CONNECTIONS INDUCED BY (1,1)-TENSOR FIELDS ON COTANGENT BUNDLES

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Abstract. On cotangent bundles the Liouville field, the Liouville 1-form ε and the canonical symplectic structure $\mathrm{d}\varepsilon$ exist. In this paper interactions between these objects and $(1,1)$-tensor fields on cotangent bundles are studied. Properties of the connections induced by the above structures are investigated.

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Introduction

We assume that all manifolds and maps in this paper are infinitely differentiable.

Let M be a manifold and (x^i) a local chart on M. Let (x^i, z_i) be the induced chart on the cotangent bundle T^*M of all 1-forms on M. Let us recall that the Liouville vector field $V = z_i\partial/\partial z_i$ the flow of which is formed by the homotheties on individual fibres of T^*M, the Liouville 1-form $\varepsilon = z_i\,\mathrm{d}x^i$, and the symplectic 2-form $\omega = \mathrm{d}\varepsilon = \mathrm{d}z_i\wedge\,\mathrm{d}x^i$ on T^*M exist. Let F be a (1,1)-tensor field on M. It is known, [4], [7] that there is no connection on M, i.e. a linear connection on TM, which could be constructed by natural operators from F only. In other words no linear connection on T^*M can be constructed from natural lifts of F on T^*M only. We deal with the connections on the vector fibre bundle $\pi\colon T^*M \to M$ which are induced by (1,1)-tensor fields α on T^*M that are very close to the natural lifts of F on T^*M. We favour almost complex structures α (ACS). First of all, two cases are

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investigated: α preserves the vertical subbundle VT^*M or not. In the former case we deal with connections Γ_α when α preserves the horizontal subbundle $H\Gamma$ and in the latter with connections Γ_α for which $\alpha(VT^*M) = H\Gamma^2_\alpha$ or $\alpha(H\Gamma^1_\alpha) = VT^*M$.

The main results are in the third part of the paper. We deal with symmetric (1,1)-tensor fields α when the form $\mathrm{d}\varepsilon^\alpha(X,Y) = \mathrm{d}\varepsilon(\alpha X, Y)$ is symmetric, and with symmetric connections Γ on T^*M when $\mathrm{d}\varepsilon|_{H\Gamma} = 0$, where $H\Gamma$ is the horizontal bundle of a connection Γ.

When α does not preserve VT^*M our investigations are concentrated on the semi-linear case of α when $B = T\pi \cdot \alpha|_{VT^*M}$ is a base morphism and the map $T\pi \cdot \alpha \cdot X\colon T^*M \to TM$ is a vector bundle morphism for any projectable linear vector field $X\colon T^*M \to TT^*M$ on T^*M. Propositions 5–7 determine sufficient conditions for the connection Γ^1_α to be linear, for the equality $\gamma^1_\alpha = TB(\Gamma^1_\alpha)$ and for the connection γ^1_α to be just the Levi-Civita connection determined by the pseudo-Riemannian structure B^{-1} on M, where γ^1_Γ is the connection on TM induced by the linear connection Γ^1_α. Propositions 8 and 9 describe some properties of the ACS $\alpha(\Gamma, B)$ which are determined by a linear symmetric connection Γ on T^*M and by a vector bundle morphism B.

In the case $\alpha(VT^*M) \subset VTM$ there are morphisms $A = T\pi \cdot \alpha\colon TT^*M \to TM$, $H = \alpha|_{VT^*M}$. Remember that the complete lift $\alpha = F^C$ of a (1, 1)-tensor field F on M preserves VT^*M, $A = H$ and it is a VB-field, i.e. for any linear projectable vector field X on T^*M the vector field $\alpha(X)$ is again linear and projectable. When α is symmetric then $A = -H$. Propositions 12 and 13 state sufficient conditions under which a symmetric (1,1)-tensor field α (especially an ACS) determines connections Γ_α on T^*M.

Our investigations are local.

Connections induced by (1,1)-and (0,2)-tensor fields on fibre bundles

Let $\pi\colon E \to M$ be a fibre bundle. Let (x^i, y^α) be a local fibre chart on Y. A connection Γ on E can be regarded as a (1,1)-tensor field h_Γ on E (called the horizontal form of Γ) such that $h_\Gamma(VE) = 0$, $T\pi \cdot h_\Gamma = T\pi$, where VE is the vector fibre bundle of the vertical vectors on E and Tf denotes the tangent prolongation of a map f. In coordinates,

$$h_\Gamma = \mathrm{d}x^i \otimes \partial/\partial x^i + \Gamma^\alpha_j(x,y)\mathrm{d}x^j \otimes \partial/\partial y^\alpha. \tag{1}$$

Denote $H\Gamma := h_\Gamma(TE) \subset TE$ the vector fibre bundle of the Γ-horizontal vectors on E, i.e. such vectors $(x^i, y^\alpha, \mathrm{d}x^i, \mathrm{d}y^\alpha)$ on E which satisfy the equation

$$\mathrm{d}y^\alpha = \Gamma^\alpha_j \mathrm{d}x^j.$$

The functions Γ^α_j are called the local components of the connection Γ. Readers are refered to [6] for more details on connections on fibre bundles.

1. Let

$$\alpha=(a^i_j(x,y)\mathrm{d}x^j+b^i_\alpha(x,y)\mathrm{d}y^\alpha)\otimes\partial/\partial x^i+(c^\alpha_j(x,y)\mathrm{d}x^j+h^\alpha_\beta(x,y)\mathrm{d}y^\beta)\otimes\partial/\partial y^\alpha$$

be a (1,1)-tensor field on E. We will briefly denote by B the vector bundle morphism $T\pi\cdot\alpha|_{VE}\colon\ VE\to TM$ over $\pi\colon E\to M$. It can be interpreted as a section $E\to V^*E\otimes TM,\ B=b^i_\alpha\mathrm{d}y^\alpha\otimes\partial/\partial x^i$.

We will say shortly that α is vertical if $B=0$, i.e. if $\alpha(VE)\subset VE$. We will recall some properties of connections connected with a (1,1)-tensor field α, see [2].

Let h_Γ be the horizontal form of a connection Γ on E given by (1). Let $Y=\eta^\alpha\partial/\partial y^\alpha\in VE$ be an arbitrary vertical vector on E and let $X=\xi^i\partial/\partial x^i+\Gamma^\alpha_i\xi^i\partial/\partial y^\alpha$ be a Γ-horizontal vector. Then $\alpha(Y)=b^i_\beta\eta^\beta\partial/\partial x^i+h^\alpha_\beta\eta^\beta\partial/\partial y^\alpha$ or $\alpha(X)=(a^i_j+b^i_\beta\Gamma^\beta_j)\xi^j\partial/\partial x^i+(c^\alpha_j+h^\alpha_\beta\Gamma^\beta_j)\xi^j\partial/\partial y^\alpha$ is Γ-horizontal for any vertical vector Y or vertical for any Γ-horizontal vector X iff

$$\Gamma^\alpha_k b^k_\beta=h^\alpha_\beta\quad\text{or}\quad a^i_j+b^i_\beta\Gamma^\beta_j=0. \tag{2}$$

We have

Lemma 1. *Let* $\dim M$ *be the dimension of the fibres on* E. *Then there is a unique connection* Γ^1_α *and a unique connection* Γ^2_α *on* E *such that* $\alpha(H\Gamma^1_\alpha)=VE,\ \alpha(VE)=H\Gamma^2_\alpha$ *if and only if the vector bundle morphism* B *is regular. Then* $-\tilde b^\alpha_k a^k_j$ *and* $h^\alpha_\beta\tilde b^\beta_j$ *are respectively the local components of* Γ^1_α *or* Γ^2_α, *and* $\Gamma^1_\alpha=\Gamma^2_\alpha$ *if and only if* α^2 *is vertical.*

We will suppose that $\dim M$ is the dimension of fibres on E.

