega(U)$. Assume that J_1 and J_2 have the same graph. Let G denote the graph of J_1 and J_2, and let $n \geqslant 0$. We will write

$$J_1 \subseteq_{(n)} J_2 \text{ (or } J_1 =_{(n)} J_2)$$

if and only if $J_1(r,s) \subseteq J_2(r,s)$ for all r, $s \in U$ such that $d_G(r,s) \leqslant n$ (or $J_1(r,s) = J_2(r,s)$ for all $r, s \in U$ such that $d_G(r,s) \leqslant n$, respectively).

Lemma 3. *Let U be a finite nonempty set, let $n \geqslant 0$, $J \in \Omega(U)$, and let G denote the graph of J. Assume that J fulfils Axioms* A–F *(for arbitrary $u, v, x, y \in U$) and that $I_G \subseteq_{(n)} J$. Then $I_G =_{(n)} J$.*

P r o o f o f L e m m a 3. Let D denote the diameter of G. Instead of d_G and I_G we write d and I, respectively. We proceed by induction on n. The case when $n \leqslant 1$ is obvious. Assume that $n \geqslant 2$. Since $I \subseteq_{(n)} J$, we have $I \subseteq_{(n-1)} J$. By the induction hypothesis, $I =_{(n-1)} J$. If $D \leqslant n-1$, then $I =_{(n)} J$. Let $D \geqslant n$.

Consider arbitrary r, $s \in U$ such that $d(r,s) = n$. We want to prove that $J(r,s) \subseteq I(r,s)$. First, assume that $z \in I(r,s)$ for each $z \in J(r,s)$ such that $\left|J(r,z)\right| = 2$. By virtue of Lemma 2, $J(r,s) \subseteq I(r,s)$. Now, assume that there exists $t \in J(r,s)$ such that $\left|J(r,t)\right| = 2$ and $t \notin I(r,s)$.

There exist $x_0, \ldots, x_n \in I(r,s)$ such that $x_0 = s$, $x_n = r$ and the sequence

$$x_n, x_{n-1}, \ldots, x_0 \tag{4}$$

is a path in G. By virtue of Remark 1, there exist $x_{n+1}, \ldots, x_{n+m} \in U$, where $m \geqslant 1$, such that $x_{n+1} = t$, $x_{n+m} = x_0$,

$$\left|J(x_n, x_{n+1})\right| = \ldots = \left|J(x_{n+m-1}, x_{n+m})\right| = 2 \tag{5}$$

and (1) holds. Since the sequence

$$x_n, x_{n+1}, \ldots, x_{n+m} \tag{6}$$

is a path in G, $m \geqslant n$. Since $x_{n+1} \notin I(x_n, x_0)$, we have $d(x_{n+1}, x_0) \geqslant n$. Hence $m \geqslant n+1$.

We will show that

$$x_{m-1} \notin J(x_{n+m}, x_m). \tag{7}$$

Since $m > n$, (1) implies that $x_m \in J(x_{m-1}, x_{n+m})$. If $x_{m-1} \in J(x_m, x_{n+m})$, then Axioms B–D imply that $x_{m-1} = x_m$, which contradicts (5). Thus (7) holds.

By virtue of (7), there exists k, $1 \leqslant k \leqslant m$, such that

$$x_{k-1} \notin J(x_{n+k}, x_k) \tag{8}$$

and if $k \geqslant 2$, then (2) holds. Recall that $d(x_{n+1}, x_0) \geqslant n$. There exists h, $0 \leqslant h \leqslant k-1$, such that

$$d(x_{n+h+1}, x_h) \geqslant n \tag{9}$$

and

$$\text{if } h \leqslant k-2, \text{ then } d(x_{n+h+2}, x_{h+1}) \leqslant n-1. \tag{10}$$

By Lemma 1, if $k \geqslant 2$, then (3) holds. Combining this fact with (1), we get

$$x_{n+h+1} \in J(x_{n+h}, x_h). \tag{11}$$

Moreover, (3) implies that

$$\text{if } h \leqslant k-2, \text{ then } x_h \in J(x_{n+h+2}, x_{h+1}). \tag{12}$$

Clearly,

$$\left|J(x_h, x_{h+1})\right| = 2 = \left|J(x_{n+h}, x_{n+h+1})\right|. \tag{13}$$

Obviously, $d(x_{n+h+1}, x_{h+1}) \leqslant n$. It follows from (9) that $d(x_{n+h+1}, x_{h+1}) \geqslant n-1$. We distinguish two cases.

