PARTIALLY ORDERED SETS HAVING SELFDUAL SYSTEM OF INTERVALS

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Abstract. In the present paper we deal with the existence of large homogeneous partially ordered sets having the property described in the title.

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1. Introduction

This note is a continuation of [9] and [10].

Let P be a partially ordered set. We apply the same notation as in [10]. Namely, we denote by $\mathrm{Int}_0\, P$ the system of all intervals of P, including the empty set. Further, let $\mathrm{Int}\, P$ be the system $\mathrm{Int}_0\, P \setminus \{\emptyset\}$. These systems are partially ordered by the set-theoretical inclusion.

In the case when P is a lattice the system $\mathrm{Int}_0\, P$ was studied in [2]–[8], [11], [13].

The class of all partially ordered sets P such that $\mathrm{Int}_0\, P$ is selfdual will be denoted by $\mathscr{S}_0$. Let $\mathscr{S}$ have an analogous meaning with $\mathrm{Int}_0\, P$ replaced by $\mathrm{Int}\, P$.

Igoshin [8] proved the following result:

A finite lattice L belongs to $\mathscr{S}_0$ if and only if either (i) $\operatorname{card} L \leqslant 2$, or (ii) $\operatorname{card} L = 4$ and L has two atoms.

In [8] the question was proposed whether there exists an infinite lattice belonging to $\mathscr{S}_0$.

In [9] it was shown that the answer is "No" and that a partially ordered set belongs to $\mathscr{S}_0$ if and only if it is a lattice satisfying some of the conditions (i) or (ii) above.

Partially ordered sets belonging to $\mathscr{S}$ were investigated in [12] and [10].

From the above mentioned result of [9] it follows that the relation $\operatorname{card} P \leqslant 4$ is valid for each $P \in \mathscr{S}_0$. A natural question arises whether an analogous situation occurs for the class $\mathscr{S}$, i.e., whether there exists a cardinal k such that for each $P \in \mathscr{S}$ the relation $\operatorname{card} P \leqslant k$ holds.

A partially ordered set P will be said to be *homogeneous* if, whenever $x_i, y_i \in P$, $x_i < y_i$ $(i = 1, 2)$, then $\operatorname{card}\,[x_1, y_1] = \operatorname{card}\,[x_2, y_2]$. There exist partially ordered sets which belong to $\mathscr{S}$ and fail to be homogeneous (cf. [12]).

In the present note the following result will be proved:

(∗) Let α be an infinite cardinal. There exists a connected partially ordered set P_α such that (i) P_α belongs to $\mathscr{S}$; (ii) $\operatorname{card} P_\alpha = \alpha$; (iii) P_α is homogeneous.

2. Construction of P_α

We need some auxiliary results.

Let $\mathbb{Z}$ be the additive group of all integers with the natural linear order. Further, let α be an infinite cardinal and let $\omega(\alpha)$ be the first ordinal whose cardinality is α. Consider a linearly ordered set I which is dually isomorphic to $\omega(\alpha)$. Then each ideal of I is isomorphic to I.

Put $H_i = \mathbb{Z}$ for each $i \in I$ and let us have the lexicographic product

$$H = \Gamma_{i \in I} H_i$$

(cf., e.g., [1]). For $h \in H$ and $i \in I$ let h_i be the component of h in H_i. Denote

$$\operatorname{supp} h = \{i \in I \colon h_i \neq 0\}.$$

We set

$$G_\alpha = \{h \in H \colon \operatorname{supp} h \text{ is finite}\}.$$

Then we clearly have

$$\operatorname{card} G_\alpha = \alpha.$$

Let $0 < h \in G_\alpha$. There exists $i_0 \in I$ such that i_0 is the least element of $\operatorname{supp} h$. We denote by $G_\alpha^{i_0}$ the set of all $g \in G_\alpha$ such that $i < i_0$ for each $i \in \operatorname{supp} g$. Then $G_\alpha^{i_0}$ is a linearly ordered group isomorphic to G_α. This yields that $\operatorname{card} G_\alpha^{i_0} = \alpha$ and also $\operatorname{card}(G_\alpha^{i_0})^+ = \alpha$. The set $(G_\alpha^{i_0})^+$ is a subset of the interval $[0, h]$ of G_α. Hence

$$\operatorname{card}\,[0, h] = \alpha.$$

If $x, y \in G$, $x < y$, then the interval $[x, y]$ of G_α is isomorphic to $[0, y - x]$. Thus we have