The coordinate conditions for α to be an almost complex structure on E, i.e. $\alpha^2=-\mathrm{Id}_{TE}$ are

$$a^i_s a^s_j+b^i_\beta c^\beta_j=-\delta^i_j,\ a^i_s b^s_\beta+b^i_\gamma h^\gamma_\beta=0,\ c^\alpha_s a^s_j+h^\alpha_\gamma c^\gamma_j=0,\ c^\alpha_s b^s_\beta-h^\alpha_\gamma h^\gamma_\beta=-\delta^\alpha_\beta. \tag{3}$$

If B is regular then the third and fourth equations of (3) are consequences of the first and the second ones.

By the equalities (2) and (3) it is easy to prove

Lemma 2. *Let* Γ *be a connection on* E. *Let* $B\colon E\to V^*E\otimes TM$ *be a vector bundle isomorphism* $VE\to TM$ *over* π. *Then there exists a unique almost complex structure* $\alpha(\Gamma,B)$ *on* E *such that* $T\pi\cdot\alpha|_{VE}=B$ *and* $\Gamma^1_\alpha=\Gamma^2_\alpha=\Gamma_\alpha$.

In the case of $E = TM$, if we choose $B = \mathrm{Id}|_{VTM}$ then the almost complex structure $\alpha(\Gamma, B)$ is the canonical almost complex structure determined by the connection Γ on TM, see [5].

Let $Q = Q^\alpha_i \mathrm{d}x^i \otimes \partial/\partial y^\alpha$ be a section $E \to T^*M \otimes VE$. Denote

$$\alpha^+ \colon Q \to Q\alpha = (Q^\alpha_k a^k_j \mathrm{d}x^j + Q^\alpha_k b^k_\beta \mathrm{d}y^\beta) \otimes \partial/\partial y^\alpha, \ \ \alpha^+ \colon T^*M \otimes VE \to T^*E \otimes VE$$
$$\alpha^- \colon Q \to \alpha Q = b^i_\beta Q^\beta_j \mathrm{d}x^j \otimes \partial/\partial x^i + h^\alpha_\beta Q^\beta_j \mathrm{d}x^j \otimes \partial/\partial y^\alpha, \alpha^- \colon T^*M \otimes VE \to T^*M \otimes TE.$$

We say that two (1,1)-tensor fields α_1, α_2 on E are (+)-equivalent or (−)-equivalent if $\alpha_1^+ = \alpha_2^+$ or $\alpha_1^- = \alpha_2^-$ respectively for any $Q \colon E \to T^*M \otimes VE$.

If B is regular then using (3) we get ([2] Proposition 6)

Lemma 3. *In every class of all* $(+)$*-equivalent* $(1,1)$*-tensor fields on* E *there is a unique almost complex structure on* E*. The same is true for the class of* $(-)$*-equivalent* $(1,1)$*-tensor fields.*

In the case when α is vertical, i.e. when $B = 0$, denote

$$A := T\pi \cdot \alpha = a^i_j \mathrm{d}x^j \otimes \partial/\partial x^i, \quad A \colon E \to T^*M \otimes_E TM,$$
$$H \colon \alpha|_{VY} = h^\alpha_\beta \mathrm{d}y^\beta \otimes \partial/\partial y^\alpha, \quad H \colon E \to V^*E \otimes VE.$$

Let $\Gamma, \overline{\Gamma}$ be two connections on E and α a vertical (1,1)-tensor field. Then

$$\overline{\Gamma}^\alpha_k a^k_j = c^\alpha_j + h^\alpha_\beta \Gamma^\beta_j \tag{4}$$

is the coordinate condition for the inclusion $\alpha(H\Gamma) \subset H\overline{\Gamma}$.

Using (3) and (4) we get (see [2], Proposition 10)

Lemma 4. *Let* $A \colon E \to T^*M \otimes_E TM$, $H \colon E \to V^*E \otimes VE$ *be sections. Let* Γ *be a connection on* E*. Then there is a unique vertical* $(1,1)$*-tensor field* $\alpha(A, H, \Gamma)$ *such that* $\alpha(H\Gamma) \subset H\Gamma$, $T\pi \cdot \alpha = A$, $\alpha|_{VE} = H$*. Moreover, if* $A^2(u) = -\mathrm{Id}_{T\pi u M}$ *for any* $u \in E$ *and* $H^2 = -\mathrm{Id}_{VTM}$ *then* $\alpha(A, H, \Gamma)$ *is an ACS on* E*.*

R e m a r k. If A is regular then for any connection Γ there exists a unique conection $\overline{\Gamma}$ on E such that $\alpha(H\Gamma) \subset H\overline{\Gamma}$.

2. Let $\omega = f_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j + f_{i\alpha} \mathrm{d}x^i \otimes \mathrm{d}y^\alpha + f_{\alpha i} \mathrm{d}y^\alpha \otimes \mathrm{d}x^i + f_{\alpha\beta} \mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta$ be a (0,2)-tensor field on E. Recall some well known facts.

Denote by ω^t the (0,2)-field transposed to ω, $\omega^t(X, Y) = \omega(Y, X)$. Then ω is symmetric or skew-symmetric if $\omega^t = \omega$ or $\omega^t = -\omega$, respectively.

Let α be a (1,1)-tensor field on E. Denote by $\omega^\alpha, \omega_\alpha, \omega\alpha$ the following (0,2)-tensor fields:

$$\omega^\alpha(X,Y) := \omega(\alpha X, Y),\ \omega_\alpha(X,Y) := \omega(X, \alpha Y),\ \omega\alpha(X,Y) := \omega(\alpha X, \alpha Y).$$

It is evident that

$$(\omega^\alpha)^t = (\omega^t)_\alpha,\ (w_\alpha)^t = (\omega^t)^\alpha,\ (\omega\alpha)^t = \omega^t\alpha \tag{5}$$

in the general case and

$$\omega^\alpha = \pm\, \omega_\alpha \Leftrightarrow \omega\alpha = \mp\, \omega,\ \omega^\alpha\alpha = -\omega_\alpha,\ \omega_\alpha\alpha = -\omega^\alpha \tag{6}$$

$$(\omega^\alpha)^t = \omega^\alpha \Leftrightarrow \omega\alpha = -\omega^t,\ (\omega^\alpha)^t = -\omega^\alpha \Leftrightarrow \omega\alpha = \omega^t \tag{7}$$

in the case of an ACS α on E.

We say that a tangent subbundle $V_2 \subset TE$ is ω-orthogonal to a subbundle $V_1 \subset TE$ if $\omega(X,Y) = 0$ for any $X \in V_1(u),\ Y \in V_2(u)$ at any $u \in E$. A tangent subbundle $V \subset TE$ is said to be ω-zero if $\omega|_V = 0$.

Let Γ_1, Γ_2 be two connections on E. We say that Γ_2 is ω-orthogonal to Γ_1 if the Γ_2-horizontal subbundle $H\Gamma_2$ is ω-orthogonal to $H\Gamma_1$.

