ON OVERDETERMINED HARDY INEQUALITIES

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Abstract. Necessary and sufficient condition on the weights will be derived under which a k-th order Hardy inequality holds on classes of functions satisfying more than k "boundary" conditions.

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0. Introduction

In this note, we will deal with the *k-th order Hardy inequality*

$$\left(\int_0^1 |u(x)|^q w_0(x)\,\mathrm{d}x\right)^{1/q} \leqslant C\left(\int_0^1 |u^{(k)}(x)|^p w(x)\,\mathrm{d}x\right)^{1/p} \tag{1}$$

on the class of functions $u \in AC^{k-1}(0,1)$ for which the right hand side in (1) is finite and which satisfy the "boundary conditions"

$$\begin{aligned} u^{(i)}(0) &= 0 \quad \text{for } i \in M_0, \\ u^{(j)}(1) &= 0 \quad \text{for } j \in M_1, \end{aligned} \tag{2}$$

where M_0, M_1 are subsets of the set $\{0, 1, \dots, k-1\}$.

The problem what choice of the couple M_0, M_1 is meaningful, i.e., for what type of conditions (2) the inequality (1) makes sense, is completely solved in [DK]—the couple M_0, M_1 has to satisfy the so-called *Pólya condition*. Such couples M_0, M_1 will be called *admissible*.

For admissible couples satisfying additionally the condition

$$\#M_0 + \#M_1 = k, \tag{3}$$

necessary and sufficient conditions on the weights w_0 and w (i.e., functions measurable and positive a.e. in (0,1)) and on the parameters p, q $(1 \leqslant p \leqslant \infty,\ 0 < q \leqslant \infty)$ are completely described in [K] and in [S]. Such couples will be called *standard*. Let us only mention that the method consists in reducing inequality (1) under conditions (2) to the weighted norm inequality

$$\left(\int_0^1 |(Tf)(x)|^q w_0(x)\,\mathrm{d}x\right)^{1/q} \leqslant C\left(\int_0^1 |f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p} \tag{4}$$

where T is the Green operator of the boundary value problem

$$u^{(k)} = f \text{ in } (0,1),\ u^{(i)}(0) = 0 \text{ for } i \in M_0,\ u^{(j)}(1) = 0 \text{ for } j \in M_1, \tag{5}$$

i.e.,

$$u(x) = (Tf)(x) = \int_0^1 K(x,t)f(t)\mathrm{d}t$$

with a suitable kernel $K(x,t)$.

The aim of this note is to find necessary and sufficient conditions for the validity of (1) in the case of *overdetermined* conditions, i.e. in the case that condition (3) is violated:

$$\#M_0 + \#M_1 > k. \tag{6}$$

For some special cases, this problem was solved by [KSim] (the case $M_0 = M_1 = \{0, 1, \ldots, k-1\}$ and the case $M_0 = \{0, 1, \ldots, k-1\}$, $M_1 = \{k-1\}$; mainly sufficient conditions), by [KSin] (the case $M_0 = \widetilde{M_0} \cup \{k-1\}$, $M_1 = \widetilde{M_1} \cup \{k-1\}$, where the couple $\widetilde{M_0}, \widetilde{M_1}$ is standard for the inequality of order $k-1$; necessary and sufficient conditions) and by [NS] (the case $k = p = q = 2$ on the semiaxis $(0, \infty)$; necessary and sufficient conditions).

Here, we will show that if we construct the *overdetermined* couple M_0, M_1 adding some new conditions to a *standard* couple $\widehat{M_0}, \widehat{M_1}$, then the necessary and sufficient conditions of the validity of (1) under the boundary conditions given by M_0, M_1 are the same as the necessary and sufficient conditions under the boundary conditions given by $\widehat{M_0}, \widehat{M_1}$ provided the weights w_0, w satisfy some additional assumptions.

For simplicity, we will explain our idea on the *particular* standard couple

$$\widehat{M_0} = \{0, 1, \ldots, k-1\}, \quad \widehat{M_1} = \emptyset \tag{7}$$

but it will be clear that the method can be extended to general standard couples (see also Example 3).

