ON AUTOMORPHISM GROUPS OF PLANAR LATTICES

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Abstract. The structure of automorphisms of planar lattices is analyzed.

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We investigate the structure of automorphisms of planar lattices. Our results make it possible to give a lattice theoretical proof of theorems on automorphism groups of planar lattices formulated by László Babai and Dwight Duffus ([1] and [2]) and George Grätzer and Csaba Szabó ([6]).

Standard planar representations

A *planar lattice* is a finite lattice which has a planar representation. First we should say what a planar representation is. Denote the real line by $\mathbb{R}$. A *planar representation* of a finite ordered set $\mathbf{L} := (L, \leqslant)$ consists of:

1. *Vertices*: We have an injective mapping $a \mapsto [a^{\mathbf{x}}, a^{\mathbf{y}}]$ of L into $\mathbb{R} \times \mathbb{R}$ such that $a^{\mathbf{y}} < b^{\mathbf{y}}$ whenever $a < b$; the point $[a^{\mathbf{x}}, a^{\mathbf{y}}]$ is called the vertex corresponding to the element a and $a^{\mathbf{y}}$ is said to be the *level* of a in the planar representation.

2. *Arcs*: We have a mapping $[a, b] \mapsto \xi_{ab}$ of the set $\{[a, b] \in L \times L \colon a \prec b\}$ into the set of all continuous real functions on closed intervals in $\mathbb{R}$ such that the domain of ξ_{ab} is $\langle a^{\mathbf{y}}, b^{\mathbf{y}} \rangle$, $c^{\mathbf{x}} = \xi_{ab}(c^{\mathbf{y}})$ iff $c \in \{a, b\}$, and whenever $\xi_{ab}(y) = \xi_{cd}(y)$, then either $b = c$ & $b^{\mathbf{y}} = y$, or $a = d$ & $a^{\mathbf{y}} = y$, or $a = c$ & $a^{\mathbf{y}} = y$, or $b = d$ & $b^{\mathbf{y}} = y$, or

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$[a,b]=[c,d]$. The set of points

$$ab := \{[\xi_{ab}(y), y] \in \mathbb{R}\times\mathbb{R} \colon y \in \langle a^{\mathbf{y}}, b^{\mathbf{y}}\rangle\}$$

is said to be the arc connecting vertices $[a^{\mathbf{x}}, a^{\mathbf{y}}]$ and $[b^{\mathbf{x}}, b^{\mathbf{y}}]$. We say that ab is *incident* with c if $a=c$ or $b=c$.

For a planar representation $\mathbf{D}$ of $\mathbf{L}$ we define

$$\operatorname{cn}(a,b) := \begin{cases} \{[\xi_{ab}(y), y] \colon a^{\mathbf{y}} < y < b^{\mathbf{y}}\} \text{ if } a \prec b \\ \{[\xi_{ba}(y), y] \colon b^{\mathbf{y}} < y < a^{\mathbf{y}}\} \text{ if } b \prec a \\ \emptyset \text{ otherwise} \end{cases}$$

whenever $a, b \in L$, and for $X \subseteq L$ we put

$$\begin{aligned} \operatorname{pt}(X) &:= \{[a^{\mathbf{x}}, a^{\mathbf{y}}] \colon a \in X\} \cup \bigcup \{\operatorname{cn}(a,b) \colon a \in X \ \&\ b \in L\}, \\ \operatorname{dg}(X) &:= \{[a^{\mathbf{x}}, a^{\mathbf{y}}] \colon a \in X\} \cup \bigcup \{\operatorname{cn}(a,b) \colon a \in X \ \&\ b \in X\}. \end{aligned}$$

If $C := \{c_o \prec \ldots \prec c_n\}$ is a maximal chain in $\langle a, b\rangle$ where $a < b$, we define

$$\eta_C(y) := \xi_{c_{i-1}c_i}(y) \text{ for } y \in \langle {c_{i-1}}^{\mathbf{y}}, {c_i}^{\mathbf{y}}\rangle,\ i \in \{1, \ldots, n\}.$$

If $C := \{c_0\}$ is a one-element chain, we put

$$\eta_C({c_0}^{\mathbf{y}}) := {c_0}^{\mathbf{x}}.$$

It is transparent that η_C is a continuous function on $\langle a^{\mathbf{y}}, b^{\mathbf{y}}\rangle$ and $\operatorname{dg} C$ is its graph. For $a, b \in L$ we write $a \blacktriangleleft b$ if $a \parallel b$ and there exists a maximal chain C in L such that $b \in C$ and $a^{\mathbf{x}} < \eta_C(a^{\mathbf{y}})$.

As usual, we denote

$$\begin{aligned} {\downarrow} a &:= \{x \in L \colon x \leqslant a\} \\ {\uparrow} a &:= \{x \in L \colon a \leqslant x\} \\ \langle a, b\rangle &:= {\uparrow} a \cap {\downarrow} b \\ \triangledown x &:= \{y \in L \colon x \prec y\} \\ \triangle x &:= \{y \in L \colon y \prec x\}. \end{aligned}$$

By π_x, π_y we denote the natural projections of $\mathbb{R}\times\mathbb{R}$ onto $\mathbb{R}$.

By an automorphism of a finite ordered set $\mathbf{L} := (L, \leqslant)$ we mean a permutation of its universe which preserves the associated strict order $<$, or, alternatively, the associated covering relation $\prec$. The set of all automorphisms of a finite lattice $\mathbf{L} := (L, \leqslant)$ will be denoted by $\operatorname{Aut}\mathbf{L}$. It is a group with respect to composition. For $A \subseteq L$ we define

$$\operatorname{Aut}_A \mathbf{L} := \{f \in \operatorname{Aut}\mathbf{L} \colon f|_{L\setminus A} = \mathrm{id}_{L\setminus A}\}.$$

It is a subgroup of $\operatorname{Aut}\mathbf{L}$. For $a \in L$ and $f \in \operatorname{Aut}\mathbf{L}$ we put

$$\begin{aligned} a^{\equiv_f} &:= \{f^n(a) \colon n \in \mathbb{N}\} \\ a^{\wedge_f} &:= \bigwedge a^{\equiv_f} \\ a^{\vee_f} &:= \bigvee a^{\equiv_f} \end{aligned}$$

where $\mathbb{N}$ is the set of all natural numbers. The cardinality of $a^{\equiv_f}$ is called the *rank* of a with respect to f.

Notice that the concept of a planar lattice is self-dual, and so we can benefit from the well-known duality principle: whenever an assertion about planar lattices is true, the dual assertion is also true. Moreover, if we turn a planar representation of $\mathbf{L}$ round a vertical axis, then we again obtain a planar representation of $\mathbf{L}$. Thus we have something like the duality principle for planar representations in both coordinates: whenever an assertion about planar representations is true, the assertion obtained by mutually interchanging the notions of left and right is also true, and so are the dual assertions.

Lemma 1. *If L is finite, then $a^{\equiv_f}$ is an antichain in $\mathbf{L}$ for each $a \in L$ and $f \in \operatorname{Aut}\mathbf{L}$.*

P r o o f. If not, say $f^k(a) < f^l(a)$ where $k < l$, then $f^k(a) < f^{k+(l-k)}(a) < \ldots < f^{k+m(l-k)}(a) < \ldots$, which is in contradiction with the finiteness of L. $\square$

We need the following result from analysis, which we present without proof.

Lemma 2. *Let f, g be continuous functions on $\langle a, b\rangle \subseteq \mathbb{R}$ such that $f(a) \leqslant g(a)$ and $g(b) \leqslant f(b)$. Then there exists $y \in \langle a, b\rangle$ such that $f(y) = g(y)$.*

In the next lemma we summarize some properties of maximal chains in planar lattices.

Lemma 3. *Consider a planar representation of a planar lattice $\mathbf{L} := (L, \leqslant)$.*

(1) Let C be a maximal chain in $\langle a, b\rangle$, and D a maximal chain in $\langle c, d\rangle$. If $y \in \langle a^{\mathbf{y}}, b^{\mathbf{y}}\rangle \cap \langle c^{\mathbf{y}}, d^{\mathbf{y}}\rangle$ is such that $\eta_C(y) = \eta_D(y)$, then either there exists $z \in C \cap D$ such that $z^{\mathbf{y}} = y$ or there exist $v, w \in C \cap D$ such that $v \prec w$ and $v^{\mathbf{y}} < y < w^{\mathbf{y}}$. In particular, if $c^{\mathbf{y}} \leqslant a^{\mathbf{y}}, b^{\mathbf{y}} \leqslant d^{\mathbf{y}}, \eta_D(a^{\mathbf{y}}) \leqslant a^{\mathbf{x}}$ and $b^{\mathbf{x}} \leqslant \eta_D(b^{\mathbf{y}})$, then $C \cap D \neq \emptyset$.

(2) *If C is a maximal chain in $\langle a,c\rangle$, D is a maximal chain in $\langle a,d\rangle$ and R is a maximal chain in $\langle \bigwedge L, b\rangle$, where $a^{\mathbf{y}} < b^{\mathbf{y}} \leqslant \min\{c^{\mathbf{y}}, d^{\mathbf{y}}\}$ and $\eta_C(b^{\mathbf{y}}) \leqslant b^{\mathbf{x}} \leqslant \eta_D(b^{\mathbf{y}})$, then $R \cap (C \cup D) \neq \emptyset$.*

(3) *Let $a, b \in L$ be incomparable, let C, C', D, D' be maximal chains in L and $a \in C \cap C'$, $b \in D \cap D'$. Then the following conditions are equivalent:*

(i) $\eta_D(a^{\mathbf{y}}) < a^{\mathbf{x}}$
(ii) $\eta_{D'}(a^{\mathbf{y}}) < a^{\mathbf{x}}$
(iii) $b^{\mathbf{x}} < \eta_C(b^{\mathbf{y}})$
(iv) $b^{\mathbf{x}} < \eta_{C'}(b^{\mathbf{y}})$.

(4) *If C, D are maximal chains in L, $c \in C$, $d \in D$ such that $c \prec b$, $d \prec b$ and $c \blacktriangleleft d$, then $\eta_C(b^{\mathbf{y}}) \leqslant b^{\mathbf{x}} \leqslant \eta_D(b^{\mathbf{y}})$.*

(5) *The relation $\blacktriangleleft$ is a linear strict order on any antichain in $\mathbf{L}$.*

(6) *The relation $\leqslant \cup \blacktriangleleft$ is a linear order on L.*

(7) *If C, D are maximal chains in L, $c \in C$, $d \in D$ such that $\eta_C(d^{\mathbf{y}}) \leqslant d^{\mathbf{x}}$ and $\eta_D(c^{\mathbf{y}}) \leqslant c^{\mathbf{x}}$, then c and d are comparable. In particular, if C, D are maximal chains in $\langle a,b\rangle$ such that $(\forall c \in C)\eta_D(c^{\mathbf{y}}) \leqslant c^{\mathbf{x}}$ and $(\forall d \in D)\eta_C(d^{\mathbf{y}}) \leqslant d^{\mathbf{x}}$, then $C = D$.*

See [4] for the basic ideas of the proof.