If $X = \xi^i\partial/\partial x^i + \Gamma^\alpha_j\xi^j\partial/\partial y^\alpha$ is Γ_1-horizontal and $\overline{X} = \overline{\xi}^i\partial/\partial x^i + \overline{\Gamma}^\alpha_j\overline{\xi}^j$ is Γ_2-horizontal then $\omega(X,\overline{X}) = (f_{ij} + f_{i\alpha}\overline{\Gamma}^\alpha_j + f_{\alpha j}\Gamma^\alpha_i + f_{\alpha\beta}\Gamma^\alpha_i\overline{\Gamma}^\beta_j)\xi^i\overline{\xi}^j$ and so the connection Γ_2 is ω-orthogonal to Γ_1 if and only if

$$f_{ij} + f_{i\alpha}\overline{\Gamma}^\alpha_j + f_{\alpha j}\Gamma^\alpha_i + f_{\alpha\beta}\Gamma^\alpha_i\overline{\Gamma}^\beta_j = 0. \tag{8}$$

Consider the following restrictions of a (0,2)-tensor field ω:

$$\begin{aligned}
\omega_1 :\ &= \omega|_{TE\times_E VE},\ \omega_1 = f_{i\alpha}\mathrm{d}x^i \otimes \mathrm{d}y^\alpha + f_{\alpha\beta}\mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta,\\
\omega_2 :\ &= \omega|_{VE\times_E TE},\ \omega_2 = f_{\alpha i}\mathrm{d}y^\alpha \otimes \mathrm{d}x^i + f_{\alpha\beta}\mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta,\\
\omega_v :\ &= \omega|_{VE\times_E VE},\ \omega_v = f_{\alpha\beta}\mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta.
\end{aligned}$$

The equality (8) immediately gives

Lemma 5. *Let Γ_1, Γ_2 be two connections on a fibre bundle $\pi\colon E \to M$. Let $\varphi_1 = f_{i\alpha}\,\mathrm{d}x^i \otimes \mathrm{d}y^\alpha + f_{\alpha\beta}\,\mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta,\ \varphi_1\colon E \to T^*E \otimes_E V^*E,\ \varphi_2 = f_{\alpha i}\,\mathrm{d}y^\alpha \otimes \mathrm{d}x^i + f_{\alpha\beta}\,\mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta,\ \varphi_2\colon E \to V^*E \otimes_E T^*E$ be two bilinear forms such that $\varphi_1|_{VE\times_E VE} = \varphi_2|_{VE\times_E VE}$. Then there is a unique $(0,2)$-tensor field $\omega(\varphi_1, \varphi_2, \Gamma_1, \Gamma_2)$ on E such that $\omega_1 = \varphi_1,\ \omega_2 = \varphi_2$ and Γ_2 is ω-orthogonal to Γ_1.*

If $\Gamma_2 = \Gamma_1 = \Gamma$ then the tensor field $\omega(\varphi_1, \varphi_2, \Gamma)$ from Lemma 5 is such that the connection Γ is ω-zero, i.e. $\omega|_{H\Gamma} = 0$.

We can find subbundles $\mathcal{H}, \mathcal{H}' \subset TE$ such that $\mathcal{H}$ is ω-orthogonal to VE and VE is ω-orthogonal to $\mathcal{H}'$. Let $Y = \eta^\alpha \partial/\partial y^\alpha$ be vertical and let $X = \mathrm{d}x^i \partial/\partial x^i + \mathrm{d}y^\alpha \partial/\partial y^\alpha$ be an arbitrary tangent vector on E. Then the equation $\omega(Y, X) = 0$ or $\omega(X, Y) = 0$ is satisfied for any vertical vector Y iff

$$
\begin{aligned}
& f_{\alpha i}\mathrm{d}x^i + f_{\alpha\beta}\mathrm{d}y^\beta = 0 \quad \text{or} \\
& f_{i\alpha}\mathrm{d}x^i + f_{\beta\alpha}\mathrm{d}y^\beta = 0.
\end{aligned} \tag{8'}
$$

This immediately gives

Lemma 6. *There exist unique connections $\Gamma_\omega, \Gamma'_\omega$ such that Γ_ω is ω-orthogonal to VE and VE is ω-orthogonal to Γ'_ω if and only if the form $\omega_v = \omega|_{VE}$ is regular . If ω is symmetric or skew-symmetric then $\Gamma_\omega = \Gamma'_\omega$. The vertical subbundle VE is ω-zero if and only if $\omega_v = 0$.*

In the following lemma we suppose that $\dim M$ is the dimension of fibres on E and that $B = T\pi \cdot \alpha|_{VE}$ is regular (so there exist connections $\Gamma^1_\alpha, \Gamma^2_\alpha, \alpha(H\Gamma^1_\alpha) = VE$, $\alpha(VE) = H\Gamma^2_\alpha$ on E).

Lemma 7. *Let ω be a $(0,2)$-tensor field and α a $(1,1)$-tensor field on E. Then*

1. *VE is ω-zero iff VE is ω^α-orthogonal to Γ^1_α or Γ^1_α is ω_α-orthogonal to V or $H\Gamma^1_\alpha$ is $\omega\alpha$-zero.*
2. *The connection Γ^1_α is ω-orthogonal to VE iff Γ^1_α is ω^α-zero.*
3. *VE is ω-orthogonal to the connection Γ^1_α iff Γ^1_α is ω_α-zero.*
4. *VE is ω^α-zero (ω_α-zero) iff Γ^1_α is $\omega^\alpha\alpha$-zero ($\omega_\alpha\alpha$-zero).*
5. *VE is $\omega\alpha$-zero iff the connection Γ^2_α is ω-zero or $H\Gamma^2_\alpha$ is ω^α-orthogonal to VE or VE is ω_α-orthogonal to $H\Gamma^2_\alpha$.*
6. *VE is ω^α-zero iff VE is ω-orthogonal to Γ^2_α.*
7. *VE is ω_α-zero iff Γ^2_α is ω-orthogonal to VE.*
8. *The connection Γ^2_α is ω^α-zero iff VE is $\omega\alpha$-orthogonal to Γ^2_α.*
9. *The connection Γ^2_α is ω_α-zero iff Γ^2_α is $\omega\alpha$-orthogonal to VE.*

In the induced chart (x^i, z_i) on T^*M a (1,1)-tensor field α on the fibre bundle $\pi\colon T^*M \to M$ is of the form

$$\alpha = (a^i_j(x,z)\mathrm{d}x^j + b^{ij}\mathrm{d}z_j) \otimes \partial/\partial x^i + (c_{ij}\mathrm{d}x^j + h^j_i\mathrm{d}z_j) \otimes \partial/\partial z_i.$$

According to the identification $VT^*M = T^*M \times_M T^*M$ the vector bundle morphism $B = T\pi \cdot \alpha|_{VT^*M}\colon\ VT^*M \to TM,\ B = b^{ij}\mathrm{d}z_j \otimes \partial/\partial x^i,$ can be interpreted as a vector bundle morphism $B\colon T^*M \times_M T^*M \to T^*M \times_M TM$, i.e. as a section $B\colon T^*M \to TM \otimes_{T^*M} TM$, i.e. as a bilinear form in VT^*M.

Definition 1. A (1,1)-tensor field α on T^*M is called v-symmetric or v-skew symmetric or v-basic if the section B is symmetric or skew symmetric or if B is the π-pullback of a section $M \to TM \otimes TM$.

