CARDINAL AND ORDINAL ARITHMETICS OF n-ARY RELATIONAL SYSTEMS AND n-ARY ORDERED SETS

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(Received March 3, 1997)

Abstract. The aim of this paper is to define and study cardinal (direct) and ordinal operations of addition, multiplication, and exponentiation for n-ary relational systems. n-ary ordered sets are defined as special n-ary relational systems by means of properties that seem to suitably generalize reflexivity, antisymmetry, and transitivity from the case of $n = 2$ or 3. The class of n-ary ordered sets is then closed under the cardinal and ordinal operations.

Keywords: n-ary relational system, n-ary ordered set, cardinal sum, cardinal product, cardinal power, ordinal sum, ordinal product, ordinal power

MSC 1991: 04A05, 06A99, 08A02

In what follows n always denotes an integer not less than 2; $(n]$ denotes the set $\{1, \ldots, n\}$, P_n, O_n, E_n denotes the set of all, of all odd, of all even permutations of the set $(n]$, respectively. By an n-ary relational system we understand a set G together with a subset of G^n. More precisely, let G be a nonempty set. Then a subset $R \subseteq G^n$ is called an n-ary relation on G, and the ordered pair $\mathbb{G} = (G, R)$ is said to be an n-ary relational system. The set G is called the carrier of $\mathbb{G}$.

Birkhoff's arithmetic of ordered sets discussed in [1], [2] and [3] has been generalized by several authors—see e.g. [4], [5], [6], [7], [8]. Especially, in [5] V. Novák deals with direct (cardinal) operations of addition, multiplication, and exponentiation for n-ary relational structures. In [8], J. Šlapal generalizes Novák's results to relational systems of any (possibly infinite) arity. In this paper we return to finite arity, but introduce and study ordinal operations and n-ary ordered sets as well.

This research was supported by Grant Agency of the Czech Republic, grant no. 201/95/0468.

1. Definition. ([5], [7], [8]) Let $\mathbb{G} = (G, R)$, $\mathbb{H} = (H, S)$ be two n-ary relational systems. Let $\varphi\colon G \to H$ be a mapping such that, for any $(a_1, \ldots, a_n) \in G^n$, $(a_1, \ldots, a_n) \in R$ implies $(\varphi(a_1), \ldots, \varphi(a_n)) \in S$. Then φ is called a homomorphism of $\mathbb{G}$ into $\mathbb{H}$. The set of all homomorphisms of $\mathbb{G}$ into $\mathbb{H}$ is denoted by $\mathrm{Hom}(\mathbb{G}, \mathbb{H})$. A homomorphism φ of $\mathbb{G}$ into $\mathbb{H}$ is called strong if, for any $(a_1, \ldots, a_n) \in G^n$, $(a_1, \ldots, a_n) \in R$ is equivalent to $(\varphi(a_1), \ldots, \varphi(a_n)) \in S$. An injective strong homomorphism of $\mathbb{G}$ into $\mathbb{H}$ is called an embedding of $\mathbb{G}$ into $\mathbb{H}$. A bijective strong homomorphism of $\mathbb{G}$ onto $\mathbb{H}$ is called an isomorphism of $\mathbb{G}$ onto $\mathbb{H}$. We write $\mathbb{G} \sim \mathbb{H}$ and say that the n-ary relational systems $\mathbb{G}$ and $\mathbb{H}$ are isomorphic if there exists an isomorphism of $\mathbb{G}$ onto $\mathbb{H}$.

Clearly, the class of all n-ary relational systems together with homomorphisms as morphisms forms a category.

2. Definition. Let $\mathbb{G} = (G, R)$ be an n-ary relational system. The n-ary relation R (and $\mathbb{G}$ itself as well) is called:

(1) discrete if $(a_1, \ldots, a_n) \in R$ is equivalent to $a_1 = \ldots = a_n$;
(2) reflexive if $(a, a, \ldots, a) \in R$ for any $a \in G$;
(3) semisymmetric if $(a_1, \ldots, a_n) \in R$, $\varepsilon \in E_n$ imply $(a_{\varepsilon(1)}, \ldots, a_{\varepsilon(n)}) \in R$;
(4) antisymmetric if $(a_1, \ldots, a_n) \in R$, $(a_{\omega(1)}, \ldots, a_{\omega(n)}) \in R$, $\omega \in O_n$ imply $a_1 = \ldots = a_n$;
(5) transitive if $(a_1, \ldots, a_n) \in R$, $(a_1, \ldots, a_{n-2}, a_n, a_{n+1}) \in R$ imply $(a_1, \ldots, a_{n-1}, a_{n+1}) \in R$;
(6) complete if for any pairwise different elements $a_1, \ldots, a_n \in G$ there exists a $\pi \in P_n$ such that $(a_{\pi(1)}, \ldots, a_{\pi(n)}) \in R$;
(7) universal if $R = G^n$;
(8) diagonal if $(a_{1j}, \ldots, a_{nj}) \in R$ for any $j \in (n]$, $(a_{i1}, \ldots, a_{in}) \in R$ for any $i \in (n]$ imply $(a_{11}, \ldots, a_{nn}) \in R$.

3. Remark. Reflexivity is defined as in [7], transitivity, however, is weaker than n-transitivity in [7].

4. Definition. Let $\mathbb{G} = (G, R)$ be an n-ary relational system. The n-ary relation R is called an n-ary order if it is reflexive, semisymmetric, antisymmetric, and transitive; $\mathbb{G}$ itself is then called an n-ary ordered set.

5. Remark. A binary order is clearly an order. A ternary order is a cyclic order in the sense of [6] enriched by the set of all triads consisting of identical elements.

6. Definition. ([5], [8]) Let $\mathbb{A} = (A, R)$, $\mathbb{B} = (B, S)$ be n-ary relational systems such that $A \cap B = \emptyset$. The cardinal (direct) sum $\mathbb{A} + \mathbb{B}$ of $\mathbb{A}$ and $\mathbb{B}$ is the n-ary relational system $\mathbb{C} = (C, T)$ where $C = A \cup B$ and $T = R \cup S$.

7. Theorem. *The cardinal sum of discrete, reflexive, semisymmetric, antisymmetric, transitive, or diagonal n-ary relational systems has the same property.*

Proof. It follows directly from the definition. Let us show, for example, the antisymmetry. Under the notation of 6, assume that $(a_1, \ldots, a_n) \in T$, at least two of the elements $a_1, \ldots, a_n$ are different, and $\omega \in O_n$. We have either $(a_1, \ldots, a_n) \in R$, or $(a_1, \ldots, a_n) \in S$. In the first case $(a_{\omega(1)}, \ldots, a_{\omega(n)}) \notin T$, for R is antisymmetric and $a_1, \ldots, a_n \notin B$. Analogously in the second case. □

8. Corollary. *The cardinal sum of n-ary ordered sets is an n-ary ordered set.*

9. Definition. ([5], [8]) Let $\mathbb{A} = (A, R)$, $\mathbb{B} = (B, S)$ be n-ary relational systems. The cardinal (direct) product $\mathbb{A}\mathbb{B}$ of $\mathbb{A}$ and $\mathbb{B}$ is the n-ary relational system $\mathbb{C} = (C, T)$ where $C = A \times B$ and T is the n-ary relation defined as follows: For any $(a_1, \ldots, a_n) \in A^n$, $(b_1, \ldots, b_n) \in B^n$, $((a_1, b_1), \ldots, (a_n, b_n)) \in T \Leftrightarrow (a_1, \ldots, a_n) \in R$, $(b_1, \ldots, b_n) \in S$.