Lemma 4. *Let φ be a strict-order preserving map of a planar lattice $\mathbf{L} := (L, \leqslant)$ into the set of all natural numbers $\mathbb{N}$ with the natural order. Then there exists a planar representation of $\mathbf{L}$ such that $a^{\mathbf{y}} = \varphi(a)$ for each $a \in L$.*

Proof. Take any planar representation $\mathbf{D}$ of $\mathbf{L}$. We will transform it by moving vertices successively to the required levels, i.e. a vertex corresponding to a will be moved to the level $\varphi(a)$. First we translate $\mathbf{D}$ into the lower halfplane: we just subtract $z^{\mathbf{y}}+1$ from each $a^{\mathbf{y}}$ where $z := \bigvee L$. We obtain a planar representation $\mathbf{D}_0$ of $\mathbf{L}$ in which $a^{\mathbf{y}} < 0$ for each $a \in L$. Further we proceed by recursion. In order to obtain $\mathbf{D}_1$ we take the greatest element z of $\mathbf{L}$ and move it to the point $[0, \varphi(z)]$. This is the new vertex corresponding to z. We know that whenever $w \prec z$, the arc connecting vertices $[w^{\mathbf{x}}, w^{\mathbf{y}}]$ and $[z^{\mathbf{x}}, z^{\mathbf{y}}]$ is a parametrized curve $\{[\xi_{wz}(y), y] \colon y \in \langle w^{\mathbf{y}}, z^{\mathbf{y}}\rangle\}$ where ξ_{wz} is continuous on $\langle w^{\mathbf{y}}, z^{\mathbf{y}}\rangle$. There exists $b \in \mathbb{R}$ such that $b < z^{\mathbf{y}}$ and $w^{\mathbf{y}} < b$ for all $w \in L \setminus \{z\}$. The new connecting arcs for $w \prec z$ will be

$$\{[\xi_{wz}(y), y] \colon y \in \langle w^{\mathbf{y}}, b\rangle\} \cup \Big\{\Big[\frac{y-\varphi(z)}{b-\varphi(z)}\xi_{wz}(b), y\Big] \colon y \in \langle b, \varphi(z)\rangle\Big\}.$$

We have obtained a planar representation $\mathbf{D}_1$ of $\mathbf{L}$ with precisely one element on the required level. Suppose that we have a planar representation $\mathbf{D}_k$ of $\mathbf{L}$ with precisely k elements on the required levels where $1 \leqslant k < |L|$. In the sequel all coordinates will be related to $\mathbf{D}_k$. Take a maximal element a of the set of all elements remaining in the original positions. Hence all elements above a are on the required levels and all elements below a are in the original positions. We shall try to move a to the required level $\varphi(a)$. Consider the set $\triangledown a$ of all elements b which cover a and the corresponding functions ξ_{ab}. All ξ_{ab} are obviously defined on $\langle a^{\mathbf{y}}, \varphi(a)+\frac{1}{2}\rangle$, and there exist $b_0, b_1 \in \triangledown a$ such that $\xi_{ab_0}(y) \leqslant \xi_{ab}(y) \leqslant \xi_{ab_1}(y)$ for each $b \in \triangledown a$ and each $y \in \langle a^{\mathbf{y}}, \varphi(a)+\frac{1}{2}\rangle$. The set

$$C := \{[x,y] \in \mathbb{R}\times\mathbb{R}\colon\ y \in \langle a^{\mathbf{y}}, \varphi(a)+\tfrac{1}{2}\rangle \ \&\ \xi_{ab_0}(y) \leqslant x \leqslant \xi_{ab_1}(y)\}$$

is bounded closed in $\mathbb{R}\times\mathbb{R}$ endowed with the Euclidean metric, and by Lemma 3(2) it does not contain any vertex except $[a^{\mathbf{x}}, a^{\mathbf{y}}]$. All arcs not incident with a are also bounded closed and moreover disjoint with C. Hence there exists a real number $\varepsilon' > 0$ such that the ε'-neighbourhood E of C does not contain any point of arcs not incident with a. Put $\varepsilon := \frac{\varepsilon'}{2}$. Consider further the arcs going from a downwards. There exists $\delta' > 0$ such that $|\xi_{wa}(y) - a^{\mathbf{x}}| < \varepsilon$ for each w covered by a and each $y \in \langle a^{\mathbf{y}} - \delta', a^{\mathbf{y}}\rangle$. Put $\delta := \min\{\delta', \varepsilon\}$. Now all points $[\xi_{wa}(y), y]$ lie in E whenever $y \in \langle a^{\mathbf{y}} - \delta, a^{\mathbf{y}}\rangle$. Define

$$\begin{aligned} H := \{[x,y] \in \mathbb{R}\times\mathbb{R}\colon\ & y \in \langle a^{\mathbf{y}} - \delta, a^{\mathbf{y}}\rangle \ \&\ x \in \langle a^{\mathbf{x}} - \varepsilon, a^{\mathbf{x}} + \varepsilon\rangle \\ & \text{or } y \in \langle a^{\mathbf{y}}, \varphi(a)+\tfrac{1}{2}\rangle \ \&\ x \in \langle \xi_{ab_0}(y) - \varepsilon, \xi_{ab_1}(y) + \varepsilon\rangle\}. \end{aligned}$$

Clearly $H \subseteq E$ and the functions

$$d(y) := \begin{cases} a^{\mathbf{x}} - \varepsilon & \text{for } y \leqslant a^{\mathbf{y}} \\ \xi_{ab_0}(y) - \varepsilon & \text{for } y \in \langle a^{\mathbf{y}}, \varphi(a)+\frac{1}{2}\rangle \end{cases}$$
$$h(y) := \begin{cases} a^{\mathbf{x}} + \varepsilon & \text{for } y \leqslant a^{\mathbf{y}} \\ \xi_{ab_1}(y) + \varepsilon & \text{for } y \in \langle a^{\mathbf{y}}, \varphi(a)+\frac{1}{2}\rangle \end{cases}$$

are continuous on $\langle a^{\mathbf{y}} - \delta, \varphi(a)+\frac{1}{2}\rangle$ and $d(y)+\varepsilon < h(y)$ there. Consider the following continuous transformation of H onto itself:

$$[x,y] \longmapsto [u,v]$$

where

$$v := \begin{cases} (a^{\mathbf{y}} - \delta) + \dfrac{\varphi(a) - (a^{\mathbf{y}} - \delta)}{\delta}(y - (a^{\mathbf{y}} - \delta)) & \text{for } y \in \langle a^{\mathbf{y}} - \delta, a^{\mathbf{y}}\rangle \\ (\varphi(a)+\frac{1}{2}) - \dfrac{\frac{1}{2}}{\varphi(a)+\frac{1}{2} - a^{\mathbf{y}}}(\varphi(a)+\frac{1}{2} - y) & \text{for } y \in \langle a^{\mathbf{y}}, \varphi(a)+\frac{1}{2}\rangle \end{cases}$$
$$u := d(v) + \frac{x - d(y)}{h(y) - d(y)}(h(v) - d(v)).$$

For $w \in \triangle a$ the new arc connecting the vertices corresponding to w and a will be

$$\begin{aligned}
&\{[\xi_{wa}(y), y] \colon y \in \langle w^{\mathbf{y}}, a^{\mathbf{y}} - \delta\rangle\} \\
&\cup \Big\{ \Big[d(y) + (h(y) - d(y)) \\
&\quad \times \frac{\xi_{wa}(a^{\mathbf{y}} + \frac{y-\varphi(a)}{\varphi(a)-a^{\mathbf{y}}+\delta} \cdot \delta) - d(a^{\mathbf{y}} + \frac{y-\varphi(a)}{\varphi(a)-a^{\mathbf{y}}+\delta} \cdot \delta)}{h(a^{\mathbf{y}} + \frac{y-\varphi(a)}{\varphi(a)-a^{\mathbf{y}}+\delta} \cdot \delta) - d(a^{\mathbf{y}} + \frac{y-\varphi(a)}{\varphi(a)-a^{\mathbf{y}}+\delta} \cdot \delta)}, y \Big] \colon y \in \langle a^{\mathbf{y}} - \delta, \varphi(a)\rangle \Big\}.
\end{aligned}$$

For $w \in \triangledown a$ the new arc connecting the vertices corresponding to a and w will be

$$\begin{aligned}
&\Big\{ \Big[d(y) + (h(y) - d(y)) \\
&\times \frac{\xi_{aw}(a^{\mathbf{y}} + (y - \varphi(a)) \cdot (\varphi(a) + \frac{1}{2} - a^{\mathbf{y}})) - d(a^{\mathbf{y}} + (y - \varphi(a)) \cdot (\varphi(a) + \frac{1}{2} - a^{\mathbf{y}}))}{h(a^{\mathbf{y}} + (y - \varphi(a)) \cdot (\varphi(a) + \frac{1}{2} - a^{\mathbf{y}})) - d(a^{\mathbf{y}} + (y - \varphi(a)) \cdot (\varphi(a) + \frac{1}{2} - a^{\mathbf{y}}))}, y \Big] \\
&\colon y \in \langle \varphi(a), \varphi(a) + \tfrac{1}{2}\rangle \Big\} \cup \{[\xi_{aw}(y), y] \colon y \in \langle \varphi(a) + \tfrac{1}{2}, w^{\mathbf{y}}\rangle\}.
\end{aligned}$$

In this way we obtain a planar representation $\mathbf{D}_{k+1}$ of $\mathbf{L}$ with precisely $k+1$ vertices on required levels. Finally, $\mathbf{D}_{|L|}$ is a planar representation of $\mathbf{L}$ with the required property. $\square$

The preceding proof is the only place in this paper where vertical bars denote the absolute value of a real number besides the cardinality of a finite set.

We say that a planar representation is *standard* if, for each $a, b \in L$, $a^{\mathbf{y}} = b^{\mathbf{y}}$ if and only if there is an automorphism f of $\mathbf{L}$ with $a = f(b)$. Of course there exist planar ordered sets without standard planar representations, for example crowns.

Proposition 5. *Every planar lattice has a standard planar representation.*

Proof. Let $\mathbf{L} := (L, \leqslant)$ be a planar lattice. Define a binary relation Φ on L by the rule

$$a \; \Phi \; b :\Longleftrightarrow (\exists f \in \operatorname{Aut} \mathbf{L}) b = f(a).$$

It is obvious that Φ is an equivalence relation on L. The ordering $\leqslant$ on L induces an ordering $\trianglelefteq$ on $L/\,\Phi$: we put

$$A \trianglelefteq B :\Longleftrightarrow (\exists a \in A)(\exists b \in B) a \leqslant b.$$

Indeed, reflexivity and transitivity of $\trianglelefteq$ are obvious. Let $A \trianglelefteq B$ & $B \trianglelefteq A$. Then

$$(\exists a_1, a_2 \in A)(\exists b \in B)(\exists f \in \operatorname{Aut} \mathbf{L}) a_1 \leqslant b \;\&\; f(b) \leqslant a_2.$$

Hence $a_1 \leqslant b = f^{-1} f(b) \leqslant f^{-1}(a_2) \in A$. In virtue of Lemma 1, A is an antichain, and hence $a_1 = f^{-1}(a_2)$. Therefore $b = a_1$, and consequently $A = B$. This new ordering can be extended to a linear one. Hence we can order the blocks of Φ into a finite sequence $A_1, \ldots, A_r$ such that $A_i \trianglelefteq A_j$ implies $i \leqslant j$. For $a \in A_i$ put $\varphi(a) := i$. By Lemma 1, φ is strict-order preserving, and the result follows by the preceding lemma. $\square$

A construction for groups

Direct products and semidirect products are two important standard constructions in group theory. We are going to utilize them also in our further considerations.

Let $\mathbf{G} := (G, \circ)$ be a group and n a positive integer. We define $n\mathbf{G}$, the n-th *symmetric power* of $\mathbf{G}$, as follows. The universe of $n\mathbf{G}$ is the set $S_n \times G^n$, where S_n is the set of all permutations of $\{1, \ldots, n\}$. Multiplication is defined by

$$[p, x_1, \ldots, x_n] \circ [q, y_1, \ldots, y_n] := [pq, x_{q(1)} \circ y_1, \ldots, x_{q(n)} \circ y_n].$$

It is clear that $n\mathbf{G}$ is a semidirect product of groups and hence a group.

For a class of finite groups $\mathcal{K}$ we define $\mathcal{P}(\mathcal{K})$ to be the least class of finite groups which is closed under the formation of finite direct products, symmetric powers and isomorphic copies, and which $\mathcal{K}$ is included in.