2.1. Lemma. *Let α and G_α be as above, $x, y \in G_\alpha$, $x < y$. Then* $\operatorname{card}[x, y] = \alpha$.

Again, let α and G_α be as in 2.1. Choose $x \in G_\alpha$, $x > 0$. Put $A = B = G_\alpha$ and consider the direct product

$$C = A \times B.$$

The elements of C will be denoted as $t = (t_a, t_b)$ with $t_a \in A$, $t_b \in B$.

Let C_1 be the set of all $(t_a, t_b) \in C$ such that

$$(t_a, t_b) \geqslant (0, 0), \quad t_a + t_b \leqslant x.$$

Further, let C_2 be the set of all $(t_a, t_b) \in C$ such that

$$(t_a, t_b) \leqslant (x, x), \quad t_a + t_b \geqslant x.$$

Next, let $C_3 = C_1 \cup C_2$. Hence C_3 is the interval $[(0, 0), (x, x)]$ of C. (Cf. Fig. 1.)

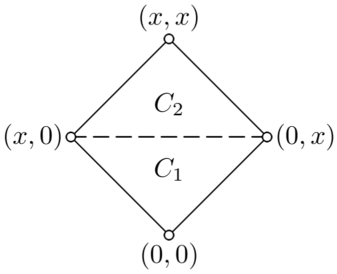

Fig. 1

2.2. Lemma. *Let $[u, v]$ be an interval of the partially ordered set C_3, $u < v$. Then* $\operatorname{card}[u, v] = \alpha$.

P r o o f. Put $u = (t_a, t_b)$, $v = (t'_a, t'_b)$. Then

$$[u, v] = [t_a, t'_a] \times [t_b, t'_b]$$

and either $t_a < t'_a$ or $t_b < t'_b$. Thus according to 2.1, $\operatorname{card}[u, v] = \alpha$. $\square$

Now suppose that we have replicas of C_1 which will be denoted by C_1^n, where n runs over the set of all integers. Similarly, let C_2^n be replicas of C_2. Hence for each $n \in \mathbb{Z}$ there exists an isomorphism φ^{n1} of C_1^n onto C_1; for $p \in C_1^n$ we denote

$$\varphi^{n1}(p) = (p_a^{n1}, p_b^{n1}).$$

Similarly, for $n \in \mathbb{Z}$ there is an isomorphism φ^{n2} of C_2^n onto C_2; for $q \in C_2^n$ we put

$$\varphi^{n2}(q) = (q_a^{n2}, q_b^{n2}).$$

All the elements p_a^{n1}, p_b^{n1}, q_a^{n2}, q_b^{n2} belong to the interval $[0, x]$ of G_α.

The following identifications will be adopted:

1) Let $p \in C_1^n$ and $q \in C_2^n$. The elements p and q will be identified if (under the notation as above) we have

$$p_a^{n1} = 0, \quad q_a^{n2} = x, \quad p_b^{n1} = q_b^{n2}.$$

2) Let p be as in 1 and $q \in C_2^{n-1}$. We identify the elements p and q if

$$p_b^{n1} = 0, \quad q_b^{(n-1)2} = x, \quad p_a^{n1} = q_b^{(n-1)2}.$$

(Cf. Fig. 2.)

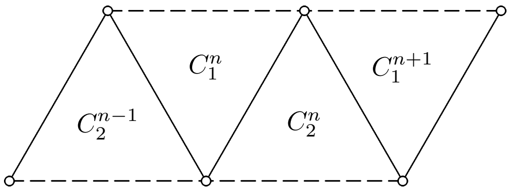

Fig. 2

Having in mind these identifications we put

$$P_\alpha = \bigcup_{n \in \mathbb{Z}} (C_1^n \cup C_2^n).$$

We define a binary relation $\leqslant$ on P_α as follows. Let $p, q \in P_\alpha$. We put $p \leqslant q$ if some of the following conditions is valid:

1) There exist $n \in \mathbb{Z}$ and $i \in \{1, 2\}$ such that both p, q belong to C_i^n and the relation $p \leqslant q$ holds in C_i^n.
2) There exists $n \in \mathbb{Z}$ such that $q \in C_1^n$ and (under the notation as above) either
   (i) $p \in C_2^{n-1}$ and $p_a^{(n-1)2} \leqslant q_a^{n1}$

   or

   (ii) $p \in C_2^n$ and $p_b^{n2} \leqslant q_b^{n1}$.