Thus, let us consider inequality (1) under the (standard) conditions

$$u(0) = u'(0) = \ldots = u^{(k-1)}(0) = 0. \tag{8}$$

Then the necessary and sufficient conditions for its validity have the form (for the case $1 < p \leqslant q < \infty$)

$$\begin{aligned} &\sup_{0<x<1} \left(\int_x^1 (t-x)^{(k-1)q} w_0(t)\mathrm{d}t \right)^{1/q} \left(\int_0^x w^{1-p'}(t)\mathrm{d}t \right)^{1/p'} < \infty, \\ &\sup_{0<x<1} \left(\int_x^1 w_0(t)\mathrm{d}t \right)^{1/q} \left(\int_0^x (x-t)^{(k-1)p'} w^{1-p'}(t)\mathrm{d}t \right)^{1/p'} < \infty, \end{aligned} \tag{9}$$

with $p' = \frac{p}{p-1}$.

Now, let us add to (8) the conditions

$$u^{(j)}(1) = 0 \quad \text{for} \quad j \in M \tag{10}$$

where M is a nonempty subset of $\{0, 1, \ldots, k-1\}$. Then we have the situation described above, with $\widehat{M}_0$ and $\widehat{M}_1$ from (7) and with $M_0 = \widehat{M}_0$, $M_1 = M$. Define an operator T by

$$(Tf)(x) = \frac{1}{(k-1)!} \int_0^x (x-t)^{k-1} f(t)\mathrm{d}t, \quad x \in (0,1). \tag{11}$$

The function $u = Tf$ obviously satisfies conditions (8) and the equation $u^{(k)} = f$ in (0,1). Moreover, since

$$u^{(j)}(x) = \frac{(-1)^j}{(k-j-1)!} \int_0^x (x-t)^{k-j-1} f(t)\mathrm{d}t, \quad 0 \leqslant j \leqslant k-1,$$

conditions (10) lead to the assumptions

$$\int_0^1 (1-t)^{k-j-1} f(t)\mathrm{d}t = 0 \quad \text{for} \quad j \in M. \tag{12}$$

If we denote by $L^p(w)$ the weighted Lebesgue space normed by

$$\|v\|_{p,w} = \left(\int_0^1 |v(x)|^p w(x)\,\mathrm{d}x \right)^{1/p}, \quad 1 < p < \infty,$$

and by W_M the subset of $L^p(w)$ of all functions f satisfying (12), we obtain immediately the following assertion:

Lemma 1. *Let M be a nonempty subset of $\{0, 1, \ldots, k-1\}$. Then the Hardy inequality* (1) *with overdetermined conditions* (8) *and* (10) *is equivalent to the weighted norm inequality*

$$\|Tf\|_{q,w_0} \leqslant C\|f\|_{p,w} \quad \textit{for all } f \in W_M \tag{13}$$

with $W_M \subset L^p(w)$ determined by conditions (12)*. If in addition the weight function w satisfies*

$$\int_0^1 (1-t)^{(k-j-1)p'} w^{1-p'}(t)\mathrm{d}t < \infty \quad \textit{for} \quad j \in M, \tag{14}$$

then W_M is a closed subspace of $L^p(w)$ with finite codimension $\#M$. Moreover,

$$W_M = F_M^{\perp} \tag{15}$$

where F_M denotes the linear hull of the functions

$$\varphi_j(t) = (1-t)^{k-j-1} w^{-1}(t), \quad j \in M, \tag{16}$$

in $L^{p'}(w)$ and $F_M^{\perp}$ its "orthogonal complement", i.e. the set of all $f \in L^p(w)$ such that

$$\int_0^1 \varphi_j(t) f(t) w(t)\mathrm{d}t = \int_0^1 (1-t)^{k-j-1} f(t)\mathrm{d}t = 0$$

for all $j \in M$.

Proof. Recalling the above explanations it is clear that the Hardy inequality (1) for u satisfying conditions (8), (10) is exactly inequality (13) for $f \in W_M$. Denote by $\langle \cdot, \cdot \rangle_w$ the duality between $L^p(w)$ and $L^{p'}(w)$:

$$\langle f, g \rangle_w = \int_0^1 f(t)g(t)w(t)\mathrm{d}t, \quad f \in L^p(w),\ g \in L^{p'}(w).$$

Conditions (14) guarantee that $\varphi_j \in L^{p'}(w)$, and assumptions (12) can be rewritten in the form

$$\langle f, \varphi_j \rangle_w = 0 \quad \text{for} \quad j \in M,$$

which gives $W_M = F_M^{\perp}$. To see that the codimension of W_M is $\#M$ consider the linear mapping $\Psi\colon L^p(w) \to \mathbb{R}^{\#M}$,

$$\Psi(f) = \{\langle f, \varphi_j \rangle_w\}_{j \in M}.$$

Clearly

$$\operatorname{Ker}\Psi = \{f \in L^p(w)\colon \langle f, \varphi_j\rangle_w = 0,\ j \in M\} = W_M$$

and

$$\Psi(L^p(w)) = \mathbb{R}^{\#M}$$

since there exist functions $\alpha_i \in L^p(w)$, $i \in M$, such that $\langle \alpha_i, \varphi_j\rangle_w = \delta_{ij}$ [Ka, Theorem 1.22]. Hence for the codimension $\dim L^p(w)/W_M$ of W_M we get

$$\dim L^p(w)/W_M = \dim L^p(w)/\operatorname{Ker}\Psi = \dim \Psi(L^p(w)) = \dim \mathbb{R}^{\#M} = \#M.$$