Let us introduce the coordinate expression of some forms and tensor fields constructed from the Liouville form $\varepsilon = z_i\mathrm{d}x^i,\ \omega \equiv \mathrm{d}\varepsilon = \mathrm{d}z_i \wedge \mathrm{d}x^i$ and α:

$$\begin{aligned} i_\alpha\varepsilon &= z_k(a^k_j\mathrm{d}x^j + b^{kj}\mathrm{d}z_j),\\ i_\alpha\mathrm{d}\varepsilon &= c_{ij}\mathrm{d}x^j \wedge \mathrm{d}x^i + (h^j_i + a^j_i)\mathrm{d}x_j \wedge \mathrm{d}x^i + b^{ij}\mathrm{d}z_i \wedge \mathrm{d}z_j,\\ i_\alpha\mathrm{d}\varepsilon|_{VT^*M} &= b^{ij}\mathrm{d}z_i \wedge \mathrm{d}z_j, \end{aligned}$$

where i_α denotes the algebraic graded derivation determined by α,

$$\begin{aligned} \mathrm{d}\varepsilon^\alpha &= c_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j + h^j_i\mathrm{d}z_j \otimes \mathrm{d}x^i - a^j_i\mathrm{d}x^i \otimes \mathrm{d}z_j - b^{ij}\mathrm{d}z_j \otimes \mathrm{d}z_i,\\ \mathrm{d}\varepsilon_\alpha &= -c_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j + a^j_i\mathrm{d}x_j \otimes \mathrm{d}x^i - h^j_i\mathrm{d}x^i \otimes \mathrm{d}z_j + b^{ji}\mathrm{d}z_j \otimes \mathrm{d}z_i,\\ \mathrm{d}\varepsilon\alpha &= c_{ti}a^t_j\mathrm{d}x^i \wedge \mathrm{d}x^j + (c_{ti}b^{tj} - a^t_i h^j_t)\mathrm{d}x^i \wedge \mathrm{d}z_j + h^i_t b^{tj}\mathrm{d}z_i \wedge \mathrm{d}z_j. \end{aligned}$$

It is evident that

(9) $\quad \mathrm{d}\varepsilon^\alpha + \mathrm{d}\varepsilon_\alpha = i_\alpha\mathrm{d}\varepsilon,$

(10) $\quad \mathrm{d}\varepsilon^\alpha - \mathrm{d}\varepsilon_\alpha = (c_{ji} + c_{ij})\mathrm{d}x^i \otimes \mathrm{d}x^j + (h^j_i - a^j_i)(\mathrm{d}x^i \otimes \mathrm{d}z_j + \mathrm{d}z_j \otimes \mathrm{d}x^i) - (b^{ij} + b^{ji})\mathrm{d}z_i \otimes \mathrm{d}x_j$ is symmetric,

(11) $\quad i_\alpha\mathrm{d}\varepsilon$ is the antisymmetrization of $\mathrm{d}\varepsilon^\alpha$,

(12) $\quad (\mathrm{d}\varepsilon^\alpha)^t = -\mathrm{d}\varepsilon_\alpha$, i.e. $(\mathrm{d}\varepsilon)^\alpha$ is symmetric or exterior iff $\mathrm{d}\varepsilon^\alpha = -\mathrm{d}\varepsilon_\alpha$ or $\mathrm{d}\varepsilon^\alpha = \mathrm{d}\varepsilon_\alpha$.

In coordinates $(\mathrm{d}\varepsilon)^\alpha$ is symmetric or exterior if

$$\begin{aligned} &c_{ij} = c_{ji},\ h^j_i = -a^j_i,\ b^{ij} = b^{ji} \quad \text{or}\\ &c_{ij} = -c_{ji},\ h^j_i = a^j_i,\ b^{ij} = -b^{ji}. \end{aligned}$$

Proposition 1. *Let α be an ACS on T^*M. Then*

1. $(i_\alpha\,\mathrm{d}\varepsilon)\alpha = -i_\alpha\,\mathrm{d}\varepsilon$,
2. $(\,\mathrm{d}\varepsilon^\alpha - \mathrm{d}\varepsilon_\alpha)\alpha = \mathrm{d}\varepsilon^\alpha - \mathrm{d}\varepsilon_\alpha$,
3. $\mathrm{d}\varepsilon^\alpha$ *is symmetric or skew symmetric iff* $\mathrm{d}\varepsilon\alpha = \mathrm{d}\varepsilon$ *or* $\mathrm{d}\varepsilon\alpha = -\,\mathrm{d}\varepsilon$, *respectively,*
4. $\mathrm{d}\varepsilon^\alpha\alpha = \mathrm{d}\varepsilon^\alpha$ *if* $\mathrm{d}\varepsilon^\alpha$ *is moreover symmetric.*

Proof. 1 and 2 follow from (6) and (9). Assertion 3 is a consequence of (7). The equalities (6) and (12) imply 4. □

Corollary. *Let α be an ACS on T^*M. Let $\mathrm{d}\varepsilon^\alpha$ be symmetric, i.e. let $\mathrm{d}\varepsilon$ be invariant under α. Then $\mathrm{d}\varepsilon^\alpha$ is a pseudo-Hermite metric on T^*M and $(T^*M,\alpha,\,\mathrm{d}\varepsilon^\alpha)$ is a pseudo-almost Kähler space,* [7].

Proof. By Proposition 1 $\mathrm{d}\varepsilon^\alpha\alpha = \mathrm{d}\varepsilon^\alpha$ so $\mathrm{d}\varepsilon^\alpha$ is a pseudo-Hermite metric. As $(\mathrm{d}\varepsilon^\alpha)^\alpha = -\mathrm{d}\varepsilon$ is exact so $(T^*M,\alpha,\mathrm{d}\varepsilon^\alpha)$ is pseudo-almost Kähler. □

Definition 2. A (1,1)-tensor field α on T^*M is called symmetric or skew symmetric if $\mathrm{d}\varepsilon^\alpha$ is symmetric or skew symmetric.

In the induced local chart (x^i, z_i) on T^*M a connection Γ on T^*M is given by the equations

$$\mathrm{d}z_i = \Gamma_{ij}(x,z)\mathrm{d}x^j.$$

As the form $\mathrm{d}\varepsilon$ is symplectic, there exists a unique connection Γ^t which is $\mathrm{d}\varepsilon$-orthogonal to Γ. By (8) its local components are $\overline{\Gamma}_{ij} = \Gamma_{ji}$. So Γ is $\mathrm{d}\varepsilon$-zero iff $\Gamma^t = \Gamma$. In this case we will say that Γ is symmetric on T^*M.

Analogously in the case when the form $i_\alpha\mathrm{d}\varepsilon$ is regular, i.e. almost-symplectic. Remember that if $\mathrm{d}\varepsilon^\alpha$ is symmetric then $i_\alpha\mathrm{d}\varepsilon = 0$.

In our futher consideration we will deal with two cases of the (1,1)-tensor field α on T^*M.

I. Let α be such that $B = T\pi\cdot\alpha|_{VT^*M} = b^{ij}\mathrm{d}z_j\otimes\partial/\partial x^i$ is regular. Then by Lemma 1 there are connections Γ^1_α and Γ^2_α such that $\alpha(H\Gamma^1_\alpha) = VT^*M$ and $\alpha(VT^*M) = H\Gamma^2_\alpha$. Then

$$\Gamma^1_{ij} = -b_{ik}a^k_j,\quad \Gamma^2_{ij} = h^k_i b_{kj},\quad b_{ik}b^{kj} = \delta^j_i \tag{14}$$

are their local components.