10. Theorem. *The cardinal product of discrete, reflexive, semisymmetric, antisymmetric, transitive, universal, or diagonal n-ary relational systems has the same property.*

Proof. We show the transitivity only, the other cases being similar. Under the notation of 9, assume that $((a_1, b_1), \ldots, (a_n, b_n)) \in T$, $((a_1, b_1), \ldots, (a_{n-2}, b_{n-2}), (a_n, b_n), (a_{n+1}, b_{n+1})) \in T$. Then $(a_1, \ldots, a_n) \in R$, $(a_1, \ldots, a_{n-2}, a_n, a_{n+1}) \in R$, $(b_1, \ldots, b_n) \in S$, $(b_1, \ldots, b_{n-2}, b_n, b_{n+1}) \in S$, so that $(a_1, \ldots, a_{n-1}, a_{n+1}) \in R$, $(b_1, \ldots, b_{n-1}, b_{n+1}) \in S$. This implies that $((a_1, b_1), \ldots, (a_{n-1}, b_{n-1}), (a_{n+1}, b_{n+1}))$ belongs to T. □

11. Corollary. *The cardinal product of n-ary ordered sets is an n-ary ordered set as well.*

12. Definition. ([5], [8]) Let $\mathbb{A} = (A, R)$, $\mathbb{B} = (B, S)$ be n-ary relational systems. The cardinal (direct) power $\mathbb{A}^{\mathbb{B}}$ with the base $\mathbb{A}$ and exponent $\mathbb{B}$ is the n-ary relational system $\mathbb{C} = (C, T)$ where $C = \mathrm{Hom}(\mathbb{B}, \mathbb{A})$ and T is the n-ary relation defined as follows: For any $f_1, \ldots, f_n \in C$,

$$(f_1, \ldots, f_n) \in T \Leftrightarrow (f_1(x), \ldots, f_n(x)) \in R \quad \text{for each} \quad x \in B.$$

13. Theorem. *The cardinal power whose base is discrete, reflexive, semisymmetric, antisymmetric, transitive, universal, or diagonal has the same property.*

Proof. As before, we show one of the assertions only, namely the semisymmetry, under the notation of 12. Let $(f_1, \ldots, f_n) \in T$ and let $\varepsilon \in E_n$. We have

$(f_1(x),\dots,f_n(x)) \in R$ for each $x \in B$, so that $(f_{\varepsilon(1)}(x),\dots,f_{\varepsilon(n)}(x)) \in R$ for each $x \in B$, and $(f_{\varepsilon(1)},\dots,f_{\varepsilon(n)}) \in T$. $\square$

14. Corollary. *The cardinal power whose base is an n-ary ordered set is an n-ary ordered set as well.*

15. Definition. Let $\mathbb{A}_j = (A_j, R_j)$ be n-ary relational systems for each $j \in (n]$ such that $A_{j_1} \cap A_{j_2} = \emptyset$ whenever $j_1, j_2 \in (n]$, $j_1 \neq j_2$. The ordinal sum $\mathbb{A}_1 \oplus \dots \oplus \mathbb{A}_n$ of $\mathbb{A}_1,\dots,\mathbb{A}_n$ is the n-ary relational system $\mathbb{B} = (B, S)$ where $B = \bigcup\limits_{j=1}^{n} A_j$ and $S = \bigcup\limits_{j=1}^{n} R_j \cup \bigcup\limits_{\varepsilon \in E_n} A_{\varepsilon(1)} \times \dots \times A_{\varepsilon(n)}$.

16. Theorem. *The ordinal sum of reflexive, semisymmetric, antisymmetric, or transitive n-ary relational systems has the same property.*

P r o o f. We limit ourselves to the case of transitivity under the notation of 15. Let $(a_1,\dots,a_n) \in S$ and $(a_1,\dots,a_{n-2},a_n,a_{n+1}) \in S$.
(i) Let there exist a j_0 such that $a_1,\dots,a_n \in A_{j_0}$. If $n = 2$, then either $a_3 \in A_{j_0}$ and then $(a_1,a_3) \in R_{j_0} \subseteq S$, or a_1, $a_2 \in A_1$, $a_3 \in A_2$, for the identity is the only even permutation of $(2]$, and then $(a_1,a_3) \in A_1 \times A_2 \subseteq S$. Let $n \geqslant 3$. Then $a_{n+1} \in A_{j_0}$, for otherwise $(a_1,\dots,a_{n-2},a_n,a_{n+1}) \notin S$, and $(a_1,\dots,a_n) \in R_{j_0}$, $(a_1,\dots,a_{n-2},a_n,a_{n+1}) \in R_{j_0}$, so that $(a_1,\dots,a_{n-1},a_{n+1}) \in R_{j_0} \subseteq S$.
(ii) Let there exist no j_0 such that $a_1,\dots,a_n \in A_{j_0}$. If $n = 2$, then $a_1 \in A_1$, $a_2 \in A_2$, hence $a_3 \in A_2$ and $(a_1,a_3) \in A_1 \times A_2 \subseteq S$. Admit that $n \geqslant 3$. We have $(a_1,\dots,a_n) \in \bigcup\limits_{\varepsilon \in E_n} A_{\varepsilon(1)} \times \dots \times A_{\varepsilon(n)}$, therefore $a_j \in A_{\varepsilon(j)}$ for each $j \in (n]$ where $\varepsilon \in E_n$. There exists no j_0 such that $a_1,\dots,a_{n-2},a_n,a_{n+1} \in A_{j_0}$, so that $(a_1,\dots,a_{n-2},a_n,a_{n+1}) \in A_{\xi(1)} \times \dots \times A_{\xi(n)}$ for some $\xi \in E_n$. Thus $a_i \in A_{\xi(i)}$ for each $i \in (n-2]$, $a_n \in A_{\xi(n-1)}$, $a_{n+1} \in A_{\xi(n)}$. This implies that $\varepsilon(i) = \xi(i)$ for each $i \in (n-2]$, $\varepsilon(n) = \xi(n-1)$, $\varepsilon(n-1) = \xi(n)$, which is a contradiction, for both ε and ξ are even. $\square$

17. R e m a r k. In contrast to the binary case, the ordinal sum of complete n-ary relational systems need not be complete if $n \geqslant 3$. Let, for example, $A_1 = \{a,b,c\}$, $A_2 = \{d,e,f\}$, $A_3 = \{g,h,i\}$, $R_1 = \{(a,b,c)\}$, $R_2 = \{(d,e,f)\}$, $R_3 = \{(g,h,i)\}$, $\mathbb{A}_j = (A_j, R_j)$ for $j = 1,2,3$. Then each of the ternary relational systems $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ is complete, but their ordinal sum $\mathbb{A}_1 \oplus \mathbb{A}_2 \oplus \mathbb{A}_3$ is not, because $S = R_1 \cup R_2 \cup R_3 \cup A_1 \times A_2 \times A_3 \cup A_2 \times A_3 \times A_1 \cup A_3 \times A_1 \times A_2$ and no permutation of the ordered triad (a,b,d) belongs to S.