□

Remark 1. (i) Thus, the investigation of inequality (1) under conditions (8), (10) can be reduced to the investigation of inequality (13) on the subset $F_M^{\perp}$ of $L^p(w)$ provided w satisfies (14).

(ii) Obviously, conditions (14) can be replaced by a *single* condition

$$\int_0^1 (1-t)^{(k-j_0-1)p'} w^{1-p'}(t)\mathrm{d}t < \infty \tag{17}$$

where

$$j_0 = \max\{j,\ j \in M\}.$$

(iii) If the set M is empty, then we have no additional conditions on w and the subset $F_M^{\perp}$ coincides with the whole space $L^p(w)$. In other words: To investigate (1) under the standard conditions (8) is equivalent to investigating (13) on the *whole* space $L^p(w)$.

1. The case $p = 2$

In this case, condition (17) reads

$$\int_0^1 (1-t)^{2(k-j_0-1)} w^{-1}(t)\mathrm{d}t < \infty \tag{18}$$

and the *Hilbert* space $L^2(w)$ can be written as the orthogonal sum

$$L^2(w) = F_M \oplus F_M^{\perp} = F_M \oplus W_M \tag{19}$$

where F_M is the linear hull of the functions φ_i from (16). Moreover, if we suppose that the weight functions w and w_0 satisfy

$$\int_0^1 \left(\int_0^x (x-t)^{k-1}(1-t)^{k-j_0-1} w^{-1}(t)\mathrm{d}t\right)^q w_0(x)\,\mathrm{d}x < \infty, \tag{20}$$

then obviously

$$T\varphi_j \in L^q(w_0) \quad \text{for} \quad j \in M$$

with T from (11). Since F_M is finite-dimensional, T maps the subspace F_M *continuously* into $L^q(w_0)$.

Thus, we are able to prove the following assertion:

Theorem 1. *Let M be a nonempty subset of $\{0, 1, \ldots, k-1\}$, $j_0 = \max\{j,\ j \in M\}$, and suppose that the weight functions w_0, w satisfy conditions* (18) *and* (20). *Let $p = 2$, $0 < q < \infty$.*

Then the k-th order Hardy inequality (1) *holds for u satisfying the overdetermined conditions* (8) *and* (10) *if and only if it holds for u satisfying the standard conditions* (8).

P r o o f. (i) If (1) holds for u satisfying (8), then it obviously holds for u satisfying (8) and (10).

(ii) If (1) holds for u satisfying (8) and (10) and w satisfies (17), then according to Lemma 1 inequality (13) holds (with $p = 2$) for all $f \in W_M$, i.e. the operator T maps W_M continuously into $L^q(w_0)$. If, moreover, (20) is satisfied, then T maps also F_M into $L^q(w_0)$, and consequently, due to (19), T maps the *whole* space $L^2(w_0)$ into $L^q(w_0)$. But this means, according to Remark 1 (iii), that (1) holds for u satisfying (8). □

2. A general $p > 1$

(a) Suppose that M contains only *one* element, $M = \{j_0\}$. Then we have only one function φ: $\varphi(t) = (1-t)^{k-j_0-1}w^{-1}(t)$, $\varphi \in L^{p'}(w)$ provided (17) holds, and the set F_M is onedimensional.