By Lemma 6 there are two connections $\Gamma_{\mathrm{d}\varepsilon^\alpha}, \Gamma'_{\mathrm{d}\varepsilon^\alpha}$ such that $\Gamma_{\mathrm{d}\varepsilon^\alpha}$ is $\mathrm{d}\varepsilon^\alpha$ orthogonal to VT^*M and VT^*M is $\mathrm{d}\varepsilon^\alpha$-orthogonal to $\Gamma'_{\mathrm{d}\varepsilon^\alpha}$. According to (8′) their local components are respectively

$$\Gamma_{ij} = h^t_j b_{ti},\quad \Gamma'_{ij} = -b_{it}a^t_j.$$

Comparing the local components of the connections $\Gamma^1_\alpha, \Gamma^2_\alpha, \Gamma_{\mathrm{d}\varepsilon^\alpha}, \Gamma'_{\mathrm{d}\varepsilon^\alpha}$ we get

Proposition 2. *If α is such a $(1,1)$-tensor field that B is regular then $\mathrm{d}\varepsilon^\alpha|_{VT^*M}$ is also regular and*

a) $\Gamma^1_\alpha = \Gamma'_{\mathrm{d}\varepsilon^\alpha}$ *and so VT^*M is $\mathrm{d}\varepsilon^\alpha$-orthogonal to Γ^1_α,*
b) $\Gamma_{\mathrm{d}\varepsilon^\alpha} = (\Gamma^2_\alpha)^t$, *i.e. the connections $\Gamma_{\mathrm{d}\varepsilon^\alpha}$ and Γ^2_α are $\mathrm{d}\varepsilon$-orthogonal and thus the connection $(\Gamma^2_\alpha)^t$ is $\mathrm{d}\varepsilon^\alpha$-orthogonal to VT^*M.*

Remark. As $(\mathrm{d}\varepsilon^\alpha)^t = -\mathrm{d}\varepsilon_\alpha$ therefore $\Gamma_{\mathrm{d}\varepsilon_\alpha} = \Gamma'_{\mathrm{d}\varepsilon^\alpha}$ and $\Gamma'_{\mathrm{d}\varepsilon_\alpha} = \Gamma_{\mathrm{d}\varepsilon^\alpha}$. So the connection Γ^1_α is $\mathrm{d}\varepsilon_\alpha$-orthogonal to VT^*M and VT^*M is $\mathrm{d}\varepsilon_\alpha$-orthogonal to the connection $(\Gamma^2_\alpha)^t$. By Lemma 7/1, Γ^1_α is $\mathrm{d}\varepsilon$-zero because VT^*M is $\mathrm{d}\varepsilon$-zero.

Proposition 3. *If α is such a $(1,1)$-tensor field that B is regular, α^2 is vertical and $d\varepsilon^\alpha$ is symmetric or skew-symmetric then the connection $\Gamma_\alpha = \Gamma^1_\alpha = \Gamma^2_\alpha$ is $\mathrm{d}\varepsilon$-zero, i.e. $(\Gamma_\alpha)^t = \Gamma_\alpha$.*

Proof. If $\mathrm{d}\varepsilon^\alpha$ is symmetric then $b_{kj} = b_{jk}, -a^k_j = h^k_j$. When α^2 is vertical then $\Gamma^1_\alpha = \Gamma^2_\alpha$, i.e. $-b_{ik}a^k_j = h^k_i b_{kj}$, i.e. $b_{ki}h^k_j = b_{kj}h^k_i$, i.e. $\Gamma_{ij} = \Gamma_{ji}$. Analogously when $-b_{kj} = b_{jk}$, $a^k_j = h^k_j$, i.e. when $\mathrm{d}\varepsilon^\alpha$ is skew symmetric. □

Let us remark that if α is an ACS then α^2 is vertical.

The inverse $B^{-1} = b_{ij}(x,z)\mathrm{d}x^i \otimes \partial/\partial z_j \colon T^*M \times_M TM \to T^*M \times_M T^*M$ can be interpreted as a semibasic bilinear form $b_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$ on T^*M, i.e. as a section $T^*M \to T^*M \otimes_{T^*M} T^*M$.

Proposition 4. *Let α be such an ACS on T^*M that B is regular and let h_{Γ_α} be the horizontal form of the connection $\Gamma_\alpha = \Gamma^1_\alpha = \Gamma^2_\alpha$. Then $\mathrm{d}\varepsilon^\alpha h_{\Gamma_\alpha} = -B^{-1}$ and $\mathrm{d}\varepsilon_\alpha h_{\Gamma_\alpha} = (B^{-1})^t$.*

Proof. $h_{\Gamma_\alpha}(X_q) = \xi^i_q\partial/\partial x^i + h^s_i\tilde{b}_{si}\xi^j_q\partial/\partial z_i$, $q = 1,2$. Then using (3) we get $\mathrm{d}\varepsilon^\alpha(h_{\Gamma_\alpha}X_1, h_{\Gamma_\alpha}X_2) = (c_{ij} + h^t_i h^s_t b_{sj} - a^t_j h^s_t b_{si} - b^{ut}h^s_t b_{sj} h^r_u b_{ri})\xi^j_1\xi^i_2 = -b_{ij}\xi^j_1\xi^i_2$. This proves the first part. The second is a consequence of the equality $(d\omega^\alpha)^t = -d\omega_\alpha$. □

Remark. As $B^{-1} = b_{ij}\mathrm{d}x^i \otimes \partial/\partial z_j$ is a semibasic 1-form with values in VT^*M then if h_Γ is the horizontal form of a connection Γ on T^*M then $h_\Gamma + B^{-1}$ is the horizontal form of the other connection on T^*M.

Recall that a projectable linear vector field X on T^*M is such a vector field that $T\pi X$ is a vector field on M and its flow is formed by linear maps of fibres on T^*M, i.e. in coordinates $X = \xi^i(x)\partial/\partial x^i + \eta^j_i(x)z_j\partial/\partial z_i$.

Definition 3. A non-vertical (1,1)-tensor field α on T^*M is said to be semi-linear if it is v-basic and for any projectable linear vector field $X\colon T^*M \to TT^*M$ on T^*M the map $T\pi \cdot \alpha X\colon T^*M \to TM$ is a vector bundle morphism.

In a local chart it is easy to see that α is semi-linear iff $a^i_j(x,z) = a^{ik}_j(x) z_k$ and $b^{ij}(x,z) = b^{ij}(x)$.

Proposition 5. *If a $(1,1)$-tensor field α on T^*M is semi-linear and such that B is regular then the connection Γ^1_α is linear.*

Proof. The local components of Γ^1_α are $\Gamma_{ij} = -b_{is} a^s_j = -b_{is}(x) a^{sk}_j(x) z_k$, i.e. Γ^1_α is linear. $\square$

Let us recall that every linear connection $\Gamma, \Gamma_{ij} = \Gamma^k_{ij} z_k$, on T^*M is induced by the linear connection γ on the tangent bundle TM with the local components $\gamma^k_i = -\Gamma^k_{ji} x^j_1$. The connection Γ is symmetric if and only if γ is symmetric.