Case 1. Let $d(x_{n+h+1}, x_{h+1}) = n$. This implies that $x_{n+h} \in I(x_{n+h+1}, x_{h+1})$. Since $I \subseteq_{(n)} J$, we have

$$x_{n+h} \in J(x_{n+h+1}, x_{h+1}). \tag{14}$$

Combining (11), (13) and (14) with Axiom F, we get

$$x_h \in J(x_{n+h+1}, x_{h+1}). \tag{15}$$

It follows from (8) that $h \leqslant k-2$. According to (12), $x_h \in J(x_{n+h+2}, x_{h+1})$. By (10), $d(x_{n+h+2}, x_{h+1}) \leqslant n-1$. Since $I =_{(n-1)} J$, $x_h \in I(x_{n+h+2}, x_{h+1})$. Therefore, $d(x_{n+h+2}, x_h) \leqslant n-2$. This implies that $d(x_{n+h+1}, x_h) < n$, which contradicts (9).

C a s e 2. Let $d(x_{n+h+1}, x_{h+1}) = n-1$. It follows from (9) that $d(x_{n+h+1}, x_h) = n$. Hence $x_{h+1} \in I(x_{n+h+1}, x_h)$. Since $I \subseteq_{(n)} J$, we get

$$x_{h+1} \in J(x_{n+h+1}, x_h). \tag{16}$$

Combining (11) and (16) with Axiom B, we get

$$x_{n+h+1} \in J(x_{n+h}, x_{h+1}). \tag{17}$$

Since $d(x_{n+h}, x_{h+1}) \leqslant n-1$ and $I =_{(n-1)} J$, we see that $x_{n+h+1} \in I(x_{n+h}, x_{h+1})$. We have $d(x_{n+h+1}, x_{h+1}) \leqslant n-2$. This means that $d(x_{n+h+1}, x_h) < n$, which contradicts (9) again.

Thus $J(r,s) \subseteq I(r,s)$. We conclude that $I =_{(n)} J$, which completes the proof of the lemma. $\square$

P r o o f o f t h e t h e o r e m. Instead of d_G and I_G we write d and I, respectively. By Proposition 1, (I) implies (II).

Now, we will prove that (II) implies (III). Suppose, to the contrary, that (II) holds but (III) does not. Then there exists $n \geqslant 2$ such that $I \subseteq_{(n-1)} J$ but it is not true that $I \subseteq_{(n)} J$. Since J fulfils Axioms A–F, Lemma 3 implies that $I =_{(n-1)} J$. Clearly, there exist r, $s \in U$ such that $d(r,s) = n$ and $I(r,s) - J(r,s) \neq \emptyset$.

First, assume that $z \in J(r,s)$ for each $z \in I(r,s)$ such that $\left|I(r,z)\right| = 2$. Then we get $I(r,s) \subseteq J(r,s)$, which is a contradiction. Now, assume that there exists $t \in I(r,s)$ such that $\left|I(r,t)\right| = 2$ and $t \notin J(r,s)$. Obviously, there exist $x_0, \ldots, x_{n-1}, x_n \in U$ such that $x_0 = s$, $x_{n-1} = t$, $x_n = r$ and (4) is a path in G. Thus $x_{n-1} \notin J(x_n, x_0)$.