Notice that we can define the members of $\mathcal{P}(\mathcal{K})$ recursively as the groups obtained by finitely many applications of the following rules:

(1) Isomorphic copies of members of $\mathcal{K}$ are members of $\mathcal{P}(\mathcal{K})$.
(2) If $\mathbf{G}_1$ and $\mathbf{G}_2$ are members of $\mathcal{P}(\mathcal{K})$, then isomorphic copies of $\mathbf{G}_1 \times \mathbf{G}_2$ are also members of $\mathcal{P}(\mathcal{K})$.
(3) If n is a positive integer and $\mathbf{G}$ is a member of $\mathcal{P}(\mathcal{K})$, then isomorphic copies of $n\mathbf{G}$ are also members of $\mathcal{P}(\mathcal{K})$.

The *linear sum* $\mathbf{L}_1 \oplus \mathbf{L}_2$ and the *disjoint sum* $\mathbf{L}_1 \uplus \ldots \uplus \mathbf{L}_n$ of ordered sets we define as usual, cf. [3]. By a *parallel sum* of $\mathbf{L}_1, \ldots, \mathbf{L}_n$ we mean $\mathbf{1} \oplus (\mathbf{L}_1 \uplus \ldots \uplus \mathbf{L}_n) \oplus \mathbf{1}$, the disjoint sum with new top and bottom elements added. All these concepts are instances of the general concept of an ordinal sum of ordered sets over an ordered set.

Let $\mathcal{AG}(\mathcal{PL})$ denote the class of all finite groups isomorphic to automorphism groups of planar lattices.

Proposition 6. $\mathcal{P}(\mathcal{AG}(\mathcal{PL})) = \mathcal{AG}(\mathcal{PL})$.

P r o o f. By definition, $\mathcal{AG}(\mathcal{PL})$ is closed under the formation of isomorphic copies, and $\mathcal{AG}(\mathcal{PL})$ is included in $\mathcal{AG}(\mathcal{PL})$.

C l a i m. If $\mathbf{L}_1$ and $\mathbf{L}_2$ are planar lattices, then $\mathbf{L}_1 \oplus \mathbf{L}_2$ is a planar lattice and

$$\operatorname{Aut} \mathbf{L}_1 \times \operatorname{Aut} \mathbf{L}_2 \cong \operatorname{Aut} (\mathbf{L}_1 \oplus \mathbf{L}_2).$$

Without loss of generality we can assume that L_1 and L_2 are disjoint. The first part of the claim is obvious. For $f \in \operatorname{Aut}(\mathbf{L}_1 \oplus \mathbf{L}_2)$ we have $f[\![L_1]\!] = L_1$ and $f[\![L_2]\!] = L_2$ in virtue of Lemma 1. Clearly $f|_{L_1} \in \operatorname{Aut}\mathbf{L}_1$ and $f|_{L_2} \in \operatorname{Aut}\mathbf{L}_2$. We define

$$\varphi\colon \operatorname{Aut}(\mathbf{L}_1 \oplus \mathbf{L}_2) \longrightarrow \operatorname{Aut}\mathbf{L}_1 \times \operatorname{Aut}\mathbf{L}_2$$

by the rule

$$\varphi(f) := [f|_{L_1}, f|_{L_2}].$$

Since $\varphi(f \circ g) = [(f \circ g)|_{L_1}, (f \circ g)|_{L_2}] = [f|_{L_1}, f|_{L_2}] \circ [g|_{L_1}, g|_{L_2}] = \varphi(f) \circ \varphi(g)$, we conclude that φ is a group homomorphism. φ is injective: $\varphi(f) = \varphi(g)$ implies that $f|_{L_1} = g|_{L_1}$ and $f|_{L_2} = g|_{L_2}$, which in turn yields $f = f|_{L_1} \cup f|_{L_2} = g|_{L_1} \cup g|_{L_2} = g$. φ is surjective: Let $f_1 \in \operatorname{Aut}\mathbf{L}_1$ and $f_2 \in \operatorname{Aut}\mathbf{L}_2$. Then $f_1 \cup f_2 \in \operatorname{Aut}(\mathbf{L}_1 \oplus \mathbf{L}_2)$. Indeed, $f_1 \cup f_2$ is a permutation, and whenever $a < b$, then either $a, b \in L_1$ and hence $(f_1 \cup f_2)(a) = f_1(a) < f_1(b) = (f_1 \cup f_2)(b)$, or $a, b \in L_2$ and hence $(f_1 \cup f_2)(a) = f_2(a) < f_2(b) = (f_1 \cup f_2)(b)$, or $a \in L_1$ and $b \in L_2$ and hence $(f_1 \cup f_2)(a) = f_1(a) \in L_1$, $(f_1 \cup f_2)(b) = f_2(b) \in L_2$, which in turn yields $(f_1 \cup f_2)(a) < (f_1 \cup f_2)(b)$. Now $\varphi(f_1 \cup f_2) = [f_1, f_2]$.

Claim. If n is a positive integer and $\mathbf{L}$ a planar lattice, then the parallel sum $\mathbf{1} \oplus (\mathbf{L} \uplus \ldots \uplus \mathbf{L}) \oplus \mathbf{1}$ of n isomorphic copies of $\mathbf{L}$ is a planar lattice and

$$n \operatorname{Aut}\mathbf{L} \cong \operatorname{Aut}(\mathbf{1} \oplus (\mathbf{L} \uplus \ldots \uplus \mathbf{L}) \oplus \mathbf{1}).$$

The first part of the claim is obvious. The universe of $\mathbf{1} \oplus (\mathbf{L} \uplus \ldots \uplus \mathbf{L}) \oplus \mathbf{1}$ is $\{0,1\} \cup L \times \{1, \ldots, n\}$. It is clear that for each $f \in \operatorname{Aut}(\mathbf{1} \oplus (\mathbf{L} \uplus \ldots \uplus \mathbf{L}) \oplus \mathbf{1})$ we have $0, 1 \in \operatorname{fix} f$. Furthermore, from $f([a_1, i]) = [b_1, j_1]$ & $f([a_2, i]) = [b_2, j_2]$ it follows that $j_1 = j_2$, and hence we have a permutation p_f of $\{1, \ldots, n\}$ and $f_{(i)}\colon L \to L$ for each $i \in \{1, \ldots, n\}$ such that $f([a, i]) = [f_{(i)}(a), p_f(i)]$. Clearly $p_{f \circ g} = p_f p_g$. Further, $a < b$ implies that $[a, i] < [b, i]$, which in turn yields $f([a, i]) < f([b, i])$, and we can conclude that $f_{(i)}(a) < f_{(i)}(b)$. $f_{(i)}$ is bijective as $f_{(i)}(a) = f_{(i)}(b)$ implies that $[a, i] = f^{-1}f([a, i]) = f^{-1}([f_{(i)}(a), p_f(i)] = f^{-1}([f_{(i)}(b), p_f(i)] = f^{-1}f([b, i]) = [b, i]$, which in turn yields $a = b$, and L is finite. Hence $f_{(i)} \in \operatorname{Aut}\mathbf{L}$. We define

$$\varphi\colon \operatorname{Aut}(\mathbf{1} \oplus (\mathbf{L} \uplus \ldots \uplus \mathbf{L}) \oplus \mathbf{1}) \longrightarrow n \operatorname{Aut}\mathbf{L}$$

by the rule

$$\varphi(f) := [p_f, f_{(1)}, \ldots, f_{(n)}].$$

By the preceding, $[(f \circ g)_{(i)}(a), p_{f \circ g}(i)] = f \circ g([a, i]) = f([g_{(i)}(a), p_g(i)]) = [f_{(p_g(i))} \circ g_{(i)}(a), p_f p_g(i)]$, and hence $\varphi(f \circ g) = [p_{f \circ g}, (f \circ g)_{(1)}, \ldots, (f \circ g)_{(n)}] = [p_f p_g, f_{(p_g(1))} \circ g_{(1)}, \ldots, f_{(p_g(n))} \circ g_{(n)}] = [p_f, f_{(1)}, \ldots, f_{(n)}] \circ [p_g, g_{(1)}, \ldots, g_{(n)}] = \varphi(f) \circ \varphi(g)$, and so

φ is a group homomorphism. φ is injective: $\varphi(f) = \varphi(g)$ implies that $p_f = p_g$ and $(\forall i)\, f_{(i)} = g_{(i)}$, and hence $f([a,i]) = [f_{(i)}(a), p_f(i)] = [g_{(i)}(a), p_g(i)] = g([a,i])$ for each $[a,i] \in L \times \{1,\dots,n\}$. φ is surjective: Let $f_1,\dots,f_n \in \operatorname{Aut}\mathbf{L}$ and $p \in S_n$. Then clearly

$$f := \mathrm{id}_{\{0,1\}} \cup ([a,i] \mapsto [f_i(a), p(i)]) \in \operatorname{Aut}(\mathbf{1} \oplus (\mathbf{L} \uplus \ldots \uplus \mathbf{L}) \oplus \mathbf{1})$$

and $\varphi(f) = [p, f_1, \dots, f_n]$. $\square$

Notice that the above statement is true for each class of finite lattices closed under the formation of linear sums and parallel sums, not only for planar lattices.

If we denote the class of all one-element groups by $\mathcal{T}$, then we have

$$\mathcal{P}(\mathcal{T}) \subseteq \mathcal{P}(\mathcal{AG}(\mathcal{PL})) = \mathcal{AG}(\mathcal{PL}).$$

In the remainder of this paper we will prove the converse inclusion.

Blocks and components

In the whole of this section $\mathbf{L} := (L, \leqslant)$ will be a planar lattice.

A sequence $[z_0,\dots,z_p]$ of elements of a subset $X \subseteq L$ is a *connection* in X if either $z_i \prec z_{i+1}$ or $z_{i+1} \prec z_i$ for each $i \in \{0,\dots,p-1\}$. In particular, $[z_0]$ is a connection. We define $a \overset{X}{\leftrightsquigarrow} b$ if there is a connection $[a = z_0,\dots,z_p = b]$ in X. A subset $X \subseteq L$ is said to be *connected* if $x \overset{X}{\leftrightsquigarrow} y$ for each $x, y \in X$, i.e. $\overset{X}{\leftrightsquigarrow} = X \times X$.

For $f \in \operatorname{Aut}\mathbf{L}$ we put

$$\begin{aligned} \operatorname{fix} f &:= \{x \in L \colon f(x) = x\}, \\ \operatorname{mov} f &:= \{x \in L \colon f(x) \neq x\}, \\ \beta_f &:= \{[x,y] \in \operatorname{mov} f \times \operatorname{mov} f \colon y = f(x)\}. \end{aligned}$$

Clearly $\overset{\operatorname{mov} f}{\leftrightsquigarrow} = \overset{\operatorname{mov} f^{-1}}{\leftrightsquigarrow}$, and $a \overset{\operatorname{mov} f}{\leftrightsquigarrow} b$ implies that $f(a) \overset{\operatorname{mov} f}{\leftrightsquigarrow} f(b)$.

We define

$$a\theta_f b :\Longleftrightarrow (\exists n \in \mathbb{N}) a \overset{\operatorname{mov} f}{\leftrightsquigarrow} f^n(b).$$

It is easy to verify that θ_f is the equivalence on $\operatorname{mov} f$ generated by $\overset{\operatorname{mov} f}{\leftrightsquigarrow} \cup \beta_f$; moreover $\theta_f = \theta_{f^{-1}}$, and $a\theta_f b$ implies $f(a)\theta_f f(b)$. It is obvious that each block of θ_f has at least two elements. We say that a subset $B \subseteq L$ is a *block* if it is a block of θ_f for an automorphism f. By a *component* of a block B we mean a maximal connected subset of B.