So, if α is semi-linear and B is regular then the connection Γ^1_α is induced by the connection γ^1_α on TM with the local components $\gamma^i_j = \gamma^i_{jk} x^k_1$, $\gamma^i_{jk} = b_{ks} a^{si}_j$. As $B\colon T^*M \to TM$, $\overline{x}^i = x^i$, $\overline{x}^i_1 = b^{ij}(x) z_j$, is a vector bundle isomorphism therefore $TB(\Gamma^1_\alpha)$ is a connection on TM. We find its functions.

$$
\begin{aligned}
TB\colon \overline{x}^i = x^i, \quad x^i_1 = b^{ij}(x) z_j,\, \mathrm{d}\overline{x}^i = \mathrm{d}x^i, \quad \mathrm{d}x^i_1 = b^{ij}_k z_j \mathrm{d}x^k + b^{ij} \mathrm{d}z_j,\\
h_{\Gamma^1_\alpha} = \mathrm{d}x^i \otimes \partial/\partial x^i - b_{is} a^{sk}_j z_k \mathrm{d}x^j \otimes \partial/\partial z_i,\\
TB \cdot h_{\Gamma_1} = \mathrm{d}x^i \otimes \partial/\partial x^i + (b^{ij}_k - b^{it} b_{ts} a^{sj}_k) z_j \mathrm{d}x^k \otimes \partial/\partial x^i.
\end{aligned}
$$

Then $\Gamma^i_j = (b^{is}_j - a^{is}_j) b_{sk} x^k_1$ establish the components of the connection $TB(\Gamma^1_\alpha)$. Therefore $\gamma^1_\alpha = TB(\Gamma^1_\alpha)$ if and only if

$$
b_{ks} a^{si}_j = (b^{is}_j - a^{is}_j) b_{sk}, \quad \text{i.e.} \quad b_{ku} a^{ui}_j b^{ks} = b^{is}_j - a^{is}_j. \tag{15}
$$

Proposition 6. *Let α be such a semilinear $(1,1)$-tensor field on T^*M that B is regular and symmetric, i.e. α is v-symmetric. Then $\gamma^1_\alpha = TB(\Gamma^1_\alpha)$ if and only if $i_V\, \mathrm{d}(i_\alpha \varepsilon) = 0$.*

Proof.

$$
\varepsilon = z_k \mathrm{d}x^k, \quad i_\alpha \varepsilon = a^{kt}_j z_k z_t \mathrm{d}x^j + b^{kj} z_k \mathrm{d}z_j
$$
$$
\mathrm{d}i_\alpha \varepsilon = a^{kt}_{ji} z_k z_t \mathrm{d}x^i \wedge \mathrm{d}x^j + a^{kt}_j (z_t \mathrm{d}z_k - z_k \mathrm{d}z_t) \wedge \mathrm{d}x^j + b^{kj}_i z_k \mathrm{d}x^i \wedge \mathrm{d}z_j + b^{kj} \mathrm{d}z_k \wedge \mathrm{d}z_j
$$

where we use the notation $f_i := \frac{\partial f}{\partial x^i}$, $f^i := \frac{\partial f}{\partial z_i}$. Then

$$
\begin{aligned}
i_V\, \mathrm{d}i_\alpha \varepsilon &= (a^{kt}_j + a^{tk}_j - b^{kt}_j) z_k z_t \mathrm{d}x^j + (b^{kj} - b^{jk}) z_k \mathrm{d}z_j\\
&= (a^{kt}_j + a^{tk}_j - b^{kt}_j) z_k z_t \mathrm{d}x^j = 0
\end{aligned}
$$

if and only if $b_j^{kt} = a_j^{kt} + a_j^{tk}$. When B is symmetric then (15) reads $a_j^{si} = b_j^{is} - a_j^{is}$. This completes our proof. $\square$

If $B\colon T^*M \to TM$ is regular and symmetric then $B^{-1} = b_{ij}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j$ determines a pseudo-Riemannian structure on M the Levi-Civita connection γ_b of which is given by the local components $C_{jk}^i = -\frac{1}{2}b^{is}(b_{skj} + b_{sjk} - b_{jks})$.

Proposition 7. *Let α be such a semi-linear $(1,1)$-tensor field α on T^*M that B is regular, $\mathrm{d}\varepsilon^\alpha$ is symmetric, α^2 is vertical and $i_V\,\mathrm{d}(i_\alpha\varepsilon) = 0$. Then the connection γ_α determined on TM by Γ_α is just the Levi-Civita connection γ_b of the pseudo-Riemannian structure on M induced by B^{-1}.*

P r o o f. By supposition $b^{ij} = b^{ji}$, $h_j^i = -a_j^{ik}z_k$, $a_s^{ij}b^{sk} = b^{is}a_s^{kj}$ or $b_{is}a_j^{st} = a_i^{st}b_{sj}$ and $b_j^{kt} = a_j^{kt} + a_j^{tk}$. Then

$$\begin{aligned} C_{jk}^i &= -\tfrac{1}{2}b^{is}(b_{skj} + b_{sjk} - b_{jks}) = \tfrac{1}{2}b_j^{is}b_{sk} + \tfrac{1}{2}b_k^{is}b_{sj} - \tfrac{1}{2}b^{is}b_{jt}b_s^{tu}b_{uk} \\ &= \tfrac{1}{2}(a_j^{is} + a_j^{si})b_{sk} + \tfrac{1}{2}(a_k^{is} + a_k^{si})b_{sj} - \tfrac{1}{2}b^{is}b_{jt}(a_s^{tu} + a_s^{ut})b_{uk} \\ &= \tfrac{1}{2}a_j^{is}b_{sk} + \tfrac{1}{2}b_{js}a_k^{si} + \tfrac{1}{2}a_k^{is}b_{sj} + \tfrac{1}{2}b_{ks}a_j^{si} - \tfrac{1}{2}a_j^{iu}b_{uk} - \tfrac{1}{2}a_k^{it}b_{jt} \\ &= \tfrac{1}{2}b_{js}a_k^{si} + \tfrac{1}{2}b_{ks}a_j^{si} = \tfrac{1}{2}a_j^{si}b_{sk} + \tfrac{1}{2}b_{ks}a_j^{si} = b_{ks}a_j^{si} = \gamma_{jk}^i. \end{aligned}$$

This completes our proof. $\square$

Recall in the sense of Lemma 2 that by $\alpha(\Gamma, B)$ we denote the almost complex structure α on T^*M determined uniquely by a connection Γ on T^*M and by a vector bundle isomorphism $B\colon VT^*M \to TM$ over $\pi\colon T^*M \to M$.

Proposition 8. *Let Γ be a symmetric connection on T^*M, i.e. $\Gamma^t = \Gamma$. Let $B\colon VT^*M \to TM$ be a symmetric vector bundle isomorphism. Then the almost complex structure $\alpha(\Gamma, B)$ is symmetric.*

P r o o f. Let $B = b^{ij}\mathrm{d}z_j \otimes \partial/\partial x_1^i$, $b^{ij} = b^{ji}$, and $\Gamma_{ij} = \Gamma_{ji}$ be the components of Γ. Let α be such a ACS on T^*M that $T\pi\alpha|_{VT^*M} = B$ and $\Gamma_\alpha^1 = \Gamma$. Using (14) we get $a_j^i = -b^{is}\Gamma_{sj}$. Then by the second equality of (3)

$$h_j^i = -b_{jt}a_s^t b^{si} = b_{jt}b^{tu}\Gamma_{us}b^{si} = b^{is}\Gamma_{sj} = -a_j^i.$$

The first equality of (3) reads $c_{ij} = -b_{ij} - b_{it}a_s^t a_j^s = -b_{ij} - b^{sr}\Gamma_{si}\Gamma_{rj}$. So $c_{ij} = c_{ji}$. This completes our proof. $\square$

Proposition 9. *Let $b = b_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ be a symmetric and regular bilinear form on TM. Let Γ_b be the connection induced on T^*M by the Levi-Civita connection γ_b*

on TM established by the pseudo-Riemannian structure b. Let $B = b^{ij}\,\mathrm{d}z_j \otimes \partial/\partial x^i$ be the inverse of b. Then the almost complex structure $\alpha(\Gamma_b, B)$ is symmetric and $i_V\,\mathrm{d}(i_\alpha\varepsilon) = 0$.