Define a function α by

$$\alpha(t) = C_0(1-t)^{(k-j_0-1)(p'-1)}w^{1-p'}(t) \tag{21}$$

with a suitable constant $C_0 > 0$. Condition (17) guarantees that

$$\alpha \in L^p(w) \quad \text{and} \quad \langle \alpha, \varphi \rangle_w > 0.$$

Indeed,

$$\int_0^1 \alpha^p(t)w(t)\mathrm{d}t = C_0^p \int_0^1 (1-t)^{(k-j_0-1)p'}w^{1-p'}(t)\mathrm{d}t < \infty$$

and

$$\langle \alpha, \varphi \rangle_w = \int_0^1 \alpha(t)\varphi(t)w(t)\mathrm{d}t = C_0 \int_0^1 (1-t)^{(k-j_0-1)p'} w^{1-p'}(t)\mathrm{d}t.$$

If we choose C_0 such that $\langle \alpha, \varphi \rangle = 1$, we can write every $g \in L^p(w)$ in the form

$$g = f + C\alpha, \quad f \in F_M^{\perp}. \tag{22}$$

Indeed, if we put $f = g - \langle g, \varphi \rangle_w \alpha$, we have $\langle f, \varphi \rangle_w = \langle g, \varphi \rangle_w - \langle g, \varphi \rangle_w \langle \alpha, \varphi \rangle_w = 0$, i.e., $f \in F_M^{\perp}$, and we have (22) with $C = \langle g, \varphi \rangle_w$.

If we now suppose that the functions w and w_0 satisfy

$$\int_0^1 \left(\int_0^x (x-t)^{k-1} (1-t)^{(k-j_0-1)(p'-1)} w^{1-p'}(t)\mathrm{d}t \right)^q w_0(x)\,\mathrm{d}x < \infty, \tag{23}$$

then we have

$$T\alpha \in L^q(w_0)$$

and T maps the onedimensional subset $\{c\alpha\}$, $c \in \mathbb{R}$, of $L^p(w)$ continuously into $L^q(w_0)$. Since T maps $F_M^{\perp}$ continuously into $L^q(w_0)$ if and only if (1) holds for u satisfying (8) and (10) (due to Lemma 1), we can repeat, in view of (22), the assertion of Theorem 1, replacing assumptions (18) and (20) by (17) and (23), respectively, and omitting the assumption $p = 2$.

Example 1. Let us consider inequality (1) under the conditions

$$u(0) = u'(0) = \ldots = u^{(k-1)}(0) = 0, \quad u(1) = 0.$$

Then we have the situation just described, with $M = \{j_0\} = \{0\}$, and we can assert that inequality (1) holds for p, q satisfying $1 < p \leqslant q < \infty$ if and only if conditions (9) are satisfied, provided that

$$\int_0^1 (1-t)^{(k-1)p'} w^{1-p'}(t)\mathrm{d}t < \infty$$

and

$$\int_0^1 \left(\int_0^x (x-t)^{k-1} (1-t)^{(k-1)(p'-1)} w^{1-p'}(t)\mathrm{d}t \right)^q w_0(x)\,\mathrm{d}x < \infty.$$

Remark 2. Of course our foregoing assertions remain true also for the case $p > q$, $0 < q < \infty$, $1 < p < \infty$, but then we have to replace conditions (9) by the corresponding conditions for the case $p > q$ (see, e.g., [OK]).

E x a m p l e 2. Let us consider inequality (1) under the conditions

$$u(0) = u'(0) = \ldots = u^{(k-1)}(0) = 0, \; u^{(k-1)}(1) = 0. \tag{24}$$

Then we have again the situation described above, with $M_0 = \{j_0\} = \{k-1\}$, and we can again assert that inequality (1) holds for p, q satisfying $1 < p \leqslant q < \infty$ if and only if conditions (9) are satisfied provided that

$$\int_0^1 w^{1-p'}(t)\mathrm{d}t < \infty \tag{25}$$

and

$$\int_0^1 \left(\int_0^x (x-t)^{k-1} w^{1-p'}(t)\mathrm{d}t \right)^q w_0(x)\,\mathrm{d}x < \infty. \tag{26}$$

The case of conditions (24) was investigated in [KSin] where the following *five* conditions have been shown to be necessary and sufficient for (1) to hold, with $z \in (0,1)$ arbitrary but fixed:

$$\begin{aligned}
&\sup_{0<x<z} \left(\int_x^z (t-x)^{(k-1)q} w_0(t)\mathrm{d}t \right)^{1/q} \left(\int_0^x w^{1-p'}(t)\mathrm{d}t \right)^{1/p'} < \infty,\\
&\sup_{0<x<z} \left(\int_x^z w_0(t)\mathrm{d}t \right)^{1/q} \left(\int_0^x (x-t)^{(k-1)p'} w^{1-p'}(t)\mathrm{d}t \right)^{1/p'} < \infty,\\
&\sup_{z<x<1} \left(\int_x^1 (t-z)^{(k-2)q} w_0(t)\mathrm{d}t \right)^{1/q} \left(\int_z^x (t-z)^{p'} w^{1-p'}(t)\mathrm{d}t \right)^{1/p'} < \infty,\\
&\sup_{z<x<1} \left(\int_z^x (t-z)^{(k-1)q} w_0(t)\mathrm{d}t \right)^{1/q} \left(\int_x^1 w^{1-p'}(t)\mathrm{d}t \right)^{1/p'} < \infty,\\
&\sup_{j=1,\ldots,k-1} \left(\int_z^1 (t-z)^{(k-j-1)q} w_0(t)\mathrm{d}t \right)^{1/q} \left(\int_0^z (z-t)^{jp'} w^{1-p'}(t)\mathrm{d}t \right)^{1/p'} < \infty.
\end{aligned} \tag{27}$$

We will come back to this case later (see Remark 4).