We put

$$\operatorname{Mov}_A \mathbf{L} := \bigcup\{\operatorname{mov} f\colon f \in \operatorname{Aut}_A \mathbf{L}\}$$

and

$$\operatorname{Mov} \mathbf{L} := \bigcup\{\operatorname{mov} f\colon f \in \operatorname{Aut} \mathbf{L}\}.$$

Lemma 7. *Let B be a block of θ_f with precisely k components, and let K be a component of B.*

(1) *$f^i[\![K]\!]$, $i \in \{1, \dots, k\}$ are all components of B.*

(2) *$x^{\vee_f}$ is the least fixpoint of f in $\uparrow x$.*

(3) *If $x \in K$, $y \in L \setminus K$ and $x < y$, then there exists an element $u \in K$ such that $x \leqslant u \prec x^{\vee_f} \leqslant y$. Moreover, $f(x) < y$ and $u^{\vee_f} = x^{\vee_f}$.*

(4) *$\bigvee B$ is the least fixpoint of f in $\uparrow \bigvee K$.*

(5) *$B \cup \{b^{\vee_f}\}$ is connected for each $b \in B$.*

(6) *If C is a connected subset of L such that $C \cap K \neq \emptyset$ and $C \setminus K \neq \emptyset$, then there exists an element $x \in C \cap K$ such that either $x \prec x^{\vee_f} \in C$ or $x \succ x^{\wedge_f} \in C$.*

P r o o f. (1): It is obvious that B is a block of f^{-1}, and $f[\![K]\!]$ is connected. There exists a component K' of B such that $f[\![K]\!] \subseteq K'$. Now $K = f^{-1}f[\![K]\!] \subseteq f^{-1}[\![K']\!]$, and since K is a component of B, we get $K = f^{-1}[\![K']\!]$. Hence $K' = f[\![K]\!]$. By induction, $f^i[\![K]\!]$ are components of B. Let conversely K' be a component of B. By the definition of θ_f, there exist elements $a \in K$ and $a' \in K'$ such that $a' \in a^{\equiv_f}$. Consequently $K' = f^i[\![K]\!]$ for some $i \in \mathbb{N}$. Now $f^k[\![K]\!] \neq K$ would imply $f^i[\![K]\!] = f^j[\![K]\!]$ for some $i < j$, $i, j \in \{1, \dots, k\}$. Then $f^{j-i}[\![K]\!] = K$ and so B would have at most $j - i$ components. Therefore $f^k[\![K]\!] = K$.

(2): Obvious.

(3): In $\langle x, y\rangle$ there exists a maximal chain $\{x = z_0 \prec \ldots \prec z_n = y\}$. Clearly $z_0 \in K$ and $z_n \in L \setminus K$. Let i be the least natural number such that $z_i \in L \setminus K$. Then $z_{i-1} \in K$ and $z_{i-1} \prec z_i$, and consequently $z_i \in \operatorname{fix} f$. This and (2) together yield $x^{\vee_f} \leqslant z_i \leqslant y$. Clearly $f(x) < f(x^{\vee_f}) = x^{\vee_f} \leqslant y$. Consider a maximal chain $\{x = w_1 \prec \ldots \prec w_p = x^{\vee_f}\}$. Put $u := w_{p-1}$. Since $u \prec x^{\vee_f} \in \operatorname{fix} f$ and $u \in K$ by (2), we obtain $u^{\vee_f} = x^{\vee_f}$.

(4): Obvious.

(5): Follows immediately from (3).

(6): Choose $x \in C \cap K$ and $y \in C \setminus K$. There exists a connection $[x = z_0, \dots, z_p = y]$ in C. Let i be the least natural number such that $z_i \in C \setminus K$. Then either $z_{i-1} \prec z_i = z_{i-1}^{\vee_f}$ or $z_{i-1} \succ z_i = z_{i-1}^{\wedge_f}$. □

Lemma 8. *If $B_1, \dots, B_n$ are pairwise disjoint blocks of $\theta_{f_1}, \dots, \theta_{f_n}$ respectively, then $g\colon L \to L$ defined by*

$$g(x) := \begin{cases} f_i(x) \text{ for } x \in B_i \\ x \text{ otherwise} \end{cases}$$

is an automorphism of $\mathbf{L}$ and $B_1, \dots, B_n$ are the only blocks of θ_g.

Proof. Since $g[\![B_i]\!] = f_i[\![B_i]\!] = B_i$, g is bijective. Let $x < y$. If x and y are elements of the same block B_i, then $g(x) = f_i(x) < f_i(y) = g(y)$. If not, we proceed as follows. First we prove that $g(x) < y$. This is obvious if $x \in \operatorname{fix} g$. Otherwise there exists $i \in \{1, \dots, n\}$ such that $x \in B_i$ and $y \notin B_i$. By Lemma 7(3), there exists $z \in \operatorname{fix} f_i$ such that $x < z \leqslant y$. Then $g(x) = f_i(x) < f_i(z) = z \leqslant y$. Now $g(x) \in B_i$ and $y \notin B_i$, and dualizing the preceding argument we obtain $g(x) < g(y)$. □

In the remainder of this section we will consider a fixed standard planar representation $\mathbf{D}$ of $\mathbf{L}$.

In view of Lemma 3(5), in any non-empty antichain X in $\mathbf{L}$ we have an element $\ell(X)$ such that $(\forall a \in X)\ell(X) \blacktriangleleft a$. We say that $\ell(X)$ is the *leftmost element* of X. Let $f \in \operatorname{Aut} \mathbf{L}$. We say that an element $x \in L$ *is situated to the left (right) of* $f(x)$ if $x^{\mathbf{x}} \leqslant f(x)^{\mathbf{x}}$ ($f(x)^{\mathbf{x}} \leqslant x^{\mathbf{x}}$).

Lemma 9. *Let $f \in \operatorname{Aut} \mathbf{L}$. Let a, b be elements of a connected subset C of L such that a is situated to the left of $f(a)$ and b is situated to the right of $f(b)$. Then f has a fixpoint in C.*

Proof. The assertion is obviously true if $a = b$. Suppose $a \neq b$. There exists a connection $[a = z_0, \dots, z_p = b]$ in C. Put

$$i := \min\{j\colon j \in \{1, \dots, p\} \;\&\; z_j \text{ is situated to the right of } f(z_j)\}.$$

Then z_{i-1} is situated to the left of $f(z_{i-1})$. Moreover, either $z_{i-1} \prec z_i$ or $z_i \prec z_{i-1}$. In the first case, we have ${z_{i-1}}^{\mathbf{x}} = \xi_{z_{i-1}z_i}({z_{i-1}}^{\mathbf{y}}) \leqslant \xi_{f(z_{i-1})f(z_i)}({z_{i-1}}^{\mathbf{y}}) = f(z_{i-1})^{\mathbf{x}}$ and $f(z_i)^{\mathbf{x}} = \xi_{f(z_{i-1})f(z_i)}({z_i}^{\mathbf{y}}) \leqslant \xi_{z_{i-1}z_i}({z_i}^{\mathbf{y}}) = {z_i}^{\mathbf{x}}$. In virtue of Lemma 2, either ${z_{i-1}}^{\mathbf{x}} = f(z_{i-1})^{\mathbf{x}}$ and hence $z_{i-1} = f(z_{i-1})$, or ${z_i}^{\mathbf{x}} = f(z_i)^{\mathbf{x}}$ and hence $z_i = f(z_i)$. Similarly for the other case. □

Lemma 10. *Let K and $f^k[\![K]\!]$ be distinct components of a block of θ_f. If an element $x \in K$ is situated to the left (right) of $f^k(x)$, then each element $y \in K$ is situated to the left (right) of $f^k(y)$.*

Proof. If not, then by Lemma 9 f^k would have a fixpoint in K, which is a contradiction. □

In this case we say that *the component K is situated to the left (right) of* $f^k[\![K]\!]$. Notice that the assertion is not true if $K = f^k[\![K]\!]$.

Lemma 11. *Every block has at least two components.*

Proof. Suppose to the contrary that the block B of θ_f is connected. Consider an arbitrary element $x \in B$. Denote by y the leftmost and by z the rightmost elements of $x^{\equiv_f}$. Then y is situated to the left of $f(y)$ and z is situated to the right of $f(z)$. By Lemma 9, f has a fixpoint in B, which is a contradiction. $\square$

We say that a block B of θ_f is *of kind* A if $(\forall b \in B)(b^{\vee_f} = \bigvee B \;\&\; b^{\wedge_f} = \bigwedge B)$. Otherwise we say that B is *of kind* Z.

Lemma 12. *Every block of kind Z has precisely two components.*

Proof. Let a block B of θ_f be of kind Z. Without loss of generality we may assume that there exists $b \in B$ such that $b^{\vee_f} < \bigvee B$. Let K denote the component of B for which $b \in K$. By Lemma 7(4), there exists $a \in K$ such that $a \nleqslant b^{\vee_f}$. We have a connection $[b = z_0, \ldots, z_n = a]$ in K. Denote by i the least integer such that $z_i \nleqslant b^{\vee_f}$. Clearly $1 \leqslant i$. Put $c := z_{i-1}$ and $d := z_i$. Then $c \prec d$, $c^{\vee_f} \leqslant b^{\vee_f}$ and $d \nleqslant c^{\vee_f}$. Assume that B has at least three components. Then there exist $k, l, m \in \mathbb{N}$ such that $f^k(c)^{\mathbf{x}} < f^l(c)^{\mathbf{x}} < f^m(c)^{\mathbf{x}}$. In view of Lemma 3(2) we have $f^l(d) \leqslant c^{\vee_f}$, which is a contradiction. $\square$

An element $x \in L$ *is situated on the left (right) of* a connected subset $C \subseteq L$ if for each $a, b \in C$ such that $a \prec b$ and $a^{\mathbf{y}} \leqslant x^{\mathbf{y}} \leqslant b^{\mathbf{y}}$ we have $x^{\mathbf{x}} \leqslant \xi_{ab}(x^{\mathbf{y}})$ ($\xi_{ab}(x^{\mathbf{y}}) \leqslant x^{\mathbf{x}}$), and there exists at least one such pair of elements a, b.

We say that a maximal chain C in $\langle a, b\rangle \subseteq L$ is the *leftmost (rightmost) maximal chain* in $\langle a, b\rangle$ if all elements of $\langle a, b\rangle$ are situated on the right (left) of C.

Lemma 13. *Let $a \in L$ and $X \subseteq L$. Then there exists precisely one chain $\{a = c_0 \prec \ldots \prec c_n\}$ such that $c_i = \ell(\triangledown c_{i-1} \cap X)$ for $i \in \{1, \ldots, n\}$, and $\triangledown c_n \cap X = \emptyset$.*

Proof. Consider the set $\mathcal{C}$ of all chains $\{a = z_0 \prec \ldots \prec z_p\}, p \in \mathbb{N}$, where $z_i = \ell(\triangledown z_{i-1} \cap X)$ for $i \in \{1, \ldots, p\}$. Since $\{a\} \in \mathcal{C}$, $\mathcal{C} \neq \emptyset$. Further, $p \leqslant |L|$ for each such chain. Hence $\mathcal{C}$ has a maximal element $\{a = c_0 \prec \ldots \prec c_n\}$. Clearly $c_i = \ell(\triangledown c_{i-1} \cap X)$ for $i \in \{1, \ldots, n\}$, and $\triangledown c_n \cap X = \emptyset$ by maximality. Uniqueness is obvious. $\square$

The chain from Lemma 13 will be denoted by $\mathrm{L}^{\triangledown}(a, X)$.

Lemma 14. *For any pair of elements $a < b$ of L there are precisely one leftmost and precisely one rightmost maximal chains in $\langle a, b\rangle$. Moreover, if $\{a = c_0 \prec \ldots \prec$*

$c_n = b\}$ *is the leftmost maximal chain in* $\langle a,b\rangle$*, then* $c_{i-1} = \ell(\triangle c_i \cap \uparrow a)$ *and* $c_i = \ell(\triangledown c_{i-1} \cap \downarrow b)$ *for each* $i \in \{1,\dots,n\}$*.*

Proof. Will be accomplished for the leftmost maximal chain. $\mathrm{L}^{\triangledown}(a,\downarrow b)$ is clearly a maximal chain in $\langle a,b\rangle$ and all elements of $\langle a,b\rangle$ are situated on the right of it. Uniqueness follows from Lemma 3(7). The rest of the proof is obvious. □

By a *left (right) boundary* of a block B we mean a maximal chain C in $\langle \bigwedge B, \bigvee B\rangle$ such that $C \setminus \{\bigwedge B, \bigvee B\} \subseteq B$ and all elements of B are situated on the right (left) of C.