P r o o f. The symmetry of $\alpha(\Gamma_b, B)$ is a consequence of Proposition 7 and the equality $i_V\,\mathrm{d}(i_\alpha\varepsilon) = 0$ follows, according to Proposition 5, from the well known fact that $Tb(\gamma_b) = \Gamma_b$, where b is interpreted as a map $b\colon TM \to T^*M$. □

It is easy to prove

Corollary. *Let $J = \mathrm{d}x^i \otimes \partial/\partial x^i_1$ be the almost tangent structure on TM (which can be identified with Id_{VTM}). Then the vector bundle isomorphism $b\colon TM \to T^*M$ is an almost complex map of the almost complex structures $\alpha(\gamma_b, J)$ and $\alpha(\Gamma_b, B)$ where we use the notation from Proposition 9.*

We turn to the second case when B vanishes.

II. Let α be such a (1,1)-tensor field that $B = T\pi \cdot \alpha|_{VT^*M} = 0$.
Now,

$$\begin{aligned} A &:= T\pi \cdot \alpha = a^i_j \mathrm{d}x^j \otimes \partial/\partial x^i \colon TT^*M \to TM,\\ H &:= \alpha|_{VT^*M} = h^j_i \mathrm{d}z_j \otimes \partial/\partial z_i \colon\ VT^*M \to VT^*M. \end{aligned}$$

So A and H can be interpreted as sections $A\colon T^*M \to TM^* \otimes_{T^*M} TM$, $H\colon T^*M \to TM \otimes_{T^*M} T^*M$. If α is a VB-(1,1)-tensor field on T^*M, i.e. $\alpha(X)$ is a linear and projectable vector field on T^*M for any projectable and linear vector field X on T^*M, then A and H are the π-pull-backs of the sections $\overline{A}\colon M \to TM^* \otimes TM$, $\overline{A} = a^i_j(x)\mathrm{d}x^j \otimes \partial/\partial x^i \colon TM \to TM$ and $\overline{H}\colon M \to TM \otimes T^*M$, $\overline{H} = h^j_i(x)\partial/\partial x^j \otimes \mathrm{d}x^i \colon T^*M \to T^*M$. It is easy to see that in the VB-case $c_{ij}(x,z) = c^k_{ij}(x) z_k$, see [1].

Let $\overline{A}^* \colon T^*M \to T^*M, \overline{z}_i = a^j_i z_j$, denote the transposed vector bundle morphism to a vector bundle morphism $\overline{A}\colon TM \to TM$ over Id_M, $\overline{A}^*(\varepsilon)(X) = \varepsilon(\overline{A}X)$. If a VB-(1,1)-tensor field on T^*M is symmetric or skew symmetric then $\overline{H} = -\overline{A}^*$ or $\overline{H} = \overline{A}^*$, respectively.

Let $\Gamma, \mathrm{d}z_i = \Gamma_{ij}(x,z)\mathrm{d}x^j$, be a connection on T^*M. It is $\mathrm{d}\varepsilon^\alpha$-zero, i.e. $\mathrm{d}\varepsilon^\alpha|_{H\Gamma} = 0$ if and only if

$$c_{ij} = \Gamma_{ti} a^t_j - \Gamma_{tj} h^t_i. \tag{16}$$

Proposition 10. *Let Γ be a symmetric connection on T^*M. Then Γ is $\mathrm{d}\varepsilon^\alpha$-zero if and only if $\alpha(H\Gamma) \subset H\Gamma$.*

P r o o f. The equalities (4) read

$$c_{ij} = \Gamma_{it} a^t_j - h^t_i \Gamma_{ij}. \tag{4'}$$

Comparing (16) with (4′) we get our assertion. □

Proposition 11. *Let α be such a vertical $(1,1)$-tensor field that A is regular. Let a connection Γ be $\mathrm{d}\varepsilon^\alpha$-zero and $\alpha(H\Gamma) \subset H\Gamma$. Then Γ is symmetric.*

P r o o f. It follows from (16) and (4) that $(\Gamma_{it} - \Gamma_{ti}) a^t_j = 0$, which completes our proof. □

In the case of a vertical almost complex structure α on T^*M the formulas (3) read

$$a^i_s a^s_j = -\delta^i_j, \quad c_{it} a^t_j + h^t_i c_{tj} = 0, \quad h^i_s h^s_j = -\delta^i_j. \tag{3'}$$

We suppose that A is the π-pull-back of a (1,1)-tensor $\overline{A}$ on M. Denote by ω the exterior derivative of the form $i_\alpha \varepsilon = z_k a^k_j(x) \mathrm{d}x^j$, i.e. $\omega := \mathrm{d} i_\alpha \varepsilon = a^k_j \mathrm{d}z_k \wedge \mathrm{d}x^j + z_k a^k_{ji} \mathrm{d}x^i \wedge \mathrm{d}x^j$. Then

$$\omega^\alpha = (a^t_j c_{ti} + z_k a^k_{jt} a^t_i - z_k a^k_{tj} a^t_i) \mathrm{d}x^i \otimes \mathrm{d}x^j + h^i_t a^t_j \mathrm{d}z_i \otimes \mathrm{d}x^j - a^t_i a^j_t \mathrm{d}x^i \otimes \mathrm{d}z_j.$$

Let $\Gamma,\ \mathrm{d}z_i = \Gamma_{ij} \mathrm{d}x^j$ be a connection on T^*M. Then

$$\omega^\alpha|_{H\Gamma} = (a^t_j c_{ti} + (a^k_{jt} - a^k_{tj}) z_k a^t_i + h^s_t a^t_j \Gamma_{si} - a^t_i a^s_t \Gamma_{sj}) \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

If α is symmetric, A is an almost complex structure on M and Γ is symmetric, then $\omega^\alpha|_{H\Gamma} = 0$ if and only if

$$a^t_j c_{ti} + (a^k_{jt} - a^k_{tj}) z_k a^t_i + 2\Gamma_{ij} = 0. \tag{17}$$

Let us recall the Nijenhuis-Frölicher bracket, see for example [7],

$$[A, A] = (a^k_{jt} - a^k_{tj}) a^t_i \mathrm{d}x^j \wedge \mathrm{d}x^i \otimes \partial/\partial x^k.$$

We conclude

Proposition 12. *Let α be such a symmetric almost complex structure that A is an integrable almost complex structure on M, i.e. $[A, A] = 0$. Then there is a unique symmetric connection Γ_1 on T^*M such that $\omega^\alpha|_{H\Gamma_1} = 0$. If α moreover is a VB-tensor field then Γ_1 is linear.*