18. Corollary. *The ordinal sum of n-ary ordered sets is an n-ary ordered set.*

19. Definition. Let $\mathbb{A} = (A, R)$, $\mathbb{B} = (B, S)$ be n-ary relational systems. The ordinal product $\mathbb{A} \circ \mathbb{B}$ of $\mathbb{A}$ and $\mathbb{B}$ is the n-ary relational system $\mathbb{C} = (C, T)$ where $C = A \times B$ and T is the n-ary relation defined as follows: For any $(a_1, \ldots, a_n) \in A^n$ and $(b_1, \ldots, b_n) \in B^n$, $((a_1, b_1), \ldots, (a_n, b_n)) \in T \Leftrightarrow$ at least two of the elements $a_1, \ldots, a_n$ are different, $(a_1, \ldots, a_n) \in R$, or $a_1 = \ldots = a_n$, $(b_1, \ldots, b_n) \in S$.

20. Theorem. *The ordinal product of discrete, reflexive, semisymmetric, antisymmetric, transitive and antisymmetric, or universal n-ary relational systems has the same property (properties).*

Proof. We show the fourth and fifth cases only. Assume the notation of 19.
(i) Antisymmetry: Let $((a_1, b_1), \ldots, (a_n, b_n)) \in T$, let at least two of the elements $(a_1, b_1), \ldots, (a_n, b_n)$ be different, and let $\omega \in O_n$. If at least two of the elements $a_1, \ldots, a_n$ are different, then $(a_1, \ldots, a_n) \in R$, consequently $(a_{\omega(1)}, \ldots, a_{\omega(n)}) \notin R$ and at least two of the elements $a_{\omega(1)}, \ldots, a_{\omega(n)}$ are different, thus $((a_{\omega(1)}, b_{\omega(1)}), \ldots, (a_{\omega(n)}, b_{\omega(n)})) \notin T$. Let $a_1 = \ldots = a_n$. Then $(b_1, \ldots, b_n) \in S$ and at least two of the elements $b_1, \ldots, b_n$ are different, hence again $((a_{\omega(1)}, b_{\omega(1)}), \ldots, (a_{\omega(n)}, b_{\omega(n)})) \notin T$.
(ii) Transitivity and antisymmetry: Antisymmetry follows from (i). Let $((a_1, b_1), \ldots, (a_n, b_n)) \in T$, $((a_1, b_1), \ldots, (a_{n-2}, b_{n-2}), (a_n, b_n), (a_{n+1}, b_{n+1})) \in T$. First of all, let at least two of the elements $a_1, \ldots, a_n$ be different, and let at least two of the elements $a_1, \ldots, a_{n-2}, a_n, a_{n+1}$ be different. Then $(a_1, \ldots, a_n) \in R$, $(a_1, \ldots, a_{n-2}, a_n, a_{n+1}) \in R$, consequently $(a_1, \ldots, a_{n-1}, a_{n+1}) \in R$. If at least two of the elements $a_1, \ldots, a_{n-1}, a_{n+1}$ are different, then $((a_1, b_1), \ldots, (a_{n-1}, b_{n-1}), (a_{n+1}, b_{n+1})) \in T$. Admit that $a_1 = \ldots = a_{n-1} = a_{n+1}$. Then $(a_1, \ldots, a_1, a_n) \in R$, $(a_1, \ldots, a_1, a_n, a_1) \in R$, $a_1 \neq a_n$, which is a contradiction to the antisymmetry of R. Let $a_1 = \ldots = a_n = a_{n+1}$. Then $(b_1, \ldots, b_n) \in S$, $(b_1, \ldots, b_{n-2}, b_n, b_{n+1}) \in S$, so that $(b_1, \ldots, b_{n-1}, b_{n+1}) \in S$ and $((a_1, b_1), \ldots, (a_{n-1}, b_{n-1}), (a_{n+1}, b_{n+1})) \in T$. Let $a_1 = \ldots = a_n \neq a_{n+1}$. Then $(a_1, \ldots, a_1, a_{n+1}) \in R$, $a_1 \neq a_{n+1}$, so that $((a_1, b_1), \ldots, (a_{n-1}, b_{n-1}), (a_{n+1}, b_{n+1})) \in T$. Finally, let $a_1 = \ldots = a_{n-2} = a_n = a_{n+1} \neq a_{n-1}$. Then $(a_1, \ldots, a_1, a_{n-1}, a_1) \in R$, $a_1 \neq a_{n-1}$, so that $((a_1, b_1), \ldots, (a_{n-1}, b_{n-1}), (a_{n+1}, b_{n+1})) \in T$. □

21. Remark. (i) For reflexivity of T, it is sufficient to assume reflexivity of S.

(ii) For transitivity of T, it is sufficient to assume transitivity of R and S and antisymmetry of R.
(iii) In contrast to the binary case, the ordinal product of complete n-ary relational systems need not be complete if $n \geqslant 3$. Let, for example, $A = \{a, b, c\}$, $B = \{d, e, f\}$, $R = \{(a, b, c)\}$, $S = \{(d, e, f)\}$, $\mathbb{A} = (A, R)$, $\mathbb{B} = (B, S)$. Then both $\mathbb{A}$ and $\mathbb{B}$ are

complete, but their ordinal product $\mathbb{A} \circ \mathbb{B}$ is not, because no permutation of the ordered triad $((a,d),(a,e),(b,f))$ belongs to T.

22. Corollary. *The ordinal product of n-ary ordered sets is an n-ary ordered set.*

23. Definition. Let $\mathbb{A} = (A, R)$ be a binary relational system, $\mathbb{B} = (B, S)$ an n-ary relational system. The ordinal power ${}^{\mathbb{A}}\mathbb{B}$ with the base $\mathbb{B}$ and exponent $\mathbb{A}$ is the n-ary relational system $\mathbb{C} = (C, T)$ where C is the set B^A of all mappings of A into B and T is the n-ary relation defined as follows: For any $f_1, \ldots, f_n \in C$, $(f_1, \ldots, f_n) \in T \Leftrightarrow$ for any $a \in A$, $(f_1(a), \ldots, f_n(a)) \notin S$ there exists a $b \in A$ such that $(b, a) \in R$, $(f_1(b), \ldots, f_n(b)) \in S$, at least two of the elements $f_1(b), \ldots, f_n(b)$ are different, and $(f_1(x), \ldots, f_n(x)) \in S$ for each $x \in A$, $(x, b) \in R$.

24. Theorem. *The ordinal power whose base is a discrete, reflexive, semisymmetric, antisymmetric, or universal n-ary relational system has the same property. The ordinal power whose base is transitive and antisymmetric and whose exponent is transitive is transitive as well.*

P r o o f. Let us show antisymmetry and transitivity only. Assume the notation of 23.