(b) If M contains more than one element from the set $\{0,1,\ldots,k-1\}$, we can proceed similarly, using again the concept of biorthogonality ([Ka, Theorem 1.22]). We denote again $j_0 = \max\{j,\ j \in M\}$ and suppose that (17) holds. Then for our set $\{\varphi_j\}_{j\in M}$—see (16)—with $\varphi_j \in L^{p'}(w)$ there exist functions $\alpha_i \in L^p(w)$, $i \in M$, such that $\langle \alpha_i, \varphi_j \rangle_w = \delta_{ij}$ and we can write, in analogy to (22), every function $g \in L^p(w)$ in the form

$$g = f + h$$

where f belongs to $F_M^\perp$ and h belongs to the linear hull of the $\alpha_i' s$, i.e., to a finite-dimensional subspace of $L^p(w)$. It can be shown that the functions α_i can be expressed as linear combinations of the functions

$$(1-t)^{(k-j-1)(p'-1)} w^{1-p'}(t),\ j \in M,$$

and if we suppose that (23) is satisfied, we obtain that

$$T\alpha_i \in L^q(w_0),\ i \in M.$$

The conclusion follows as in part (a).

3. A general standard couple

We explained our approach using the special standard couple $\widehat{M_0}, \widehat{M_1}$ from (7), but the method can be used for any other standard couple. Let us describe shortly the approach:

We start with a standard couple $\widehat{M_0}, \widehat{M_1}$, for which the necessary and sufficient conditions of the validity are known. Then we can express the function u with help of an integral operator T,

$$u(x) = (Tf)(x) = \int_0^1 K(x,t) f(t) \mathrm{d}t \tag{28}$$

where the kernel K is known. If our overdetermined couple M_0, M_1 is determined by some additional conditions of the type

$$u^{(\alpha)}(0) = 0, \quad u^{(\beta)}(1) = 0,$$

we simply use these conditions in (28) and obtain conditions on f:

$$\int_0^1 \frac{\partial^\alpha K}{\partial x^\alpha}(0,t) f(t) \mathrm{d}t = 0, \quad \int_0^1 \frac{\partial^\beta K}{\partial x^\beta}(1,t) f(t) \mathrm{d}t = 0. \tag{29}$$

Supposing additionally that the weight function w satisfies

$$\int_0^1 \left| \frac{\partial^\alpha K}{\partial x^\alpha}(0,t) \right|^{p'} w^{1-p'}(t) \mathrm{d}t < \infty, \ \int_0^1 \left| \frac{\partial^\beta K}{\partial x^\beta}(1,t) \right|^{p'} w^{1-p'}(t) \mathrm{d}t < \infty,$$

then the conditions (29) describe a closed subspace of $L^p(w)$ (with finite codimension), and we consider the weighted norm inequality (4) for f from the "orthogonal complement" of this subset.

Finally, we try to find conditions on w and w_0 which would allow to extend inequality (4) to the whole space $L^p(w)$.

Without going into details, we explain our idea on a simple example.

E x a m p l e 3. We consider the third order Hardy inequality

$$\left(\int_0^1 |u(x)|^q w_0(x)\,\mathrm{d}x\right)^{1/q} \leqslant C\left(\int_0^1 |u'''(x)|^p w(x)\,\mathrm{d}x\right)^{1/p} \tag{30}$$

for functions u satisfying

$$u(0)=0,\ u(1)=0,\ u'(0)=0,\ u''(1)=0. \tag{31}$$

We start with the standard conditions

$$u(0)=0,\ u'(0)=0,\ u(1)=0, \tag{32}$$

i.e., $\widehat{M_0}=\{0,1\}$, $\widehat{M_1}=\{0\}$, and the necessary and sufficient conditions of the validity of inequality (30) for functions u satisfying (32) have (for $1<p\leqslant q<\infty$) the form

$$\begin{gathered}\sup_{0<x<1}\left(\int_x^1 t^q(1-t)^q w_0(t)\mathrm{d}t\right)^{1/q}\left(\int_0^x t^{p'}(1-t)^{p'}w^{1-p'}(t)\mathrm{d}t\right)^{1/p'}<\infty,\\ \sup_{0<x<1}\left(\int_0^x t^{2q}w_0(t)\mathrm{d}t\right)^{1/q}\left(\int_x^1 (1-t)^{2p'}w^{1-p'}(t)\mathrm{d}t\right)^{1/p'}<\infty\end{gathered} \tag{33}$$