Let $\overline{H}\colon T^*M \to T^*M$ be a vector bundle isomorphism and Γ a linear connection on T^*M, $\overline{H} = h_i^j(x)\partial/\partial x^i \otimes \mathrm{d}x_j$, $\mathrm{d}z_i = \Gamma_{ij}^k(x) z_k \mathrm{d}x^j$. Then

$$T\overline{H}\colon TT^*M \to TT^*M\,;\ \overline{x}^i = x^i,\ \overline{z}_i = h_i^j z_j,\ \mathrm{d}\overline{x}^i = \mathrm{d}x^i,\ \mathrm{d}\overline{z}_i = h_{ij}^t z_t \mathrm{d}x^j + h_i^j \mathrm{d}z_j,$$
$$T\overline{H}\cdot h_\Gamma = \mathrm{d}x^i \otimes \partial/\partial x^i + (h_{ij}^t z_t + h_i^t \Gamma_{tj}^k z_k)\mathrm{d}x^j \otimes \partial/\partial z_i.$$

Therefore $T\overline{H}(H\Gamma) \subset H\Gamma$ at any $(x^i, \overline{z}_i = h_i^t z_t)$ if and only if

$$\Gamma_{ij}^t h_t^k = h_{ij}^k + \Gamma_{tj}^k h_i^t. \tag{17'}$$

Let α be such a symmetric (1,1)-tensor field on T^*M that $A = T\pi\cdot\alpha$ is the π-pull-back of a regular (1,1)-tensor field $\overline{A}$ on M. Then $\overline{H} = -\overline{A}^*$. Let Γ be an arbitrary connection on T^*M. Denote $\beta := \overline{H}^*\mathrm{d}\varepsilon = \mathrm{d}\varepsilon T\overline{H} = h_{ij}^t z_t \mathrm{d}x^j \wedge \mathrm{d}x^i + h_i^j \mathrm{d}z_j \wedge \mathrm{d}x^i$. Then $\beta|_{H\Gamma} = (h_{ij}^t z_t + h_i^t \Gamma_{tj})\mathrm{d}x^j \wedge \mathrm{d}x^i$ and $\beta|_{H\Gamma} = 0$ if and only if

$$(h_{ij}^t - h_{ji}^t) z_t + h_i^t \Gamma_{tj} - h_j^t \Gamma_{ti} = 0. \tag{18}$$

The equalities (16) and (18), using $a_j^i = -h_j^i$, give

$$c_{ij} + (h_{ij}^t - h_{ji}^t) z_t + 2h_i^t \Gamma_{tj} = 0, \quad \text{i.e.} \quad \Gamma_{ij} = -\tfrac{1}{2}\tilde{h}_i^t (c_{tj} + (h_{tj}^k - h_{jt}^k) z_k). \tag{19}$$

These functions Γ_{ij} satisfy (16) and (18).

We conclude

Proposition 13. *Let α be such a symmetric $(1,1)$-tensor field on T^*M that $A = T\pi\alpha$ is the π-pull-back of a regular $(1,1)$-tensor field $\overline{A}$ on M. Then there exists a unique connection Γ_α such that $(\mathrm{d}\varepsilon TH)|_{H\Gamma_\alpha} = 0$ and $\mathrm{d}\varepsilon^\alpha|_{H\Gamma} = 0$. If moreover*

1. *α is a VB-tensor field then Γ_α is linear,*
2. *α is such a VB-almost complex structure that the almost complex structure $\overline{A}$ on M is integrable then $\Gamma_\alpha = \Gamma_1$.*

P r o o f. The first part of Proposition 13 is evident. The equality of the functions of the connections Γ_1, Γ_α follows from the equalities (17), (19), (3′) and $[\overline{A}, \overline{A}] = 0$. $\square$

Corollary. *If α is such a symmetric VB-almost complex structure that $[\overline{A}, \overline{A}] = 0$ then by Proposition 10, $\alpha(H\Gamma_\alpha) = H\Gamma_\alpha$.*

R e m a r k. The connections Γ_α, Γ_1 cannot be constructed when $\overline{H} = \overline{A}^*$, for example when α is skew symmetric. Let us recall that if α is the so-called complete lift of a (1,1)-tensor field F on M then it is skew symmetric, see [8]. In a more general case when α is the first order natural lift of F then $\overline{H} = \overline{A}^*$, see [3], and so the connections Γ_α, Γ_1 do not exist.

Propositon 14. *Let α be such a symmetric VB-almost complex structure that $[\overline{A},\overline{A}] = 0$. Then $T\overline{H}(H\Gamma_\alpha) \subset H\Gamma_\alpha$ if and only if the Nijenhuis tensor $[\alpha,\alpha]$ is semibasic with values in VT^*M.*

Proof. By virtue of $[\overline{A},\overline{A}] = 0$ we get $\frac{1}{2}[\alpha,\alpha] = (B^k_{ij}\mathrm{d}z_k \wedge \mathrm{d}x^j + D^i_{kj}\mathrm{d}x^k \wedge \mathrm{d}x^j) \otimes \partial/\partial z_i$, where $B^k_{ij} = c^u_{ij}h^k_u + h^u_i h^k_{uj} + h^k_{iu}h^u_j - h^u_i c^k_{uj}$. Using (3′) we obtain $B^k_{ij} = (c^u_{ij} - h^u_{ij})h^k_u + (h^k_{iu} - c^k_{iu})h^u_j$. The relation $T\overline{H}(H\Gamma_\alpha) \subset H\Gamma_\alpha$ is true iff the functions Γ^k_{ij} established by (19) satisfy the equality (17′). Putting Γ^k_{ij} in (17′) and using (3′) we get $(c^u_{ij} - h^u_{ij})h^k_u = h^u_j(c^k_{iu} - h^k_{iu})$. So $T\overline{H}(H\Gamma_\alpha) \subset H\Gamma_\alpha$ if and only if $B^k_{ij} = 0$, i.e. iff $[\alpha,\alpha]$ is semibasic with values in VT^*M. The proof is complete. □

Remark. It is easy to show that the condition $T\overline{H}(H\Gamma_\alpha) \subset H\Gamma_\alpha$ is equivalent to the one that $\nabla_\gamma A = 0$, i.e. the (1,1)-tensor field A on M is constant with respect to the covariant derivative established by the linear connection γ_α on TM which induces the connection Γ_α on T^*M.

References

[1] *Cabras, A., Kolář, I.*: Special tangent valued forms and the Frölicher-Nijenhuis bracket. Arch. Math. (Brno) *29* (1993), 71–82.

[2] *Dekrét, A.*: On almost complex structures on fibre bundles. Acta Univ. M. Belii Math. *3* (1995), 3–8.

[3] *Doupovec, M., Kurek, J.*: Liftings of tensor fields to the cotangent bundle. Proc. Conf. Diff. Geometry Brno 1995. Masaryk University, Brno, 1996, p. 141–150.

[4] *Janyška, J.*: Remarks on the Nijenhuis tensor and almost complex connections. Arch. Math. (Brno) *26* (1990), 229–240.

[5] *Klein, J.*: On variational second order differential equations Polynomial case. Proc. Conf. Diff. Geometry and its Applications 1992. Silesian University, Opava, 1993, p. 449–459.

[6] *Kolář, I., Michor, P.W., Slovák, J.*: Natural Operations in Differential Geometry. Springer-Verlag, Berlin, 1993.

[7] *Yano, K.*: Differential Geometry on Complex and Almost Complex Spaces. Pergamon Press, New York, 1965.

[8] *Yano, K., Ishihara, S.*: Tangent and Cotangent Bundles. M. Dekker Inc., New York, 1973.

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