(i) Antisymmetry: Let $(f_1, \ldots, f_n) \in T$, $(f_{\omega(1)}, \ldots, f_{\omega(n)}) \in T$ for some $\omega \in O_n$. Admit that there exists an $a \in A$ for which $(f_1(a), \ldots, f_n(a)) \notin S$. Then there exists a $b \in A$ such that $(b, a) \in R$, $(f_1(b), \ldots, f_n(b)) \in S$, at least two of the elements $f_1(b), \ldots, f_n(b)$ are different. Consequently $(f_{\omega(1)}(b), \ldots, f_{\omega(n)}(b)) \notin S$. There exists again a $c \in A$ such that $(c, b) \in R$, $(f_{\omega(1)}(c), \ldots, f_{\omega(n)}(c)) \in S$, at least two of the elements $f_{\omega(1)}(c), \ldots, f_{\omega(n)}(c)$ are different. But $\omega^{-1} \in O_n$ as well, so that $(f_1(c), \ldots, f_n(c)) \notin S$. For each $x \in A$ such that $(x, b) \in R$ we have, however, $(f_1(x), \ldots, f_n(x)) \in S$, which is a contradiction. Hence there is no $a \in A$ such that $(f_1(a), \ldots, f_n(a)) \notin S$. Consequently $(f_1(a), \ldots, f_n(a)) \in S$ for each $a \in A$. Similarly $(f_{\omega(1)}(a), \ldots, f_{\omega(n)}(a)) \in S$ for each $a \in A$. As S is antisymmetric, we have $f_1(a) = \ldots = f_n(a)$ for each $a \in A$, thus $f_1 = \ldots = f_n$ and T is antisymmetric.

(ii) Transitivity: Let $(f_1, \ldots, f_n) \in T$, $(f_1, \ldots, f_{n-2}, f_n, f_{n+1}) \in T$, and let $a \in A$ be such that $(f_1(a), \ldots, f_{n-1}(a), f_{n+1}(a)) \notin S$. Then either $(f_1(a), \ldots, f_n(a)) \notin S$ or $(f_1(a), \ldots, f_{n-2}(a), f_n(a), f_{n+1}(a)) \notin S$ because of transitivity of S. Let the first case occur. As $(f_1, \ldots, f_n) \in T$, there exists a $b \in A$ such that $(b, a) \in R$, $(f_1(b), \ldots, f_n(b)) \in S$, at least two of the elements $f_1(b), \ldots, f_n(b)$ are different, and for each $x \in A$, $(x, b) \in R$ we have $(f_1(x), \ldots, f_n(x)) \in S$. Now, four possibilities (a), (b), (c), (d) can occur:

(a) $(f_1(b), \ldots, f_{n-2}(b), f_n(b), f_{n+1}(b)) \in S$, at least two of the elements $f_1(b), \ldots, f_{n-2}(b), f_n(b), f_{n+1}(b)$ are different, and $(f_1(x), \ldots, f_{n-2}(x), f_n(x), f_{n+1}(x)) \in S$ for

each $x \in A$, $(x,b) \in R$. Considering transitivity of S, we have $(f_1(b), \dots, f_{n-1}(b), f_{n+1}(b)) \in S$, and $(f_1(x), \dots, f_{n-1}(x), f_{n+1}(x)) \in S$ for each $x \in A$, $(x,b) \in R$. If at least two of the elements $f_1(b), \dots, f_{n-1}(b)$, $f_{n+1}(b)$ are different, we have $(f_1, \dots, f_{n-1}, f_{n+1}) \in T$. Admit that $f_1(b) = \dots = f_{n-1}(b) = f_{n+1}(b)$. Then $f_n(b) \neq f_j(b)$ for each $j = 1, \dots, n-1$, $n+1$. Hence $(f_1(b), \dots, f_n(b)) = (f_1(b), \dots, f_1(b), f_n(b)) \in S$, $(f_1(b), \dots, f_{n-2}(b), f_n(b), f_{n+1}(b)) = (f_1(b), \dots, f_1(b), f_n(b), f_1(b)) \in S$, a contradiction to antisymmetry of S.
(b) $f_1(b) = \dots = f_{n-2}(b) = f_n(b) = f_{n+1}(b)$, $(f_1(x), \dots, f_{n-2}(x), f_n(x), f_{n+1}(x)) \in S$ for each $x \in A$, $(x,b) \in R$. Then $(f_1(b), \dots, f_{n-1}(b), f_{n+1}(b)) = (f_1(b), \dots, f_n(b)) \in S$, at least two of the elements $f_1(b), \dots, f_{n-1}(b)$, $f_{n+1}(b)$ are different, $(f_1(x), \dots, f_{n-1}(x), f_{n+1}(x)) \in S$ for each $x \in A$, $(x,b) \in R$ due to transitivity of S, so that $(f_1, \dots, f_{n-1}, f_{n+1}) \in T$.
(c) $(f_1(b), \dots, f_{n-2}(b), f_n(b), f_{n+1}(b)) \notin S$. As $(f_1, \dots, f_{n-2}, f_n, f_{n+1}) \in T$, there exists a $c \in A$, $(c,b) \in R$ such that $(f_1(c), \dots, f_{n-2}(c), f_n(c), f_{n+1}(c)) \in S$, at least two of the elements $f_1(c), \dots, f_{n-2}(c)$, $f_n(c)$, $f_{n+1}(c)$ are different, and $(f_1(x), \dots, f_{n-2}(x), f_n(x), f_{n+1}(x)) \in S$ for each $x \in A$, $(x,c) \in R$. Considering transitivity of R we have $(c,a) \in R$. By the preceding, we have $(f_1(c), \dots, f_n(c)) \in S$ (for $(c,b) \in R$) and $(f_1(x), \dots, f_n(x)) \in S$ for each $x \in A$, $(x,c) \in R$ (by transitivity of R), consequently, regarding transitivity of S, $(f_1(c), \dots, f_{n-1}(c), f_{n+1}(c)) \in S$ and $(f_1(x), \dots, f_{n-1}(x), f_{n+1}(x)) \in S$ for each $x \in A$, $(x,c) \in R$. If at least two of the elements $f_1(c), \dots, f_{n-1}(c)$, $f_{n+1}(c)$ are different, we are done. Admit that $f_1(c) = \dots = f_{n-1}(c) = f_{n+1}(c)$. Then $(f_1(c), \dots, f_n(c)) = (f_1(c), \dots, f_1(c), f_n(c)) \in S$, $(f_1(c), \dots, f_{n-2}(c), f_n(c), f_{n+1}(c)) = (f_1(c), \dots, f_1(c), f_n(c), f_1(c)) \in S$, $f_1(c) \neq f_n(c)$, which is a contradiction to antisymmetry of S.
(d) $(f_1(b), \dots, f_{n-2}(b), f_n(b), f_{n+1}(b)) \in S$, there exists an $x \in A$, $(x,b) \in R$ such that $(f_1(x), \dots, f_{n-2}(x), f_n(x), f_{n+1}(x)) \notin S$. As $(f_1, \dots, f_{n-2}, f_n, f_{n+1}) \in T$, there exists a $c \in A$, $(c,x) \in R$ such that $(f_1(c), \dots, f_{n-2}(c), f_n(c), f_{n+1}(c)) \in S$, at least two of the elements $f_1(c), \dots, f_{n-2}(c)$, $f_n(c)$, $f_{n+1}(c)$ are different, and $(f_1(y), \dots, f_{n-2}(y), f_n(y), f_{n+1}(y)) \in S$ for each $y \in A$, $(y,c) \in R$. Considering transitivity of R we have $(c,a) \in R$. By the preceding, we have $(f_1(c), \dots, f_n(c)) \in S$ (for $(c,b) \in R$) and $(f_1(y), \dots, f_n(y)) \in S$ for each $y \in A$, $(y,c) \in R$. Regarding transitivity of S, we obtain $(f_1(c), \dots, f_{n-1}(c), f_{n+1}(c)) \in S$ and $(f_1(y), \dots, f_{n-1}(y), f_{n+1}(y)) \in S$ for each $y \in A$, $(y,c) \in R$. If at least two of the elements $f_1(c), \dots, f_{n-1}(c)$, $f_{n+1}(c)$ are different, we are done. The case of $f_1(c) = \dots = f_{n-1}(c) = f_{n+1}(c)$, however, leads to a contradiction to antisymmetry of S similarly as in (c). The case of $(f_1(a), \dots, f_{n-2}(a), f_n(a), f_{n+1}(a)) \notin S$ is analogous. $\square$