(see, e.g., [K] or [S]). In this case we can write

$$(Tf)(x)=u(x)=\tfrac{1}{2}\int_0^x t(2x^2-x^2t-2x+t)f(t)\mathrm{d}t-\tfrac{1}{2}x^2\int_x^1(1-t)^2f(t)\mathrm{d}t. \tag{34}$$

The *additional* condition $u''(1)=0$ leads to the condition

$$\int_0^1 t(2-t)f(t)\mathrm{d}t=0. \tag{35}$$

Thus inequality (30) holds for u satisfying (31) if the inequality $\|Tf\|_{q,w_0}\leqslant C\|f\|_{p,w}$ holds with T from (34) for all $f\in L^p(w)$ satisfying (35).

But condition (35) can be rewritten in the form $\langle f,\varphi\rangle_w=0$ with $\varphi(t)=t(2-t)w^{-1}(t)$ provided φ belongs to $L^{p'}(w)$, which means that w satisfies the assumption

$$\int_0^1 t^{p'}w^{1-p'}(t)\mathrm{d}t<\infty. \tag{36}$$

Then, similarly as in Section 2(a), we can construct a function $\alpha \in L^p(w)$, $\alpha(t) = c_0 t^{p'-1}(2-t)^{p'-1} w^{1-p'}(t)$ and decompose the space $L^p(w)$. If, moreover,

$$\int_0^1 \left(x(1-x) \int_0^x t^{p'}(1-t) w^{1-p'}(t) \mathrm{d}t \right. \\ \left. + x^2 \int_x^1 (1-t)^2 t^{p'-1} w^{1-p'}(t) \mathrm{d}t \right)^q w_0(x) \, \mathrm{d}x < \infty \tag{37}$$

then $T\alpha \in L^q(w_0)$ and we can conclude that the conditions (33) are necessary and sufficient for (30) to hold also in the overdetermined case (31) provided w and w_0 satisfy assumptions (36) and (37).

4. Another approach

Now, let us consider inequality (1) for functions u satisfying

$$u^{(i)}(0) = u^{(i)}(1) = 0 \quad \text{for} \quad i = 0, 1, \ldots, k-1, \tag{38}$$

i.e., for the case that in (2) we have $M_0 = M_1 = \{0, 1, \ldots, k-1\}$.

Let us choose $z \in (0,1)$ arbitrary but fixed and introduce operators $S_{1,z}$, $S_{2,z}$ by

$$\begin{aligned} (S_{1,z}f)(x) &= \frac{1}{(k-1)!} \int_0^x (x-t)^{k-1} f(t) \mathrm{d}t, \quad x \in (0,z), \\ (S_{2,z}f)(x) &= \frac{1}{(k-1)!} \int_x^1 (t-x)^{k-1} f(t) \mathrm{d}t, \quad x \in (z,1). \end{aligned} \tag{39}$$

If we define

$$u(x) = \chi_{(0,z)}(x)(S_{1,z}f)(x) + (-1)^k \chi_{(z,1)}(x)(S_{2,z}f)(x) \tag{40}$$

then u satisfies conditions (38) and we have that $u^{(k)}(x) = f(x)$ on $(0,z) \cup (z,1)$. If f satisfies the assumptions

$$\int_0^1 t^i f(t) \mathrm{d}t = 0, \quad i = 0, 1, \ldots, k-1 \tag{41}$$

then $u^{(i)}(z+0) = u^{(i)}(z-0)$ and we immediately have

Lemma 2. *The Hardy inequality* (1) *with overdetermined conditions* (38) *is equivalent to the weighted norm inequality*

$$\|T_z f\|_{q,w_0} \leqslant C \|f\|_{p,w} \tag{42}$$

for all $f \in L^p(w)$ *satisfying conditions* (41) *with* $z \in (0,1)$ *fixed and* $T_z = S_{1,z} + (-1)^k S_{2,z}$.