Lemma 15. *Each block has precisely one left and precisely one right boundaries.*

Proof. Will be accomplished for the left boundary of a block B of θ_f. The set M of all maximal elements in B is non-empty as B is non-empty finite. By the preceding lemma, the leftmost maximal chain $\{\bigwedge B = c_0 \prec \ldots \prec c_p = \ell(M)\}$ in $\langle \bigwedge B, \ell(M)\rangle$ and the leftmost maximal chain $\{\ell(M) = c_p \prec \ldots \prec c_n = \bigvee B\}$ in $\langle \ell(M), \bigvee B\rangle$ exist. Clearly $c_p \in B$. Denote by i the least natural number such that $c_j \in B$ for each $j \in \{i,\dots,p\}$. Then $c_{p+1} = c_p^{\vee f} \in \operatorname{fix} f$ and $c_{i-1} = c_i^{\wedge f} \in \operatorname{fix} f$ by Lemma 7. Let K be the component of B for which $\ell(M) \in K$.

Claim. All elements of B are situated on the right of $C := \{c_0 \prec \ldots \prec c_n\}$.

If $b \in B$, then $b \leqslant \ell(M)$ or $b \parallel \ell(M)$. In the former case $b \in \langle \bigwedge B, \ell(M)\rangle$, and hence $\eta_C(b^{\mathbf{y}}) \leqslant b^{\mathbf{x}}$. In the latter case, there is an element $c \in M \cap \uparrow b$, and $\ell(M) \blacktriangleleft c$ by assumption. From Lemma 3(1) it follows that if $b^{\mathbf{x}} < \eta_C(b^{\mathbf{y}})$, then either $b \leqslant \ell(M)$ or $\ell(M) \leqslant c$, which is a contradiction.

Claim. $K \subseteq \langle c_{i-1}, c_{p+1}\rangle \setminus \{c_{i-1}, c_{p+1}\}$.

Clearly $K \cap \{c_{i-1}, c_{p+1}\} = \emptyset$ because $c_{i-1}, c_{p+1} \in \operatorname{fix} f$. Let $[\ell(M) = z_0, \dots, z_q]$ be a connection in K. We will verify by induction on k that

$$(*) \qquad c_{i-1} < z_k < c_{p+1} \;\&\; \eta_C({z_k}^{\mathbf{y}}) \leqslant {z_k}^{\mathbf{x}} < \eta_{f[\![C]\!]}({z_k}^{\mathbf{y}}).$$

This is obviously true for $k = 0$. Let $k \geqslant 1$. Assume that $(*)$ is satisfied for $k-1$. Suppose first that $z_{k-1} \prec z_k$. Then $c_{i-1} < z_{k-1} < z_k$. Since $z_k \in B$, by the preceding claim we have $\eta_C({z_k}^{\mathbf{y}}) \leqslant {z_k}^{\mathbf{x}}$. Since ${z_{k-1}}^{\mathbf{x}} < \eta_{f[\![C]\!]}({z_{k-1}}^{\mathbf{y}})$ and $z_{k-1} \prec z_k$, it follows from Lemma 3(1) that ${z_k}^{\mathbf{x}} \leqslant \eta_{f[\![C]\!]}({z_k}^{\mathbf{y}})$. But ${z_k}^{\mathbf{x}} = \eta_{f[\![C]\!]}({z_k}^{\mathbf{y}})$ would yield either $c_{p+1} \leqslant z_k$ or $z_k \in f[\![\{c_i,\dots,c_p\}]\!]$, which is a contradiction. Now ${z_k}^{\mathbf{y}} \leqslant {c_{p+1}}^{\mathbf{y}}$ implies that $z_k < c_{p+1}$ by Lemma 3(2). If ${c_{p+1}}^{\mathbf{y}} < {z_k}^{\mathbf{y}}$, then $c_{p+1} < z_k$ by the dual argument, but this is impossible. Similarly for $z_k \prec z_{k-1}$.

Claim. $\bigwedge B = c_{i-1}$ and $\bigvee B = c_{p+1}$.

From Lemma 7(4) and the preceding claim it follows that $c_{i-1} \leqslant \bigwedge B < c_i \in B$, and hence $\bigwedge B = c_{i-1}$. Analogously for the latter identity.

Uniqueness follows from Lemma 3(7). □

The component of B which contains the left boundary of B is called the *leftmost component* of B.

Let C, D be maximal chains in $\langle a, b\rangle$ such that $\eta_C(y) \leqslant \eta_D(y)$ for each $y \in \langle a^{\mathbf{y}}, b^{\mathbf{y}}\rangle$. We say that $c \in L$ is *inside the region bounded by* C, D if $c^{\mathbf{y}} \in \langle a^{\mathbf{y}}, b^{\mathbf{y}}\rangle$ & $\eta_C(c^{\mathbf{y}}) \leqslant c^{\mathbf{x}} \leqslant \eta_D(c^{\mathbf{y}})$. Otherwise we say that c is *outside the region bounded by* C, D. It follows from Lemma 3(2) that whenever c is inside the region bounded by C, D, then $c \in \langle a, b\rangle$.

Lemma 16. *Let B be a block.*

(1) *Let $x, y \in L, \bigwedge B \neq x < y$, x inside and y outside the region bounded by the boundaries of B. Then $\bigvee B \leqslant y$.*

(2) *If X is connected and $X \cap \{\bigwedge B, \bigvee B\} = \emptyset$, then X is completely (i.e. each $c \in X$ is) inside or completely outside the region bounded by the boundaries of B.*

(3) *$B \cup \{\bigvee B\}$ and $B \cup \{\bigwedge B\}$ are connected.*

Proof. Denote by S, T the left and right boundaries of B respectively.

(1): Take a maximal chain $\{x = z_0 \prec \ldots \prec z_p = y\}$ in $\langle x, y\rangle$. There exists $i \in \{0, \ldots, p-1\}$ such that z_i is inside and z_{i+1} outside the region bounded by the boundaries of B. Clearly $z_i \neq \bigwedge B$. The only possibility is that z_i is an element of a boundary, say the left one; see Lemma 3(2). If $z_i \neq \bigvee B$, then obviously $z_{i+1} = z_i^{\vee f} \leqslant \bigvee B$. Clearly $f(z_i) \prec z_{i+1}$ and $z_i \blacktriangleleft f(z_i)$. By Lemma 3(4) we obtain $\eta_S(z_{i+1}{}^{\mathbf{y}}) \leqslant z_{i+1}{}^{\mathbf{x}} \leqslant \eta_T(z_{i+1}{}^{\mathbf{y}})$, which is a contradiction. Hence $z_i = \bigvee B$.

(2): The assertion follows immediately from (1).

(3): Denote by K the leftmost component of B. Since $S \setminus \{\bigwedge B, \bigvee B\} \subseteq K$, $K \cup \{\bigvee B\}$ is connected. By Lemma 7, each component of B is of the form $f^k[\![K]\!]$. Hence $f^k[\![K]\!] \cup \{\bigvee B\}$ is also connected. Consequently, $B \cup \{\bigvee B\}$ is connected. $\square$

Lemma 17. *Let K be a component of a block of θ_f. If $c \in L$ and $[u, c^{\mathbf{y}}], [v, c^{\mathbf{y}}] \in$ $\operatorname{pt} K$ such that $u < c^{\mathbf{x}} < v$, then $c \in K$.*

Proof. There exist $a, b \in K$ such that $[u, c^{\mathbf{y}}] \in \operatorname{pt}\{a\}$ and $[v, c^{\mathbf{y}}] \in \operatorname{pt}\{b\}$. Moreover, there exists a connection $Z := [a = c_0, \ldots, c_p = b]$ in K. Choose a shortest one. Its diagram $\operatorname{dg} Z$ can be regarded as a parametrized curve in the real plane. If $[u, c^{\mathbf{y}}] \notin \operatorname{dg} Z$, then we can extend $\operatorname{dg} Z$ by adding the adjacent part of the arc containing $[u, c^{\mathbf{y}}]$, similarly for $[v, c^{\mathbf{y}}]$. In this way we obtain a curve $C \subseteq \operatorname{pt} K$ connecting $[u, c^{\mathbf{y}}]$ and $[v, c^{\mathbf{y}}]$. Let $[u, c^{\mathbf{y}}] = d_0, \ldots, d_q = [v, c^{\mathbf{y}}]$ be the sequence of all points of C the y-coordinates of which are equal to $c^{\mathbf{y}}$, in the natural order. Clearly $q \leqslant p+1$. There exists $i \in \{1, \ldots, q\}$ such that either $\pi_x(d_{i-1}) < c^{\mathbf{x}} < \pi_x(d_i)$ or $\pi_x(d_i) < c^{\mathbf{x}} < \pi_x(d_{i-1})$ or $\pi_x(d_i) = c^{\mathbf{x}}$. If $\pi_x(d_i) = c^{\mathbf{x}}$, then $d_i = [c^{\mathbf{x}}, c^{\mathbf{y}}]$ and hence $c \in K$. Further, one of the conditions $\pi_y(d) < c^{\mathbf{y}}$ and $c^{\mathbf{y}} < \pi_y(d)$ is satisfied for all points $d \in C$ between d_{i-1} and d_i. Denote $C' := \{d \in C : d \text{ is between } d_{i-1} \text{ and } d_i\}$. Without loss of generality we may assume that $(\forall d \in C')c^{\mathbf{y}} < \pi_y(d)$, $\pi_x(d_{i-1}) < c^{\mathbf{x}} < \pi_x(d_i)$ and $f[\![K]\!]$ is situated to the right of K. Denote by $Z' := [c_k, \ldots, c_l]$ the largest part of Z with $\operatorname{dg} Z' \subseteq C'$. Consider the set

$$X := \{e \in \uparrow c \cap (L \setminus K) \colon \forall \text{ maximal chain } D \text{ in } \langle e, \bigvee L\rangle \; D \cap Z' \neq \emptyset\}.$$

Suppose $c \in X$. Choose first a maximal element m in X and then z in $\uparrow m \cap Z'$ with the least possible level. Then, by Lemma 7, $m < f(z)$, and we have a maximal chain D in $\langle m, \bigvee L\rangle$ such that $f(z) \in D$. Since $D \cap Z' \neq \emptyset$ by assumption, there exists an element $w \in D \cap Z'$ such that either $m < w < f(z)$ or $m < f(z) < w$. Choose a minimal one. The former case is impossible as the levels of z and $f(z)$ are the same. In the latter case we obtain $m < z < z^{\vee_f} = f(z)^{\vee_f} < w$ by Lemma 7, and $z^{\vee_f} \in \uparrow c \cap \operatorname{fix} f \subseteq \uparrow c \cap (L \setminus K)$. Since m is maximal in X, there exists a maximal chain D in $\langle z^{\vee_f}, \bigvee L\rangle$ such that $D \cap Z' = \emptyset$. But for each pair of maximal chains D', D'' in $\langle m, f(z)\rangle, \langle f(z), z^{\vee_f}\rangle$ respectively we have $D' \cap Z' = \emptyset$, $D'' \cap Z' = \emptyset$, and hence $(D' \cup D'' \cup D) \cap Z' = \emptyset$. This contradicts the assumption, and therefore $c \notin X$. Now c_k is situated on the left and c_l is situated on the right of each maximal chain D in $\langle c, \bigvee L\rangle$, and hence $D \cap Z' \neq \emptyset$. Therefore $c \notin \uparrow c \cap (L \setminus K)$ and consequently $c \in K$. □

Lemma 18. *If $a \prec b$ and $f \in \operatorname{Aut} \mathbf{L}$, then $a^{\mathbf{x}} < f(a)^{\mathbf{x}}$ implies $b^{\mathbf{x}} \leqslant f(b)^{\mathbf{x}}$.*

P r o o f. Follows immediately from Lemma 2. □

For $Y \subseteq L$ and $x \in L$ we define $Y \rightharpoondown x := \{y \in Y : y^{\mathbf{y}} = x^{\mathbf{y}}\}$.