25. Corollary. *The ordinal power whose base is an n-ary ordered set and whose exponent is a transitive binary relational system is an n-ary ordered set.*

26. Definition. Let $\mathbb{A} = (A, R)$ be an n-ary relational system. The dual n-ary relational system $\widetilde{\mathbb{A}}$ to $\mathbb{A}$ is the n-ary relational system $\mathbb{B} = (A, S)$ where S is the n-ary relation defined as follows: For any $(a_1, \ldots, a_n) \in A^n$,

$$(a_1, \ldots, a_n) \in S \Leftrightarrow (a_n, \ldots, a_1) \in R.$$

27. Theorem. *The dual n-ary relational system to a discrete, reflexive, semisymmetric, antisymmetric, semisymmetric and transitive, complete, universal, or diagonal n-ary relational system has the same property (properties).*

P r o o f. For brevity, we show the third and the fifth property only. Assume the notation of 26.
(i) Semisymmetry: Let $(a_1, \ldots, a_n) \in S$, $\varepsilon \in E_n$. Then $(a_n, \ldots, a_1) \in R$, thus $(a_{n+1-\varepsilon(1)}, \ldots, a_{n+1-\varepsilon(n)}) \in R$, so that $(a_{\varepsilon(1)}, \ldots, a_{\varepsilon(n)}) \in S$.
(ii) Semisymmetry and transitivity: Semisymmetry follows from (i). Let $(a_1, \ldots, a_n) \in S$, $(a_1, \ldots, a_{n-2}, a_n, a_{n+1}) \in S$. Then $(a_n, \ldots, a_1) \in R$, $(a_{n+1}, a_n, a_{n-2}, \ldots, a_1) \in R$. The case of $n = 2$ is evident. Let $n \geqslant 3$. Clearly $(3, 4, \ldots, n, 1, 2) \in E_n$. Considering semisymmetry of R, we have $(a_{n-2}, \ldots, a_1, a_n, a_{n-1}) \in R$, $(a_{n-2}, \ldots, a_1, a_{n+1}, a_n) \in R$. Due to transitivity of R, it follows that $(a_{n-2}, \ldots, a_1, a_{n+1}, a_{n-1}) \in R$, so that, by semisymmetry of R, $(a_{n+1}, a_{n-1}, \ldots, a_1) \in R$. Hence $(a_1, \ldots, a_{n-1}, a_{n+1}) \in S$. □

28. Corollary. *The dual to a (complete) n-ary ordered set is a (complete) n-ary ordered set.*

29. Theorem. *For any n-ary relational systems $\mathbb{A}, \mathbb{B}$ with disjoint carriers and for any n-ary relational system $\mathbb{C}$ we have*

$$\begin{aligned} &\mathbb{A} + \mathbb{B} = \mathbb{B} + \mathbb{A}, \\ &(\mathbb{A} + \mathbb{B})\mathbb{C} = \mathbb{A}\mathbb{C} + \mathbb{B}\mathbb{C}, \\ &\mathbb{C}(\mathbb{A} + \mathbb{B}) = \mathbb{C}\mathbb{A} + \mathbb{C}\mathbb{B}. \end{aligned}$$

P r o o f. The first equality is trivial. The second and third are special cases of Theorem 4 in [8]. □

30. Theorem. *For any n-ary relational systems $\mathbb{A}, \mathbb{B}, \mathbb{C}$ with pairwise disjoint carriers we have*

$$\mathbb{A} + (\mathbb{B} + \mathbb{C}) = (\mathbb{A} + \mathbb{B}) + \mathbb{C}.$$

P r o o f. It follows from [8], Theorem 1. □

31. Theorem. *For any n-ary relational systems $\mathbb{A}, \mathbb{B}, \mathbb{C}$ we have*

$$\begin{aligned} &\mathbb{A}\mathbb{B} \sim \mathbb{B}\mathbb{A}, \\ &\mathbb{A}(\mathbb{B}\mathbb{C}) \sim (\mathbb{A}\mathbb{B})\mathbb{C}. \end{aligned}$$

Proof. In the first case, under the notation of 9, it suffices to take the natural bijection $\varphi\colon A \times B \to B \times A$ defined by $\varphi((a,b)) = (b,a)$ for any $a \in A$, $b \in B$; φ is clearly an isomorphism of $\mathbb{A}\mathbb{B}$ onto $\mathbb{B}\mathbb{A}$. The second case follows from [8], Theorem 2. $\square$

32. Theorem. *For any n-ary relational systems $\mathbb{B}, \mathbb{C}$ with disjoint carriers and for any n-ary relational system $\mathbb{A}$ we have*

$$\mathbb{A}^{\mathbb{B}+\mathbb{C}} \sim \mathbb{A}^{\mathbb{B}}\,\mathbb{A}^{\mathbb{C}}.$$

Proof. It follows from [8], Theorem 8. $\square$

33. Theorem. *For any n-ary relational systems $\mathbb{A}, \mathbb{B}, \mathbb{C}$ we have*

$$(\mathbb{A}\mathbb{B})^{\mathbb{C}} \sim \mathbb{A}^{\mathbb{C}}\,\mathbb{B}^{\mathbb{C}}.$$

Proof. It follows from [8], Theorem 7. $\square$

34. Theorem. *Let $\mathbb{A}, \mathbb{B}, \mathbb{C}$ be n-ary relational systems such that $\mathbb{A}$ is universal or $\mathbb{A}$ is reflexive and both $\mathbb{B}$ and $\mathbb{C}$ are discrete. Then*

$$\mathbb{A}^{\mathbb{B}\mathbb{C}} \sim (\mathbb{A}^{\mathbb{B}})^{\mathbb{C}}.$$

Proof. It follows from [8], Corollary of Theorem 5. $\square$

35. Theorem. *Let $\mathbb{A}, \mathbb{B}, \mathbb{C}$ be n-ary relational systems such that $\mathbb{B}$ and $\mathbb{C}$ are reflexive. Then there exists an embedding of $\mathbb{A}^{\mathbb{B}\mathbb{C}}$ into $(\mathbb{A}^{\mathbb{B}})^{\mathbb{C}}$. If, moreover, $\mathbb{A}$ is diagonal, then*

$$\mathbb{A}^{\mathbb{B}\mathbb{C}} \sim (\mathbb{A}^{\mathbb{B}})^{\mathbb{C}}.$$

Proof. It follows from [8], Theorem 9. $\square$

36. Corollary. *Let $\mathbb{A}, \mathbb{B}, \mathbb{C}$ be n-ary ordered sets such that $\mathbb{A}$ is diagonal or both $\mathbb{B}$ and $\mathbb{C}$ are discrete. Then*

$$\mathbb{A}^{\mathbb{B}\mathbb{C}} \sim (\mathbb{A}^{\mathbb{B}})^{\mathbb{C}}.$$