2.3. Lemma. *The relation $\leqslant$ is a partial order on the set P_α and under this partial order, P_α is connected.*

The proof is a routine, it will be omitted.

For each partially ordered set P we denote by $\operatorname{Max} P$ and $\operatorname{Min} P$ the set of all maximal elements of P or the set of all minimal elements of P, respectively.

For each integer n we have

$$\operatorname{Max} C_1^n = \{t \in C_1^n : t_a^{n1} + t_b^{n1} = x\},$$
$$\operatorname{Min} C_2^n = \{t \in C_2^n : t_a^{n2} + t_b^{n2} = x\}.$$

Further, we have

$$\operatorname{Max} P_\alpha = \bigcup_{n \in \mathbb{Z}} \operatorname{Max} C_1^n,$$
$$\operatorname{Min} P_\alpha = \bigcup_{n \in \mathbb{Z}} \operatorname{Min} C_2^n.$$

3. Proof of (*)

If Θ is an equivalence relation on a partially ordered set P and $a \in P$, then we put $[a]\Theta = \{b \in P \colon b\Theta a\}$. The symbol id denotes the least equivalence relation on P. Let $\mathscr{D}$ be the class of all discrete partially ordered sets, i.e., the class of all partially ordered sets P such that each bounded chain in P is finite.

We shall apply the following result (cf. [10]):

3.1. Theorem. *A partially ordered set P belongs to $\mathscr{S}$ if and only if there exist equivalence relations Θ_1 and Θ_2 on P such that*

(i) *for each $a \in P$ there are elements $u_1, u_2 \in \operatorname{Min} P$ and $v_1, v_2 \in \operatorname{Max} P$ such that $[a]\Theta_1 = [u_1, v_1]$ and $[a]\Theta_2 = [u_2, v_2]$;*
(ii) *$\Theta_1 \wedge \Theta_2 = \mathrm{id}$;*
(iii) *whenever a and b are elements of P with $a \leqslant b$, then there exist $z_1, z_2 \in [a, b]$ such that $a\Theta_1 z_1 \Theta_2 b$ and $a\Theta_2 z_2 \Theta_1 b$.*

In [12] this result was proved under the additional hypothesis that P belongs to $\mathscr{D}$.

Let P_α be as in Section 2. We define binary relations Θ_1 and Θ_2 on P_α as follows.

Let p and q be elements of P_α with $p \in C_i^n$, $q \in C_j^m$ $(m, n \in \mathbb{Z};\, i, j \in \{1, 2\})$; for p and q we apply the notation as in Section 2.

We put $p\Theta_1 q$ if

$$m = n \quad \text{and} \quad p_b^{ni} = q_b^{mj}.$$

Further, we put $p\Theta_2 q$ if $p_a^{ni} = q_a^{mj}$ and either

$$n = m, \quad i = j,$$

or

$$m = n + 1 \quad \text{and} \quad i \neq j.$$

From the definitions of Θ_1 and Θ_2 we immediately obtain

3.2. Lemma. *Θ_1 and Θ_2 are equivalence relations on P such that $\Theta_1 \wedge \Theta_2 = \mathrm{id}$.*

3.3. Lemma. *Θ_1 and Θ_2 satisfy the condition* (i) *from* 3.1.

Proof. Let $p \in P_\alpha$. First suppose that there is an integer n such that $p \in C_1^n$. Hence $\varphi^{n1}(p) = (p_a^{n1}, p_b^{n1})$. There exist $v_1, v_2 \in C_1^n$ such that

$$v_{1b}^{n1} = p_b^{n1}, \quad v_{1b}^{n1} + v_{1a}^{n1} = x, \tag{1}$$

$$v_{2b}^{n1} = p_a^{n1}, \quad v_{2a}^{n1} + v_{2b}^{n1} = x. \tag{2}$$

From the first relation in (1) we infer that $p\Theta_1 v_1$ is valid; the second relation in (1) yields that $v_1 \in \operatorname{Max} P_\alpha$ (cf. the formulas at the end of Section 2). Analogously, from (2) we obtain that $p\Theta_2 v_2$ and $v_2 \in \operatorname{Max} P_\alpha$.