By virtue of Remark 1, there exist $x_{n+1}, \ldots, x_{n+m} \in U$, where $m \geqslant 1$, such that $x_{n+m} = x_0$, and (1) and (5) hold. Since (6) is a path in G, we have $m \geqslant n$. If $m = n$, then (7) holds. If $m > n$, then similarly as in the proof of Lemma 3 we get (7) again.

There exists k, $1 \leqslant k \leqslant m$, such that (8) holds and if $k \geqslant 2$, then (2) holds. Recall that $x_{n-1} \notin J(x_n, x_0)$ and $d(x_n, x_0) = n$. This implies that there exists h, $0 \leqslant h \leqslant k-1$, such that

$$x_{n+h-1} \notin J(x_{n+h}, x_h) \text{ and } d(x_{n+h}, x_h) = n \tag{18}$$

and

$$\text{if } h \leqslant k-2, \text{ then either } x_{n+h} \in J(x_{n+h+1}, x_{h+1}) \\ \text{or } d(x_{n+h+1}, x_{h+1}) \leqslant n-1. \tag{19}$$

By Lemma 1, if $k \geqslant 2$, then (3) holds. Combining this fact with (1), we get (11). Moreover, it is easy to see that (13) holds.

By (18), $d(x_{n+h}, x_h) = n$. Thus $x_{n+h-1} \in I(x_{n+h}, x_{h+1})$. Since $I \subseteq_{(n-1)} J$, we get $x_{n+h-1} \in J(x_{n+h}, x_{h+1})$. If $x_{h+1} \in J(x_{n+h}, x_h)$, then, combining Axioms A and E, we get $x_{n+h-1} \in J(x_{n+h}, x_h)$, which contradicts (18). Thus

$$x_{h+1} \notin J(x_{n+h}, x_h). \tag{20}$$

Obviously, $d(x_{n+h+1}, x_{h+1}) \leqslant n$. We distinguish two cases.

Case 1. Let (15) hold. As follows from (8), $h \leqslant k - 2$.

First, assume that $d(x_{n+h+1}, x_{h+1}) = n$. By virtue of (19), (14) holds. Combining (11), (13) and (14) with Axioms E and F, we get $x_{h+1} \in J(x_{n+h}, x_h)$, which contradicts (20).

Now, assume that $d(x_{n+h+1}, x_{h+1}) \leqslant n - 1$. Combining (15) with the fact that $J \subseteq_{(n-1)} I$, we get $x_h \in J(x_{n+h+1}, x_{h+1})$. Therefore, $d(x_{n+h+1}, x_h) \leqslant n - 2$. This implies that $d(x_{n+h}, x_h) < n$, which contradicts (18).

Case 2. Let $x_h \notin J(x_{n+h+1}, x_{h+1})$. Combining this fact with (11), (13), (20) and Axiom G, we see that (17) holds. Since $d(x_{n+h}, x_{h+1}) = n - 1$, the fact that $J \subseteq_{(n-1)} I$ implies that $x_{n+h+1} \in I(x_{n+h}, x_{h+1})$. Thus $d(x_{n+h+1}, x_{h+1}) = n-2$. This means that $d(x_{n+h+1}, x_h) \leqslant n - 1$. It follows from (18) that $d(x_{n+h+1}, x_h) = n - 1$. This implies that $x_{h+1} \in I(x_{n+h+1}, x_h)$. Since $I \subseteq_{(n-1)} J$, (16) holds. Combining (11) and (16) with Axiom A, we see that $x_{h+1} \in J(x_{n+h}, x_h)$, which contradicts (20).

Thus $I(r, s) \subseteq J(r, s)$, which is a contradiction. We conclude that (II) implies (III).

By virtue of Lemma 3, (III) implies (I), which completes the proof of the theorem. □

Remark 3. Let G be a connected graph. An axiomatic characterization of the set of all ordered triples (u, v, w) of vertices in G with the properties that $d_G(u, v) = 1$ and $d_G(u, w) = d_G(v, w) + 1$ was given by the present author in [6].

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