37. Theorem. *For any n-ary relational systems $\mathbb{A}, \mathbb{B}$ with disjoint carriers we have*

$$\widetilde{\mathbb{A}+\mathbb{B}} = \widetilde{\mathbb{A}} + \widetilde{\mathbb{B}}.$$

Proof. It follows directly from the definitions. □

38. Theorem. *For any n-ary relational systems $\mathbb{A}, \mathbb{B}$ we have*

$$\begin{aligned} \widetilde{\mathbb{A}\mathbb{B}} &= \widetilde{\mathbb{A}}\widetilde{\mathbb{B}}, \\ \widetilde{\mathbb{A}^{\mathbb{B}}} &= \widetilde{\mathbb{A}}^{\widetilde{\mathbb{B}}}. \end{aligned}$$

Proof. Let us show the second equality only. Let $\mathbb{A} = (A, R)$, $\mathbb{B} = (B, S)$. It is easy to prove that $\mathrm{Hom}(\mathbb{B}, \mathbb{A}) = \mathrm{Hom}(\widetilde{\mathbb{B}}, \widetilde{\mathbb{A}})$. Hence both sides of the equality have the same carrier $\mathrm{Hom}(\mathbb{B}, \mathbb{A})$. Denote $\widetilde{\mathbb{A}} = (A, T)$, $\mathbb{A}^{\mathbb{B}} = (\mathrm{Hom}(\mathbb{B}, \mathbb{A}), U)$, $\widetilde{\mathbb{A}^{\mathbb{B}}} = (\mathrm{Hom}(\mathbb{B}, \mathbb{A}), V)$, $\widetilde{\mathbb{A}}^{\widetilde{\mathbb{B}}} = (\mathrm{Hom}(\mathbb{B}, \mathbb{A}), W)$. Let $(f_1, \dots, f_n) \in V$. Then $(f_n, \dots, f_1) \in U$, thus $(f_n(x), \dots, f_1(x)) \in R$ for each $x \in B$, consequently $(f_1(x), \dots, f_n(x)) \in T$ for each $x \in B$ and $(f_1, \dots, f_n) \in W$, so that $V \subseteq W$. Similarly $W \subseteq V$. □

39. Theorem. *For any n-ary relational systems $\mathbb{A}, \mathbb{B}, \mathbb{C}$ we have*

$$\mathbb{A} \circ (\mathbb{B} \circ \mathbb{C}) \sim (\mathbb{A} \circ \mathbb{B}) \circ \mathbb{C}.$$

Proof. Denote $\mathbb{A} = (A, R)$, $\mathbb{B} = (B, S)$, $\mathbb{C} = (C, T)$, $\mathbb{A} \circ \mathbb{B} = (A \times B, U)$, $\mathbb{B} \circ \mathbb{C} = (B \times C, V)$, $\mathbb{A} \circ (\mathbb{B} \circ \mathbb{C}) = (A \times (B \times C), W)$, $(\mathbb{A} \circ \mathbb{B}) \circ \mathbb{C} = ((A \times B) \times C, X)$. Let φ be the natural bijection of $A \times (B \times C)$ onto $(A \times B) \times C$ defined by $\varphi((a, (b, c))) = ((a, b), c)$ for any $a \in A$, $b \in B$, $c \in C$. Let $((a_1, (b_1, c_1)), \dots, (a_n, (b_n, c_n))) \in W$. Then at least two of the elements $a_1, \dots, a_n$ are different, $(a_1, \dots, a_n) \in R$ or $a_1 = \dots = a_n$, $((b_1, c_1), \dots, (b_n, c_n)) \in V$, so that at least two of the elements $a_1, \dots, a_n$ are different, $(a_1, \dots, a_n) \in R$ or $a_1 = \dots = a_n$, at least two of the elements $b_1, \dots, b_n$ are different, $(b_1, \dots, b_n) \in S$ or $a_1 = \dots = a_n$, $b_1 = \dots = b_n$, $(c_1, \dots, c_n) \in T$, consequently at least two of the elements $(a_1, b_1), \dots, (a_n, b_n)$ are different, $((a_1, b_1), \dots, (a_n, b_n)) \in U$ or $(a_1, b_1) = \dots = (a_n, b_n)$, $(c_1, \dots, c_n) \in T$, hence $(\varphi((a_1, (b_1, c_1))), \dots, \varphi((a_n, (b_n, c_n)))) = (((a_1, b_1), c_1), \dots ((a_n, b_n), c_n)) \in X$ and φ is a homomorphism of $\mathbb{A} \circ (\mathbb{B} \circ \mathbb{C})$ onto $(\mathbb{A} \circ \mathbb{B}) \circ \mathbb{C}$. Similarly $(\varphi((a_1, (b_1, c_1))), \dots, \varphi((a_n, (b_n, c_n)))) \in X$ implies $((a_1, (b_1, c_1), \dots, (a_n, (b_n, c_n))) \in W$ and φ is an isomorphism of $\mathbb{A} \circ (\mathbb{B} \circ \mathbb{C})$ onto $(\mathbb{A} \circ \mathbb{B}) \circ \mathbb{C}$. □

40. Theorem. *Let $\mathbb{A}_1, \ldots, \mathbb{A}_n$ be n-ary relational systems with pairwise disjoint carriers. Then, for any n-ary relational system $\mathbb{B}$, we have*

$$(\mathbb{A}_1 \oplus \ldots \oplus \mathbb{A}_n) \circ \mathbb{B} = \mathbb{A}_1 \circ \mathbb{B} \oplus \ldots \oplus \mathbb{A}_n \circ \mathbb{B}.$$

Proof. As the n-ary relational systems $\mathbb{A}_1, \ldots, \mathbb{A}_n$ have pairwise disjoint carriers, the n-ary relational systems $\mathbb{A}_1 \circ \mathbb{B}, \ldots, \mathbb{A}_n \circ \mathbb{B}$ have pairwise disjoint carriers as well. Let $\mathbb{A}_j = (A_j, R_j)$ for $j \in (n]$, $\mathbb{B} = (B, S)$, $\mathbb{A}_1 \oplus \ldots \oplus \mathbb{A}_n = (\bigcup\limits_{j=1}^{n} A_j, T)$, $(\mathbb{A}_1 \oplus \ldots \oplus \mathbb{A}_n) \circ \mathbb{B} = ((\bigcup\limits_{j=1}^{n} A_j) \times B, U)$, $\mathbb{A}_1 \circ \mathbb{B} \oplus \ldots \oplus \mathbb{A}_n \circ \mathbb{B} = ((\bigcup\limits_{j=1}^{n} A_j) \times B, V)$. Assume that $((a_1, b_1), \ldots, (a_n, b_n)) \in U$. Then at least two of the elements $a_1, \ldots, a_n$ are different, $(a_1, \ldots, a_n) \in T$ or $a_1 = \ldots = a_n$, $(b_1, \ldots, b_n) \in S$. Hence at least two of the elements $a_1, \ldots, a_n$ are different, $a_1, \ldots, a_n \in A_j$, $(a_1, \ldots, a_n) \in R_j$ for some $j \in (n]$ or $a_1 = \ldots = a_n$, $(b_1, \ldots, b_n) \in S$ or there exists an $\varepsilon \in E_n$ such that $a_i \in A_{\varepsilon(i)}$ for each $i \in (n]$ and $((a_1, b_1), \ldots, (a_n, b_n)) \in V$. Similarly $V \subseteq U$. □

41. Remark. The second distributive law, of course, does not hold in general, because it is even not valid for binary relational systems.