Further, there exist elements $u_1 \in C_2^{n-1}$ and $u_2 \in C_2^n$ such that

$$u_{1a}^{(n-1)2} = p_a^{n1}, \quad u_{1b}^{(n-1)2} + u_{1a}^{(n-1)2} = x, \tag{1'}$$

$$u_{2b}^{n2} = p_b^{n1}, \quad u_{2a}^{n2} + u_{2b}^{n2} = x. \tag{2'}$$

Then $p\Theta_2 u_1$, $p\Theta_1 u_2$ and $u_1, u_2 \in \operatorname{Min} P_\alpha$.

The case when $p \in C_2^n$ for some $n \in \mathbb{Z}$ is analogous. □

3.4. Lemma. *Θ_1 and Θ_2 satisfy the condition* (iii) *from* 3.1.

Proof. Let $p, q \in P_\alpha$, $p \leqslant q$.

a) Suppose that $p \in C_1^n$ for some $n \in \mathbb{Z}$. Then q also belongs to C_1^n. Thus

$$p_a^{n1} \leqslant q_a^{n1}, \quad p_b^{n1} \leqslant q_b^{n1}.$$

There exist $z_1, z_2 \in C_1^n$ such that

$$\varphi^{n1}(z_1) = (q_a^{n1}, p_b^{n1}), \quad \varphi^{n1}(z_2) = (p_a^{n1}, q_b^{n2}).$$

Then $z_1, z_2 \in [p, q]$ and

$$p\Theta_1 z_1 \Theta_2 q, \quad p\Theta_2 z_2 \Theta_1 q.$$

b) Now suppose that $p \in C_2^n$ for some $n \in \mathbb{Z}$. Then we have three possibilities for the element q, namely

$$q \in C_2^n, \quad q \in C_1^n, \quad q \in C_1^{n+1}.$$

In the first case we proceed as in a). In the second case we have analogous relations as in a) with the distinction that in the components of p we write the index 2 instead of 1; the conclusion is the same as in a). The third case is similar to the second. $\square$

3.5. Lemma. *P_α belongs to $\mathscr{S}$.*

Proof. This is a consequence of 3.1–3.4. $\square$

Under the notation as in Section 2 we have

$$\operatorname{card} A = \operatorname{card} B = \alpha,$$

whence $\operatorname{card} C = \alpha$. Since $C_3 \subseteq C$, according to 2.3 we get $\operatorname{card} C_3 = \alpha$. Clearly $\operatorname{card} C_1 = \operatorname{card} C_2 = \operatorname{card} C_3$ and hence $\operatorname{card} C_i = \alpha$ $(i = 1, 2)$. Thus in view of the definition of P_α we obtain

$$\operatorname{card} P_\alpha = \alpha. \tag{3}$$

3.6. Lemma. *Let $p, q \in P_\alpha$, $p < q$. Then* $\operatorname{card}[p, q] = \alpha$.

Proof. In view of 3.4 there is $z_1 \in P_\alpha$ such that $p\Theta_1 z_1 \Theta_2 q$, $z_1 \in [p, q]$. We have either $p < z_1$ or $z_1 < q$. Suppose that $p < z_1$. In view of the definition of Θ_1, the interval $[p, z_1]$ of P_α is isomorphic to some interval of G_α. Hence $\operatorname{card}[p, z_1] = \alpha$. Then in view of (3) the relation $\operatorname{card}[p, q] = \alpha$ is valid. The case $z_1 < q$ is analogous. $\square$

Now, $(*)$ is a consequence of 3.5, (3) and 3.6. $\square$

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