Lemma 19. *Let K_1 and K_2 be components of blocks B_1 and B_2 respectively such that $K_1 \cap K_2 \neq \emptyset$. Then $B_1 \cup B_2$ is a block and precisely one of the following conditions is satisfied:*

$B_1 \subset K_2$;

$B_2 \subset K_1$;

$K_1 = K_2$ and K is a component of $B_1 \cup B_2$ if and only if it is a component of B_1 or B_2.

P r o o f. Let B_1, B_2 be blocks of $\theta_{f_1}, \theta_{f_2}$ respectively. The assertion of the lemma is obviously satisfied if $B_1 \subseteq K_2$ or $B_2 \subseteq K_1$. In view of Lemma 11, $B_1 \neq K_2$ and $B_2 \neq K_1$. Assume that $K_1 \cap K_2 \neq \emptyset$, $B_1 \setminus K_2 \neq \emptyset$ and $B_2 \setminus K_1 \neq \emptyset$.

C l a i m. $\{\bigvee B_1, \bigvee B_2, \bigwedge B_1, \bigwedge B_2\} \subseteq \operatorname{fix} f_1 \cap \operatorname{fix} f_2$.

It suffices to verify that $\bigvee B_1 \in \operatorname{fix} f_2$. Suppose to the contrary that $\bigvee B_1 \in \operatorname{mov} f_2$. For $z \in B_1 \cap \operatorname{fix} f_2$ we would have $z = f_2(z) < f_2(\bigvee B_1)$, and by Lemma 16(1) we would obtain $\bigvee B_1 \leqslant f_2(\bigvee B_1)$, which is a contradiction. Hence $B_1 \cap \operatorname{fix} f_2 = \emptyset$ and, again by Lemma 16(3), $B_1 \subseteq B_1 \cup \{\bigvee B_1\} \subseteq K_2$ as $B_1 \cup \{\bigvee B_1\}$ is connected. This is again a contradiction.

Claim. $\bigvee B_1 = \bigvee B_2$ and $\bigwedge B_1 = \bigwedge B_2$.

It suffices to verify that $\bigvee B_1 \leqslant \bigvee B_2$. As K_1 is connected, and by the preceding claim $K_1 \cap \{\bigwedge B_2, \bigvee B_2\} = \emptyset$, K_1 is inside the region bounded by the boundaries of B_2 by Lemma 16(2). Hence $\bigvee K_1 \leqslant \bigvee B_2$, and since $\bigvee B_2 \in \operatorname{fix} f_1$, we conclude that $\bigvee B_1 \leqslant \bigvee B_2$.

Claim. If B_1 is of kind A, then also B_2 is of kind A and $K_1 = K_2$.

This follows from Lemma 7(6). Clearly $K_2 \subseteq K_1$, otherwise $\bigvee B_1 \in K_2$ or $\bigwedge B_1 \in K_2$. If B_2 were of kind Z, say there are $y \in K_2$ and $z \in \operatorname{fix} f_2$ such that $y \prec z < \bigvee B_2$, then we would have $K_1 \ni y \prec z < \bigvee B_2 = \bigvee B_1$, and hence $z \in K_1$. Now $\bigwedge B_1 = \bigwedge B_2 < f_2(y) \prec z \in K_1$, and therefore $f_2(y) \in K_1 \cap f_2[\![K_2]\!]$. Hence $f_2[\![K_2]\!] \subseteq K_1$, which would yield $B_2 = K_2 \cup f_2[\![K_2]\!] \subseteq K_1$, which contradicts the assumption. Thus B_2 is of kind A, and by the same argument as above $K_1 \subseteq K_2$. Hence $K_1 = K_2$.

Claim. If B_1 and B_2 are of kind A, then $B_1 \cup B_2$ is a block.

Order the set of all components of B_1 and B_2 in a sequence $K'_1, \ldots, K'_s$. In view of the preceding claim, they are pairwise disjoint and isomorphic. Take an arbitrary isomorphism of K'_i onto K'_{i+1} for $f|_{K'_i}$ if $i \in \{1, \ldots, s-1\}$, and put $f|_{K'_s} := (f|K'_1)^{-1} \circ \ldots \circ (f|K'_{s-1})^{-1}$, $f|_{L \setminus (B_1 \cup B_2)} := \mathrm{id}_{L \setminus (B_1 \cup B_2)}$. Then $f \in \operatorname{Aut} \mathbf{L}$ and $B_1 \cup B_2$ is its block.

Claim. If B_1 and B_2 are of kind Z, then B_1 and B_2 have the same boundaries.

There exists an element $b \in B_1$ such that $b^{\vee f} < \bigvee B_1$ or $\bigwedge B_1 < b^{\wedge f}$. Without loss of generality we can consider the former case. By Lemma 7(5), $B_1 \cup \{b^{\vee f}\}$ is connected, and moreover $(B_1 \cup \{b^{\vee f}\}) \cap \{\bigwedge B_2, \bigvee B_2\} = \emptyset$. In virtue of Lemma 16(2), $B_1 \cup \{b^{\vee f}\}$ is completely inside the region bounded by the boundaries of B_2, and so is B_1. By an analogous argument we obtain that B_2 is completely inside the region bounded by the boundaries of B_1. From Lemma 3(7) it follows that B_1 and B_2 have the same boundaries.

Claim. If B_1 and B_2 are of kind Z, then K_1 and K_2 are simultaneously left or right components in the respective blocks.

Take $x \in K_1 \cap K_2$. If K_1 were left and K_2 right, then $|f_2[\![K_2]\!] \rightharpoondown x| < |K_1 \rightharpoondown x| = |f_1[\![K_1]\!] \rightharpoondown x| < |K_2 \rightharpoondown x| = |f_2[\![K_2]\!] \rightharpoondown x|$ by Lemma 17, which is a contradiction.

Claim. If B_1 and B_2 are of kind Z, then $K_1 = K_2$.

If not, one of the following possibilities must occur in virtue of Lemma 7(6):

(1) $(\exists x \in K_1 \cap K_2)(\exists y \in K_2 \setminus K_1) x \prec y$,
(2) $(\exists x \in K_1 \cap K_2)(\exists y \in K_2 \setminus K_1) y \prec x$,
(3) $(\exists x \in K_1 \cap K_2)(\exists y \in K_1 \setminus K_2) x \prec y$,

(4) $(\exists x \in K_1 \cap K_2)(\exists y \in K_1 \setminus K_2) y \prec x$.

Without loss of generality we may assume the first one and suppose that K_1, K_2 are left components. Then $y \in \operatorname{fix} f_1$. As $f_1(x) \prec f_1(y) = y \in K_2$, we have $f_1(x) \notin f_2[\![K_2]\!]$ by Lemma 7(3). If $f_1(x) \in K_2$, then $|K_1 \rightharpoondown x| < |K_2 \rightharpoondown x| = |f_2[\![K_2]\!] \rightharpoondown x| < |f_1[\![K_1]\!] \rightharpoondown x| = |K_1 \rightharpoondown x|$ in virtue of Lemma 17, which is a contradiction. Hence $f_1(x) \notin K_2$. Again by Lemma 7(3), $f_1(x) \in \operatorname{fix} f_2$. As $x \prec y$ and $f_2 f_1(x) = f_1(x) \prec f_1(y) = y$, for each $k \in \mathbb{N}$ we have $(f_2 f_1)^k(x) \prec (f_2 f_1)^k(y) \succ (f_2 f_1)^{k+1}(x)$. We shall prove by induction on k that

$$(\mathrm{I}_k) \qquad \begin{aligned} (f_2 f_1)^k(x)^{\mathbf{x}} &< (f_2 f_1)^{k+1}(x)^{\mathbf{x}}, \\ (f_2 f_1)^k(y)^{\mathbf{x}} &< (f_2 f_1)^{k+1}(y)^{\mathbf{x}} \end{aligned}$$

is true for each $k \in \mathbb{N}$. Since K_1, K_2 are left components, (I_0) is true. Suppose $(\mathrm{I}_0), \ldots, (\mathrm{I}_k)$ are true. Then $(f_2 f_1)^k(y) \succ (f_2 f_1)^{k+1}(x) \prec (f_2 f_1)^{k+1}(y) \succ (f_2 f_1)^{k+2}(x) \prec (f_2 f_1)^{k+2}(y)$. By Lemma 18, $(f_2 f_1)^{k+1}(x)^{\mathbf{x}} \leqslant (f_2 f_1)^{k+2}(x)^{\mathbf{x}}$. If $(f_2 f_1)^{k+1}(x)^{\mathbf{x}} = (f_2 f_1)^{k+2}(x)^{\mathbf{x}}$, then $(f_2 f_1)^{k+1}(x) = (f_2 f_1)^{k+2}(x)$, and consequently $x = f_2 f_1(x) = f_1(x)$, which is a contradiction. Hence $(f_2 f_1)^{k+1}(x)^{\mathbf{x}} < (f_2 f_1)^{k+2}(x)^{\mathbf{x}}$. Now we can proceed analogously for y. We have obtained an infinite subset of L, and this is not possible.

Claim. If B_1 and B_2 are of kind Z, then $B_1 = B_2 = B_1 \cup B_2$.

By the preceding claims, $K_1 = K_2$ and $f_1[\![K_1]\!] = f_2[\![K_2]\!]$, which yields $B_1 = B_2 = B_1 \cup B_2$. $\square$

Proposition 20. *Let $A \subseteq L$. The set $\{\theta_f : f \in \operatorname{Aut}_A \mathbf{L}\}$ has the greatest element with respect to set inclusion.*

Proof. Consider an arbitrary element $x \in \operatorname{Mov}_A \mathbf{L}$. Since $\mathbf{L}$ is finite, the set of all blocks from $\wp(A)$ which contain x has a maximal element B with respect to set inclusion. Now, if D is a block from $\wp(A)$ containing x, then $B \cup D$ is also a block from $\wp(A)$ containing x in virtue of the preceding lemma, and the maximality of B implies that $D \subseteq B$. Hence B is the greatest block from $\wp(A)$ containing x. Denote it by B_x. By Lemma 8, there exists $g \in \operatorname{Aut}_A \mathbf{L}$ such that $\bigcup\{B_x \times B_x : x \in \operatorname{Mov}_A \mathbf{L}\} = \theta_g$. It is clear that θ_g is the greatest element of $\{\theta_f : f \in \operatorname{Aut}_A \mathbf{L}\}$. $\square$

In fact, $\{\theta_f \cup \Delta_L : f \in \operatorname{Aut} \mathbf{L}\}$ is a $\vee$-subsemilattice of the lattice of all equivalences on L.

We denote the greatest element of $\{\theta_f\colon f \in \operatorname{Aut}\mathbf{L}\}$ by $\Theta_{\mathbf{L}}$.

Lemma 21. (1) *Each $f \in \operatorname{Aut}\mathbf{L}$ maps any component of a block of $\Theta_{\mathbf{L}}$ onto a component of the same block.*

(2) *If B is a block of $\Theta_{\mathbf{L}}$ and $f, g \in \operatorname{Aut}\mathbf{L}$, then $f|_B \cup g|_{L\setminus B} \in \operatorname{Aut}\mathbf{L}$.*

Proof. (1) Let K be a component of a block B of $\Theta_{\mathbf{L}}$. For each $a \in K$ we have either $f(a) = a$ and hence $f(a) \in B$, or a and $f(a)$ are elements of the same block of θ_f and hence $f(a) \in B$. Therefore $f[\![K]\!]$ is a connected subset of B with the same cardinality as K, and so a component of B.

(2) The proof follows immediately from Lemma 8, since each block of f and g is included either in B or in $L \setminus B$. $\square$

Lemma 22. *For any planar lattice $\mathbf{L} := (L, \leqslant)$ and $A \subseteq L$ there exists a planar lattice $\mathbf{M} := (M, \leqslant)$ such that $\operatorname{Aut}_A \mathbf{L} \cong \operatorname{Aut}\mathbf{M}$.*

Proof. By induction on the cardinality of $L \setminus A$.