42. Theorem. *For any binary relational systems $\mathbb{A}, \mathbb{B}$ with disjoint carriers and for any n-ary relational system $\mathbb{C}$ we have*

$$^{\mathbb{A}+\mathbb{B}}\,\mathbb{C} \sim (^{\mathbb{A}}\mathbb{C})(^{\mathbb{B}}\mathbb{C}).$$

Proof. Let $\mathbb{A} = (A, R)$, $\mathbb{B} = (B, S)$, $\mathbb{C} = (C, T)$, $^{\mathbb{A}}\mathbb{C} = (C^A, U)$, $^{\mathbb{B}}\mathbb{C} = (C^B, V)$, $^{\mathbb{A}+\mathbb{B}}\,\mathbb{C} = (C^{A \cup B}, W)$, $(^{\mathbb{A}}\mathbb{C})(^{\mathbb{B}}\mathbb{C}) = (C^A \times C^B, X)$. Let φ be the natural bijection of $C^{A \cup B}$ onto $C^A \times C^B$ defined by $\varphi(f) = (f{\restriction}A, f{\restriction}B)$. Assume that $(f_1, \ldots, f_n) \in W$. In order to show that $(\varphi(f_1), \ldots, \varphi(f_n)) = ((f_1{\restriction}A, f_1{\restriction}B), \ldots, (f_n{\restriction}A, f_n{\restriction}B)) \in X$, we must verify that $(f_1{\restriction}A, \ldots, f_n{\restriction}A) \in U$, $(f_1{\restriction}B, \ldots, f_n{\restriction}B) \in V$. Let $((f_1{\restriction}A)(a), \ldots, (f_n{\restriction}A)(a)) \notin T$ for some $a \in A$. Then $(f_1(a), \ldots, f_n(a)) \notin T$ and there exists a $b \in A \cup B$ such that $(b, a) \in R \cup S$, $(f_1(b), \ldots, f_n(b)) \in T$, at least two of the elements $f_1(b), \ldots, f_n(b)$ are different, and $(f_1(x), \ldots, f_n(x)) \in T$ for each $x \in A \cup B$, $(x, b) \in R \cup S$. But $(b, a) \in R \cup S$ implies $b \in A$ and $(x, b) \in R \cup S$ implies $x \in A$. Hence there exists a $b \in A$ such that $(b, a) \in R$, $((f_1{\restriction}A)(b), \ldots, (f_n{\restriction}A)(b)) \in T$, at least two of the elements $(f_1{\restriction}A)(b), \ldots, (f_n{\restriction}A)(b)$ are different, and $((f_1{\restriction}A)(x), \ldots, (f_n{\restriction}A)(x)) \in T$ for each $x \in A$, $(x, b) \in R$, so that $(f_1{\restriction}A, \ldots, f_n{\restriction}A) \in U$. Similarly $(f_1{\restriction}B, \ldots, f_n{\restriction}B) \in V$ and $(\varphi(f_1), \ldots, \varphi(f_n)) \in X$. Analogously $(\varphi(f_1), \ldots, \varphi(f_n)) \in X$ implies $(f_1, \ldots, f_n) \in W$. □

43. Theorem. *For any binary relational systems $\mathbb{A}, \mathbb{B}$ with disjoint carriers and for any reflexive n-ary relational system $\mathbb{C}$ we have*

$$^{\mathbb{A}\oplus\mathbb{B}}\,\mathbb{C} \sim (^{\mathbb{A}}\mathbb{C}) \circ (^{\mathbb{B}}\mathbb{C}).$$

Proof. Let $\mathbb{A} = (A, R)$, $\mathbb{B} = (B, S)$, $\mathbb{C} = (C, T)$, $\mathbb{A} \oplus \mathbb{B} = (A \cup B, U)$, $^{\mathbb{A}\oplus\mathbb{B}}\,\mathbb{C} = (C^{A\cup B}, V)$, $^{\mathbb{A}}\mathbb{C} = (C^A, W)$, $^{\mathbb{B}}\mathbb{C} = (C^B, X)$, $(^{\mathbb{A}}\mathbb{C}) \circ (^{\mathbb{B}}\mathbb{C}) = (C^A \times C^B, Y)$. Again, let φ be the natural bijection of $C^{A\cup B}$ onto $C^A \times C^B$ defined by $\varphi(f) = (f{\restriction}A, f{\restriction}B)$. Assume that $(\varphi(f_1), \dots, \varphi(f_n)) = ((f_1{\restriction}A, f_1{\restriction}B), \dots, (f_n{\restriction}A, f_n{\restriction}B)) \in Y$, i.e. either at least two of the elements $f_1{\restriction}A, \dots, f_n{\restriction}A$ are different, $(f_1{\restriction}A, \dots, f_n{\restriction}A) \in W$, or $f_1{\restriction}A = \dots = f_n{\restriction}A$, $(f_1{\restriction}B, \dots, f_n{\restriction}B) \in X$. Now, let there exist an $a \in A \cup B$ such that $(f_1(a), \dots, f_n(a)) \notin T$. If $a \in A$, then $((f_1{\restriction}A)(a), \dots, (f_n{\restriction}A)(a)) \notin T$ and, because of reflexivity of T, at least two of the elements $f_1{\restriction}A, \dots, f_n{\restriction}A$ are different, thus $(f_1{\restriction}A, \dots, f_n{\restriction}A) \in W$. Hence there exists a $b \in A$ such that $(b, a) \in R$, $((f_1{\restriction}A)(b), \dots, (f_n{\restriction}A)(b)) \in T$, at least two of the elements $(f_1{\restriction}A)(b), \dots, (f_n{\restriction}A)(b)$ are different, and for each $x \in A$, $(x, b) \in R$ we have $((f_1{\restriction}A)(x), \dots, (f_n{\restriction}A)(x)) \in T$. Consequently, there exists a $b \in A \cup B$ such that $(b, a) \in U$, $(f_1(b), \dots, f_n(b)) \in T$, at least two of the elements $f_1(b), \dots, f_n(b)$ are different, and for each $x \in A \cup B$, $(x, b) \in U$ we have $(f_1(x), \dots, f_n(x)) \in T$. If $a \in B$, then $((f_1{\restriction}B)(a), \dots, (f_n{\restriction}B)(a)) \notin T$. If, moreover, at least two of the mappings $f_1{\restriction}A, \dots, f_n{\restriction}A$ are different, then $(f_1{\restriction}A, \dots, f_n{\restriction}A) \in W$. Let there exist a $b \in A$ such that $((f_1{\restriction}A)(b), \dots, (f_n{\restriction}A)(b)) \notin T$. Then there exists a $c \in A$ such that $(c, b) \in R$, $((f_1{\restriction}A)(c), \dots, (f_n{\restriction}A)(c)) \in T$, at least two of the elements $(f_1{\restriction}A)(c), \dots, (f_n{\restriction}A)(c)$ are different, and for each $x \in A$, $(x, c) \in R$ we have $((f_1{\restriction}A)(x), \dots, (f_n{\restriction}A)(x)) \in T$. Hence there exists a $c \in A \cup B$ such that $(c, a) \in U$, $(f_1(c), \dots, f_n(c)) \in T$, at least two of the elements $f_1(c), \dots, f_n(c)$ are different, and for each $x \in A \cup B$, $(x, c) \in U$ we have $(f_1(x), \dots, f_n(x)) \in T$. On the contrary, let there exist no $b \in A$ such that $((f_1{\restriction}A)(b), \dots, (f_n{\restriction}A)(b)) \notin T$. Then for any $b \in A$ such that at least two of the elements $(f_1{\restriction}A)(b), \dots, (f_n{\restriction}A)(b)$ are different, we have $b \in A \cup B$, $(b, a) \in U$, $(f_1(b), \dots, f_n(b)) \in T$, at least two of the elements $f_1(b), \dots, f_n(b)$ are different, and for each $x \in A \cup B$, $(x, b) \in U$ we have $(f_1(x), \dots, f_n(x)) \in T$. Finally, if $a \in B$, $((f_1{\restriction}B)(a), \dots, (f_n{\restriction}B)(a)) \notin T$ and $f_1{\restriction}A = \dots = f_n{\restriction}A$, then $(f_1{\restriction}B, \dots, f_n{\restriction}B) \in X$, thus there exists a $b \in B$ such that $(b, a) \in S$, $((f_1{\restriction}B)(b), \dots, (f_n{\restriction}B)(b)) \in T$, at least two of the elements $(f_1{\restriction}B)(b), \dots, (f_n{\restriction}B)(b)$ are different, and for each $x \in B$, $(x, b) \in S$ we have $((f_1{\restriction}B)(x), \dots, (f_n{\restriction}B)(x)) \in T$. Hence there exists a $b \in A \cup B$ such that $(b, a) \in U$, $(f_1(b), \dots, f_n(b)) \in T$, at least two of the elements $f_1(b), \dots, f_n(b)$ are different, and for each $x \in A \cup B$, $(x, b) \in U$ we have $(f_1(x), \dots, f_n(x)) \in T$ (for $x \in B$ this follows from the preceding, for $x \in A$ from the reflexivity of T). In all cases we have $(f_1,\dots,f_n)\in V$. Similarly $(f_1,\dots,f_n)\in V$ implies $(\varphi(f_1),\dots,\varphi(f_n))\in Y$. $\square$