So, we again have a situation similar to that of the foregoing sections: We have reduced the Hardy inequality (1) to the special weighted norm inequality (42) on a *subset* of the space $L^p(w)$, and now, we will try to extend T_z to the whole space $L^p(w)$. To this purpose, we introduce functions

$$\varphi_i(t) = t^i w^{-1}(t), \quad i = 0, 1, \ldots, k-1. \tag{43}$$

The *additional* assumption

$$\int_0^1 w^{1-p'}(t)\mathrm{d}t < \infty \tag{44}$$

guarantees that $\varphi_i \in L^{p'}(w)$, and conditions (41) can be rewritten as

$$\langle f, \varphi_i \rangle_w = 0, \quad i = 0, 1, \ldots, k-1.$$

Similarly as in Section 2, we introduce functions $\alpha_j \in L^p(w)$ such that $\langle \alpha_j, \varphi_i \rangle_w = \delta_{ji}$, in terms of the functions

$$\beta_i(t) = t^{i(p'-1)} w^{1-p'}(t).$$

Condition (44) guarantees that $\beta_i \in L^p(w)$, and the additional assumptions

$$\begin{aligned} &\int_0^z \left(\int_0^x (x-t)^{k-1} t^{i(p'-1)} w^{1-p'}(t)\mathrm{d}t \right)^q w_0(x)\,\mathrm{d}x < \infty, \\ &\int_z^1 \left(\int_x^1 (t-x)^{k-1} t^{i(p'-1)} w^{1-p'}(t)\mathrm{d}t \right)^q w_0(x)\,\mathrm{d}x < \infty \end{aligned} \tag{45}$$

guarantee that $T_z \alpha_i \in L^q(w_0)$. Consequently, T_z maps the whole space $L^p(w)$ continuously into $L^q(w_0)$ provided the Hardy inequality (1) holds for u satisfying (38) and the weight functions w and w_0 satisfy (44) and (45).

Now let us look for *necessary* conditions. Assume that (42) holds for all $f \in L^p(w)$.

(a) If the number k is *even*, then T_z is the *sum of positive operators* $S_{1,z}$, $S_{2,z}$ and we have that

$$|S_{i,z} f| \leqslant S_{i,z}|f| \leqslant (S_{1,z} + S_{2,z})|f| = T_z|f|, \quad i = 1, 2,$$

and

$$\|S_{i,z} f\|_{q,w_0} \leqslant \|T_z\ |f|\ \|_{q,w_0} \leqslant C\|\ |f|\ \|_{p,w} = C\|f\|_{p,w} \tag{46}$$

for every $f \in L^p(w)$, $i = 1, 2$. Consequently, the *necessary* conditions

$$
\begin{aligned}
&\sup_{0<x<z} \Big(\int_x^z (t-x)^{(k-1)q} w_0(t) \mathrm{d}t \Big)^{1/q} \Big(\int_0^x w^{1-p'}(t) \mathrm{d}t \Big)^{1/p'} < \infty, \\
&\sup_{0<x<z} \Big(\int_x^z w_0(t) \mathrm{d}t \Big)^{1/q} \Big(\int_0^x (x-t)^{(k-1)p'} w^{1-p'}(t) \mathrm{d}t \Big)^{1/p'} < \infty
\end{aligned} \tag{47}
$$

(for $i = 1$, i.e., on $(0, z)$) and

$$
\begin{aligned}
&\sup_{z<x<1} \Big(\int_z^x (x-t)^{(k-1)q} w_0(t) \mathrm{d}t \Big)^{1/q} \Big(\int_x^1 w^{1-p'}(t) \mathrm{d}t \Big)^{1/p'} < \infty, \\
&\sup_{z<x<1} \Big(\int_z^x w_0(t) \mathrm{d}t \Big)^{1/q} \Big(\int_x^1 (t-x)^{(k-1)p'} w^{1-p'}(t) \mathrm{d}t \Big)^{1/p'} < \infty
\end{aligned} \tag{48}
$$

(for $i = 2$, i.e., on $(z, 1)$) have to be satisfied.

(b) If the number k is *odd*, then T_z is the *difference* of positive operators $S_{1,z}$, $S_{2,z}$. For $f \in L^p(w)$, define g by

$$
g(x) = \begin{cases} f(x) & \text{for} \quad x \in (0, z), \\ -f(x) & \text{for} \quad x \in (z, 1). \end{cases} \tag{49}
$$

Then $g \in L^p(w)$, $\|g\|_{p,w} = \|f\|_{p,w}$ and $\|S_{i,z} f\|_{q,w_0} = \|S_{i,z} g\|_{q,w_0}$ since $S_{1,z}$ is concentrated on $(0, z)$ and $S_{2,z}$ on $(z, 1)$. Since $T_z f = S_{1,z} g + S_{2,z} g$, we again obtain (46) and the necessary conditions (47) and (48).