Basic step. $|L \setminus A| = 0$. Then of course $\operatorname{Aut}_A \mathbf{L} = \operatorname{Aut}\mathbf{L}$ and we may take $\mathbf{L}$ for $\mathbf{M}$.

Induction step. Let $m > 0$. Suppose the assertion is true whenever $|L \setminus A| < m$. Let $\mathbf{L} := (L, \leqslant)$ be a planar lattice and let $A \subseteq L$ be such that $|L \setminus A| = m$. Denote $n := |L|$. Choose $a \in L \setminus A$ and put $\mathbf{C}_a := \{c_0 < \ldots < c_n\}$ where $\{c_0, \ldots, c_n\} =: C_a$ is disjoint with L, and $\mathbf{C}_x := \{x\}$ for $x \in L \setminus \{a\}$. Let $\mathbf{K} := (K, \leqslant)$ be the ordinal sum $\sum_{x\in\mathbf{L}} \mathbf{C}_x$. Notice that we have just replaced a with an n-element chain. It is clear that $\mathbf{K}$ is a planar lattice. We first prove that for each $f \in \operatorname{Aut}_{A\cup C_a} \mathbf{K}$ we have $C_a \subseteq \operatorname{fix} f$. Since $|C_a| > |L|$, there exists $b \in C_a$ such that $f(b) \in C_a$. This in turn yields $b = f(b)$ because $\{b, f(b)\}$ is an antichain by Lemma 1. Further, each $c \in C_a$ is comparable with b and hence $f(c)$ is comparable with $b = f(b)$. By the definition of $\mathbf{K}$, $f(c)$ is comparable with c as well, and therefore $c = f(c)$. We claim that $\operatorname{Aut}_A \mathbf{L} \cong \operatorname{Aut}_{A\cup C_a} \mathbf{K}$. For each $f \in \operatorname{Aut}_A \mathbf{L}$ put $\varphi(f) := f|_{L\setminus\{a\}} \cup \mathrm{id}_{C_a}$. Clearly $\varphi(f) \in \operatorname{Aut}_{A\cup C_a} \mathbf{K}$. Then of course

$$\begin{aligned}\varphi(f \circ g) &= (f \circ g)|_{L\setminus\{a\}} \cup \mathrm{id}_{C_a} = f|_{L\setminus\{a\}} \circ g|_{L\setminus\{a\}} \cup \mathrm{id}_{C_a}\\ &= (f|_{L\setminus\{a\}} \cup \mathrm{id}_{C_a}) \circ (g|_{L\setminus\{a\}} \cup \mathrm{id}_{C_a}) = \varphi(f) \circ \varphi(g).\end{aligned}$$

Injectivity: $\varphi(f) = \varphi(g)$ implies $f|_{L\setminus\{a\}} = g|_{L\setminus\{a\}}$, which in turn yields $f = g$. Surjectivity: Let $f' \in \operatorname{Aut}_{A\cup C_a} \mathbf{K}$. Put $f := f'|_{K\setminus C_a} \cup \mathrm{id}_{\{a\}} = f'|_{L\setminus\{a\}} \cup \mathrm{id}_{\{a\}}$. Clearly $f \in \operatorname{Aut}_A \mathbf{L}$. Then

$$f' = f'|_{K\setminus C_a} \cup \mathrm{id}_{C_a} = f'|_{L\setminus\{a\}} \cup \mathrm{id}_{C_a} = (f'|_{L\setminus\{a\}} \cup \mathrm{id}_{\{a\}})|_{L\setminus\{a\}} \cup \mathrm{id}_{C_a} = \varphi(f).$$

Now $|K \setminus (A \cup C_a)| = m - 1$, and by the induction hypothesis there exists a planar lattice $\mathbf{M}$ such that $\operatorname{Aut}_{A \cup C_a} \mathbf{K} \cong \operatorname{Aut} \mathbf{M}$. $\square$

Theorem 23. $\mathcal{P}(\mathcal{T}) = \mathcal{AG}(\mathcal{PL})$.

Proof. We have already noticed that $\mathcal{P}(\mathcal{T}) \subseteq \mathcal{AG}(\mathcal{PL})$ in virtue of Proposition 6. We must verify the converse inclusion. We will prove by induction on the cardinality of $\mathbf{G}$ that $\mathbf{G} \in \mathcal{AG}(\mathcal{PL})$ implies $\mathbf{G} \in \mathcal{P}(\mathcal{T})$. We have only to show that for each $\mathbf{G} \in \mathcal{AG}(\mathcal{PL})$ one of the following conditions is fulfilled:

(1) $\mathbf{G} \in \mathcal{T}$.

(2) There exist $\mathbf{G}_1$ and $\mathbf{G}_2$ in $\mathcal{AG}(\mathcal{PL})$ such that $\mathbf{G} \cong \mathbf{G}_1 \times \mathbf{G}_2$ and $|G_1| < |G|$, $|G_2| < |G|$.

(3) There exist an integer $n \geqslant 2$ and $\mathbf{H}$ in $\mathcal{AG}(\mathcal{PL})$ such that $\mathbf{G} \cong n\mathbf{H}$; then of course $|H| < |G|$.

Without loss of generality we can assume that there exists a planar lattice $\mathbf{L}$ such that $\mathbf{G} = \operatorname{Aut} \mathbf{L}$.

Claim. If $\Theta_{\mathbf{L}}$ has no block, then $\mathbf{G} \in \mathcal{T}$.

The proof is obvious.

Claim. If $\Theta_{\mathbf{L}}$ has at least two blocks, then there exist planar lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ such that $\mathbf{G} = \operatorname{Aut} \mathbf{L} \cong \operatorname{Aut} \mathbf{L}_1 \times \operatorname{Aut} \mathbf{L}_2$ and $|\operatorname{Aut} \mathbf{L}_1| < |\operatorname{Aut} \mathbf{L}|$, $|\operatorname{Aut} \mathbf{L}_2| < |\operatorname{Aut} \mathbf{L}|$.

Take a block B of $\Theta_{\mathbf{L}}$. In view of Lemma 22, it suffices to show that

$$\operatorname{Aut} \mathbf{L} \cong \operatorname{Aut}_B \mathbf{L} \times \operatorname{Aut}_{L \setminus B} \mathbf{L}$$

and $|\operatorname{Aut}_B \mathbf{L}| < |\operatorname{Aut} \mathbf{L}|$, $|\operatorname{Aut}_{L \setminus B}| < |\operatorname{Aut} \mathbf{L}|$. This follows immediately from Lemma 21(2): we define

$$\varphi\colon \operatorname{Aut} \mathbf{L} \longrightarrow \operatorname{Aut}_B \mathbf{L} \times \operatorname{Aut}_{L \setminus B} \mathbf{L}$$

by the rule

$$\varphi(f) := [f|_B \cup \mathrm{id}_{L \setminus B}, \mathrm{id}_B \cup f|_{L \setminus B}].$$

Then clearly

$$\begin{aligned}\varphi(f \circ g) &= [(f \circ g)|_B \cup \mathrm{id}_{L \setminus B}, \mathrm{id}_B \cup (f \circ g)|_{L \setminus B}] \\ &= [f|_B \cup \mathrm{id}_{L \setminus B}, \mathrm{id}_B \cup f|_{L \setminus B}] \circ [g|_B \cup \mathrm{id}_{L \setminus B}, \mathrm{id}_B \cup g|_{L \setminus B}] = \varphi(f) \circ \varphi(g).\end{aligned}$$

Injectivity: $\varphi(f) = \varphi(g)$ implies that $f|_B = g|_B$ and $f|_{L \setminus B} = g|_{L \setminus B}$, which in turn yields $f = g$. Surjectivity: Let $f \in \operatorname{Aut}_B \mathbf{L}$ and $g \in \operatorname{Aut}_{L \setminus B} \mathbf{L}$. Then $f|_B \cup g|_{L \setminus B} \in \operatorname{Aut} \mathbf{L}$ by Lemma 21(2) and $[f, g] = \varphi(f|_B \cup g|_{L \setminus B})$.

C l a i m. If $\Theta_{\mathbf{L}}$ has precisely one block, then there exist an integer $n \geqslant 2$ and a planar lattice $\mathbf{L}_1$ such that $\mathbf{G} = \operatorname{Aut}\mathbf{L} \cong n \operatorname{Aut}\mathbf{L}_1$.

By definition, there exists $h \in \operatorname{Aut}\mathbf{L}$ such that $\Theta_{\mathbf{L}} = \theta_h$. Take the number of components of the unique block B of $\Theta_{\mathbf{L}}$ for n. By Lemma 11, $n \geqslant 2$. In view of Lemma 22, it suffices to show that

$$\operatorname{Aut}\mathbf{L} \cong n \operatorname{Aut}_K \mathbf{L}$$

where K is a component of B. We can denote the components of B by $K_1, \dots, K_n$ in such a way that $K_i = h^i[\![K]\!]$, see Lemma 7(1). For each $f \in \operatorname{Aut}\mathbf{L}$ we define $p_f \in S_n$ by $p_f(i) := j$ where $f[\![K_i]\!] = K_j$, see Lemma 21(1). Clearly $p_{f\circ g} = p_f p_g$. Denote $f' := h^{-p_f(i)} \circ f \circ h^i$. Then $f'[\![K]\!] = K$ and whenever $x \prec y$, $x, y \in \operatorname{mov} f'$, $x \in K$, then $y \in K$ because K is a component of a block of $\Theta_{\mathbf{L}} = \theta_h$ and $\operatorname{mov} f' \subseteq \operatorname{mov} h$. Therefore each block of $\theta_{f'}$ is included either in K or in $L \setminus K$. In virtue of Lemma 8, $h^{-p_f(i)} \circ f \circ h^i|_K \cup \mathrm{id}_{L\setminus K}$ is an automorphism of $\mathbf{L}$. We define

$$\varphi\colon \operatorname{Aut}\mathbf{L} \longrightarrow n \operatorname{Aut}_K \mathbf{L}$$

by the rule

$$\varphi(f) := [p_f; \dots, h^{-p_f(i)} \circ f \circ h^i|_K \cup \mathrm{id}_{L\setminus K}, \dots].$$

It is transparent that $\varphi(f \circ g) = \varphi(f) \circ \varphi(g)$. Injectivity: $\varphi(f) = \varphi(g)$ implies $p_f = p_g =: p$ and $\forall i\; h^{-p(i)} \circ f \circ h^i|_K = h^{-p(i)} \circ g \circ h^i|_K$, i.e. $f|_{K_i} = g|_{K_i}$. But this yields $f|_B = g|_B$ and consequently $f = g$. Surjectivity: Let $f_i \in \operatorname{Aut}_K \mathbf{L}$ $(i \in \{1, \dots, n\})$ and $p \in S_n$. We define $f\colon L \to L$ by

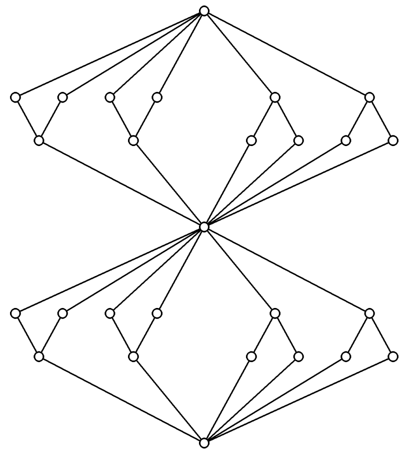
$$f := \mathrm{id}_{L\setminus B} \cup \bigcup_i h^{p(i)} \circ f_i \circ h^{-i}|_{K_i}.$$

It is clear that f is bijective. We have to show that f is an automorphism of $\mathbf{L}$. Take $x \prec y$. If $x, y \in K_i$, then $f(x) = h^{p(i)} \circ f_i \circ h^{-i}(x) \prec h^{p(i)} \circ f_i \circ h^{-i}(y) = f(y)$. If $x \in K_i$ & $y \notin K_i$, then $y \notin B$, and consequently $f(x) = h^{p(i)} \circ f_i \circ h^{-i}(x) \prec h^{p(i)} \circ f_i \circ h^{-i}(y) = y = f(y)$. Similarly for $y \in K_i$ & $x \notin K_i$. If $x \notin B$ and $y \notin B$, then $f(x) = x \prec y = f(y)$. Clearly $\varphi(f) = [p; \dots, f_i, \dots]$. $\square$

A note is hidden in [2] which says that the groups isomorphic to the automorphism groups of planar lattices are precisely the groups isomorphic to the automorphism groups of trees. This is proved as a particular case in [1] in terms of graph theory. Since it can be easily seen that Theorem 23 describes also the automorphism groups of trees, see also [5], we have in fact proved the assertion of the note mentioned above in terms of lattice theory.