44. Theorem. *For any n-ary relational systems $\mathbb{A}_1,\dots,\mathbb{A}_n$ with pairwise disjoint carriers we have*

$$\widetilde{\mathbb{A}_1\oplus\dots\oplus\mathbb{A}_n}=\widetilde{A}_n\oplus\dots\oplus\widetilde{A}_1.$$

Proof. Let $\mathbb{A}_j=(A_j,R_j)$, $\widetilde{\mathbb{A}}_j=(A_j,S_j)$ for $j\in(n]$, $\mathbb{A}_1\oplus\dots\oplus\mathbb{A}_n=(\bigcup\limits_{j=1}^{n}A_j,T)$, $\widetilde{\mathbb{A}_1\oplus\dots\oplus\mathbb{A}_n}=(\bigcup\limits_{j=1}^{n}A_j,U)$, $\widetilde{\mathbb{A}}_n\oplus\dots\oplus\widetilde{A}_1=(\bigcup\limits_{j=1}^{n}A_j,V)$. Let $(a_1,\dots,a_n)\in U$. Then $(a_n,\dots,a_1)\in T$, thus either $(a_n,\dots,a_1)\in R_j$ for some $j\in(n]$ or there exists an $\varepsilon\in E_n$ such that $a_{n+1-j}\in A_{\varepsilon(j)}$ for each $j\in(n]$. In the first case we have $(a_1,\dots,a_n)\in S_j\subseteq V$. In the second case we have $a_j\in A_{\varepsilon(n+1-j)}$ for each $j\in(n]$. Hence $(a_1,\dots,a_n)\in A_{\varepsilon(n)}\times\dots\times A_{\varepsilon(1)}\subseteq V$. Consequently $U\subseteq V$. Similarly $V\subseteq U$. $\square$

45. Theorem. *For any n-ary relational systems $\mathbb{A},\mathbb{B}$ we have*

$$\widetilde{\mathbb{A}\circ\mathbb{B}}=\widetilde{\mathbb{A}}\circ\widetilde{\mathbb{B}}.$$

Proof. Let $\mathbb{A}=(A,R)$, $\mathbb{B}=(B,S)$, $\widetilde{\mathbb{A}}=(A,T)$, $\widetilde{B}=(B,U)$, $\mathbb{A}\circ\mathbb{B}=(A\times B,V)$, $\widetilde{\mathbb{A}\circ\mathbb{B}}=(A\times B,W)$, $\widetilde{\mathbb{A}}\circ\widetilde{\mathbb{B}}=(A\times B,X)$. Let $((a_1,b_1),\dots,(a_n,b_n))\in W$. Then $((a_n,b_n),\dots,(a_1,b_1))\in V$, thus either at least two of the elements $a_n,\dots,a_1$ are different, $(a_n,\dots,a_1)\in R$ or $a_n=\dots=a_1$, $(b_n,\dots,b_1)\in S$. This implies that either at least two of the elements $a_1,\dots,a_n$ are different, $(a_1,\dots,a_n)\in T$ or $a_1=\dots=a_n$, $(b_1,\dots,b_n)\in U$, so that $((a_1,b_1),\dots,(a_n,b_n))\in X$, and $W\subseteq X$. Similarly $X\subseteq W$. $\square$

46. Theorem. *For any binary relational system $\mathbb{A}$ and any n-ary relational system $\mathbb{B}$ we have*

$$\widetilde{{}^{\mathbb{A}}\mathbb{B}}={}^{\mathbb{A}}\widetilde{\mathbb{B}}.$$

Proof. Let $\mathbb{A}=(A,R)$, $\mathbb{B}=(B,S)$, $\widetilde{\mathbb{B}}=(B,T)$, ${}^{\mathbb{A}}\mathbb{B}=(B^A,U)$, $\widetilde{{}^{\mathbb{A}}\mathbb{B}}=(B^A,V)$, ${}^{\mathbb{A}}\widetilde{\mathbb{B}}=(B^A,W)$. Let $(f_1,\dots,f_n)\in V$. Then $(f_n,\dots,f_1)\in U$. Suppose that $(f_1(a),\dots,f_n(a))\notin T$ for some $a\in A$. Then $(f_n(a),\dots,f_1(a))\notin S$, thus there exists a $b\in A$ such that $(b,a)\in R$, $(f_n(b),\dots,f_1(b))\in S$, at least two of the elements $f_n(b),\dots,f_1(b)$ are different, and for each $x\in A$, $(x,b)\in R$ we have $(f_n(x),\dots,f_1(x))\in S$. Hence there exists a $b\in A$, $(b,a)\in R$ such that $(f_1(b),\dots,f_n(b))\in T$, at least two of the elements $f_1(b),\dots,f_n(b)$ are different, and for each $x\in A$, $(x,b)\in R$ we have $(f_1(x),\dots,f_n(x))\in T$. Consequently $(f_1,\dots,f_n)\in W$ and $V\subseteq W$. Similarly $W\subseteq V$. $\square$

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