On the other hand, it follows from (47) that

$$
\Big(\int_0^z |(S_{1,z} f)(x)|^q w_0(x) \, \mathrm{d}x \Big)^{1/q} \leqslant C \Big(\int_0^z |f(x)|^p w(x) \, \mathrm{d}x \Big)^{1/p} \leqslant C \|f\|_{p,w} \tag{50}
$$

and it follows from (48) that

$$
\Big(\int_z^1 |(S_{2,z} f)(x)|^q w_0(x) \, \mathrm{d}x \Big)^{1/q} \leqslant C \Big(\int_z^1 |f(x)|^p w(x) \, \mathrm{d}x \Big)^{1/p} \leqslant C \|f\|_{p,w} \tag{51}
$$

and consequently, conditions (47) and (48) are also sufficient for (42) to hold on $L^p(w)$.

So, we have proved

Theorem 2. *Let $z \in (0, 1)$ be arbitrary but fixed. Let $1 < p \leqslant q < \infty$. Then conditions* (47) *and* (48) *are necessary and sufficient for the Hardy inequality* (1) *to* *hold on the overdetermined class of functions u satisfying conditions* (38), *provided the weight functions w and w_0 satisfy assumptions* (44) *and* (45).

Remark 3. Obviously, an analogous assertion can be derived also for the case $p > q$.

Remark 4. Let us consider the special case $k = 1$. Then it was shown in [KSin] that for w such that

$$\int_0^1 w^{1-p'}(t)\mathrm{d}t < \infty \tag{52}$$

the following pair of conditions is necessary and sufficient for (1) to hold:

$$\begin{aligned} &\sup_{0<x<z} \left(\int_x^z w_0(t)\mathrm{d}t \right)^{1/q} \left(\int_0^x w^{1-p'}(t)\mathrm{d}t \right)^{1/p'} < \infty, \\ &\sup_{z<x<1} \left(\int_z^x w_0(t)\mathrm{d}t \right)^{1/q} \left(\int_x^1 w^{1-p'}(t)\mathrm{d}t \right)^{1/p'} < \infty, \end{aligned} \tag{53}$$

with a fixed $z \in (0,1)$. (In fact, (53) are the conditions (27) which for $k = 1$ reduce to *two* conditions only.)

Conditions (53) are the conditions (47), (48) above (for $k = 1$!). Since (52) is in fact condition (44), it seems that—in contrary to [KSin]—the *additional* restrictive condition (45) appears in Theorem 2. Let us show that this is *not* the case since (for $k = 1$) condition (45) is satisfied automatically.

Without loss of generality, we can assume that (for z fixed)

$$\int_0^z w^{1-p'}(t)\mathrm{d}t \leqslant \int_z^1 w^{1-p'}(t)\mathrm{d}t.$$

Define f by

$$f(t) = \begin{cases} w^{1-p'}(t) + c & \text{for } t \in (0,z), \\ w^{1-p'}(t) & \text{for } t \in (z,1) \end{cases}$$

where the constant $c \geqslant 0$ is chosen so that

$$\int_0^z f(t)\mathrm{d}t = \int_z^1 f(t)\mathrm{d}t,$$

and define g by (49). Then $\int_0^1 g(t)\mathrm{d}t = 0$, i.e., g satisfies (41) (note that $k = 1$, i.e. $i = 0$).

According to Lemma 2, we have $\|T_z g\|_{q,w_0} \leqslant C\|g\|_{p,w}$, and consequently (50), (51). But due to the positivity of the operator $S_{1,z}$, we have from (50) that

$$\begin{aligned}\int_0^z \left(\int_0^x w^{1-p'}(t)\mathrm{d}t\right)^q w_0(x)\,\mathrm{d}x &= \int_0^z |(S_{1,z}w^{1-p'})(x)|^q w_0(x)\,\mathrm{d}x \\ &\leqslant \int_0^z |S_{1,z}(w^{1-p'}+c)(x)|^q w_0(x)\,\mathrm{d}x = \int_0^z |(S_{1,z}f)(x)|^q w_0(x)\,\mathrm{d}x \\ &\leqslant C\|f\|_{p,w}^q < \infty,\end{aligned}$$

which is the first condition in (45) (for $k=1$) while the other follows analogously from (51).

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