Proposition 24. *It is not true that every automorphism group of a planar lattice is identical with the automorphism group of a tree.*

P r o o f. Here is a counterexample:



There is no tree with the same vertices as the planar lattice visualized above such that they both have the same automorphism group. □

We say that a block B is *saturated* if for each block C either $B \cap C = \emptyset$ or $B \subseteq C$ or $C \subseteq B$. Notice that any two saturated blocks are either disjoint or in inclusion.

Lemma 25. (1) *For each block B the set of all saturated blocks that include B has the least element.*

(2) *For each element $a \in \operatorname{Mov} \mathbf{L}$ the set of all saturated blocks that contain a has the least element.*

P r o o f. (1): Since $\mathbf{L}$ is finite, each block is included in a maximal one. In view of Lemma 19, each maximal block is saturated. Hence the set of all saturated blocks that include B forms a finite nonempty chain with respect to inclusion. It follows that it has the least element.

(2) follows immediately from (1). □

The saturated blocks from Lemma 25 will be denoted by $s(B)$ and $s(\{a\})$ respectively.

Lemma 26. *All components of a block B are also components of $s(B)$.*

P r o o f. Since $\mathbf{L}$ is finite, there exists a maximal element D of the set of all blocks such that $B \subseteq D$ and all components of B are also components of D. If C is a block with $C \cap D \neq \emptyset$ and $D \nsubseteq C$, then by Lemma 19, $C \cup D$ is a block and all components of B are also components of $C \cup D$. The maximality of D yields $C \subseteq D$. Hence D is saturated. □

In fact $D = s(B)$.

We say that $(X, \sqsubset)$ is a *tree-ordered set* if $\sqsubset$ is a strict order on X with a top element, and $c \sqsubset a$ & $c \sqsubset b$ implies that either $a = b$ or $a \sqsubset b$ or $b \sqsubset a$.

Proposition 27. *Consider a fixed standard planar representation of* $\mathbf{L}$. *Put* $r(a) := s(a)$ *for* $a \in \operatorname{Mov}\mathbf{L}$ *and* $r(a) := L$ *for* $a \in L \setminus \operatorname{Mov}\mathbf{L}$. *Define a binary relation* $\sqsubset$ *on* L *by the rule*

$$a \sqsubset b :\Longleftrightarrow\ a \overset{r(b)}{\leftrightsquigarrow} b \ \&\ (r(a) \subset r(b) \text{ or } (r(a) = r(b) \ \&\ a^{\mathbf{y}} < b^{\mathbf{y}})).$$

Then $(L, \sqsubset, {}^{\mathbf{y}})$ *is a colored tree-ordered set with the same automorphism group as* $\mathbf{L}$.

Proof. Notice that, for $f \in \operatorname{Aut}\mathbf{L}$, it follows from $a \overset{r(a)}{\leftrightsquigarrow} f(a)$ that $a = f(a)$. In other words, $a \overset{r(a)}{\leftrightsquigarrow} b$ & $a \neq b$ implies that $a^{\mathbf{y}} \neq b^{\mathbf{y}}$. Indeed, in view of Lemma 11 $a^{\mathbf{y}} = b^{\mathbf{y}}$ would imply that there is a block B with components $K_1, K_2, \ldots$, $K_1 \neq K_2$ such that $a \in K_1$, $b \in K_2$. By Lemma 26 K_1 and K_2 are also components of $s(B)$. But $a \overset{r(b)}{\leftrightsquigarrow} b$ yields $a \overset{s(B)}{\leftrightsquigarrow} b$ because $r(b) \subseteq s(B)$. This is a contradiction.

First we prove that $\sqsubset$ is a tree order. Irreflexivity is obvious. The top element in $\mathbf{L}$ becomes the top element in $(L, \sqsubset)$.

Transitivity: $a \sqsubset b \ \&\ b \sqsubset c \Longrightarrow (r(a) \subset r(b) \text{ or } (r(a) = r(b) \ \&\ a^{\mathbf{y}} < b^{\mathbf{y}})) \ \&\ (r(b) \subset r(c) \text{ or } (r(b) = r(c) \ \&\ b^{\mathbf{y}} < c^{\mathbf{y}})) \ \&\ a \overset{r(b)}{\leftrightsquigarrow} b \ \&\ b \overset{r(c)}{\leftrightsquigarrow} c \Longrightarrow (r(a) \subset r(c) \text{ or } (r(a) = r(c) \ \&\ a^{\mathbf{y}} < c^{\mathbf{y}})) \ \&\ a \overset{r(c)}{\leftrightsquigarrow} c \Longrightarrow a \sqsubset c$.

Now let $c \sqsubset a$ & $c \sqsubset b$. Then $r(c) \subseteq r(a)$ & $r(c) \subseteq r(b)$ and therefore $c \in r(a) \cap r(b)$, which in turn yields $r(a) \subseteq r(b)$ or $r(b) \subseteq r(a)$. Without loss of generality we may assume that $r(a) \subseteq r(b)$. Then obviously $a \overset{r(b)}{\leftrightsquigarrow} c$ & $c \overset{r(b)}{\leftrightsquigarrow} b$ and therefore $a \overset{r(b)}{\leftrightsquigarrow} b$. Suppose that $a \neq b$. It is easy to see that $r(a) \subset r(b)$ yields $a \sqsubset b$. Assume that $r(a) = r(b)$. Hence $a^{\mathbf{y}} < b^{\mathbf{y}}$ or $b^{\mathbf{y}} < a^{\mathbf{y}}$ and consequently $a \sqsubset b$ or $b \sqsubset a$.

Let $f \in \operatorname{Aut}\mathbf{L}$. Then obviously $(\forall x \in L)\, r(f(x)) = f[\![r(x)]\!]$ & $f(x)^{\mathbf{y}} = x^{\mathbf{y}}$. If $a \sqsubset b$, then $(r(a) \subset r(b)$ or $(r(a) = r(b) \ \&\ a^{\mathbf{y}} < b^{\mathbf{y}})$ & $a \overset{r(b)}{\leftrightsquigarrow} b$. This implies that $(r(f(a)) = f[\![r(a)]\!] \subset f[\![r(b)]\!] = r(f(b))$ or $(r(f(a)) = f[\![r(a)]\!] = f[\![r(b)]\!] = r(f(b))$ & $f(a)^{\mathbf{y}} = a^{\mathbf{y}} < b^{\mathbf{y}} = f(b)^{\mathbf{y}})$ & $f(a) \overset{r(f(b))}{\leftrightsquigarrow} f(b)$, i.e. $f(a) \sqsubset f(b)$. It follows that f is an automorphism of $(L, \sqsubset, {}^{\mathbf{y}})$.

Let conversely f be an automorphism of $(L, \sqsubset, {}^{\mathbf{y}})$. Let $a \prec b$. Then $f(a)^{\mathbf{y}} = a^{\mathbf{y}} < b^{\mathbf{y}} = f(b)^{\mathbf{y}}$, and therefore either $f(a) < f(b)$ or $f(a) \parallel f(b)$. Consider the latter case. Since $f(a)^{\mathbf{y}} = a^{\mathbf{y}}$ and $f(b)^{\mathbf{y}} = b^{\mathbf{y}}$, there exist $g, h \in \operatorname{Aut}\mathbf{L}$ such that $g(b) = f(b)$ and $f(a) = h(a)$. Clearly $h(a) \neq g(a)$ and $h(b) \neq g(b)$. Let us investigate all possibilities of the mutual position of $r(a)$ and $r(b)$.

1. $r(a) \cap r(b) = \emptyset$: It is impossible by Lemma 7(3) and Lemma 8.

2. $r(a) \subset r(b)$: From $a \prec b$ we immediately obtain that $a \overset{r(b)}{\leftrightsquigarrow} b$. Then $a \sqsubset b$ and hence $h(a) = f(a) \sqsubset f(b) = g(b)$. Thus $h(a) \in r(h(b)) \cap r(g(b))$, and therefore $r(h(b)) = r(g(b)) = r(f(b))$. Hence $h(b) \overset{r(f(b))}{\leftrightsquigarrow} h(a) \overset{r(f(b))}{\leftrightsquigarrow} g(b)$. This is a contradiction.

3. $r(b) \subset r(a)$: From $a \prec b$ we immediately obtain that $a \overset{r(a)}{\leftrightsquigarrow} b$. Then $b \sqsubset a$ and hence $g(b) = f(b) \sqsubset f(a) = h(a)$. Thus $g(b) \in r(g(a)) \cap r(h(a))$, and therefore $r(h(a)) = r(g(a)) = r(f(a))$. Hence $h(b) \overset{r(f(a))}{\leftrightsquigarrow} h(a) \overset{r(f(a))}{\leftrightsquigarrow} g(b)$. This is a contradiction.

4. $r(b) = r(a)$: From $a \prec b$ we immediately obtain that $a \overset{r(b)}{\leftrightsquigarrow} b$. Since $a^{\mathbf{y}} < b^{\mathbf{y}}$, we have $a \sqsubset b$ and hence $h(a) = f(a) \sqsubset f(b) = g(b)$. Thus $h(a) \in r(g(b)) \cap r(h(b))$, and therefore $r(h(b)) = r(g(b)) = r(f(b))$. Hence $h(b) \overset{r(f(b))}{\leftrightsquigarrow} h(a) \overset{r(f(b))}{\leftrightsquigarrow} g(b)$. This is a contradiction.

Summing up we obtain that $f(a) < f(b)$. It follows that $f \in \operatorname{Aut} \mathbf{L}$. □

As an immediate consequence we obtain the following recent result of George Grätzer and Csaba Szabó.

Corollary 28. *Every automorphism group of a planar lattice is identical with the automorphism group of a colored tree.*

Problems and concluding remarks

It is clear that $\mathcal{AG}(\mathcal{PL})$ is decidable. We must only construct the trees of all possible decompositions of a given finite group into symmetric powers and direct products of subgroups. This algorithm is finite but rather complex. Its complexity would be lower if the following hypotheses were true.

(1) Is it true that for $m, n > 1$, $n\mathbf{G} \cong m\mathbf{H} \Longrightarrow m = n \ \& \ \mathbf{G} \cong \mathbf{H}$?
(2) Is it true that $n\mathbf{G} \cong \mathbf{G}_1 \times \mathbf{G}_2 \Longrightarrow \min\{|G_1|, |G_2|, n\} = 1$?

We have defined components via the definition of blocks. Some people may prefer to define a component of $\operatorname{mov} f$ as a maximal connected subset of $\operatorname{mov} f$, then to check that for each component K of $\operatorname{mov} f$ its image $f[\![K]\!]$ is also a component of $\operatorname{mov} f$, and finally to say that a block is the smallest union of components which is closed under f.

We could also define $\Theta_{\mathbf{L}}$ as the equivalence on $\operatorname{Mov} \mathbf{L}$ generated by $\bigcup_f \theta_f$. The crucial point is to verify that each block of $\Theta_{\mathbf{L}}$ has at least two components. We have done this in Lemmas 9–19 and Proposition 20 with help of Lemmas 1–4 and Proposition 5.

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