REMARKS ON GENERALIZED SOLUTIONS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS IN THE COLOMBEAU ALGEBRA

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Abstract. In this paper first order linear ordinary differential equations are considered. It is shown that the Cauchy problem for these systems has a unique solution in $\mathcal{G}^n(\mathbb{R})$, where $\mathcal{G}(\mathbb{R})$ denotes the Colombeau algebra.

Keywords: generalized ordinary differential equations, the Cauchy problem, distributions, Colombeau algebra

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1. Introduction

We consider the Cauchy problem

$$x'_k(t) = \sum_{j=1}^{n} A_{kj}(t)x_j(t) + f_k(t), \tag{1.0}$$

$$x_k(t_0) = x_{0k}, \quad t_0 \in \mathbb{R}, \quad k = 1, \dots, n, \tag{1.1}$$

where A_{kj}, x_j and f_k are elements of the Colombeau algebra $\mathcal{G}(\mathbb{R})$, x_{0k} are known elements of the Colombeau algebra $\overline{\mathbb{C}}$ of generalized complex numbers, $x_k(t_0)$ is understood as the value of the generalized function x_k at the point t_0 and $k = 1, \dots, n$ (see [1]–[2]). Elements A_{kj} and f_k are given, elements x_k are unknown (for $k, j = 1, \dots, n$). Multiplication, derivative, sum and equality is meant in the Colombeau algebra sense. We prove theorems on existence and uniqueness of solutions of the Cauchy problem for the system (1.0). Our theorems generalize some results given in [1], [13].

2. Notation

Let $\mathcal{D}(\mathbb{R})$ be the space of all C^∞ functions $\mathbb{R} \to \mathbb{C}$ with compact support. For $q = 1, 2, \ldots$ we denote by $\mathcal{A}_q$ the set of all functions $\varphi \in \mathcal{D}(\mathbb{R})$ such that the relations

$$(2.1)\qquad \int_{-\infty}^{\infty} \varphi(t)\,\mathrm{d}t = 1, \quad \int_{-\infty}^{\infty} t^k\varphi(t)\,\mathrm{d}t = 0, \qquad 1 \leqslant k \leqslant q$$

hold.

Next, $\mathcal{E}[\mathbb{R}]$ is the set of all functions $R\colon \mathcal{A}_1 \times \mathbb{R} \to \mathbb{C}$ such that $R(\varphi, t) \in C^\infty(\mathbb{R})$ for each fixed $\varphi \in \mathcal{A}_1$.

If $R \in \mathcal{E}[\mathbb{R}]$, then $D_kR(\varphi,t)$ for any fixed φ denotes a differential operator in t (i.e. $D_kR(\varphi,t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}(R(\varphi,t))$ for $k \geqslant 1$ and $D_0R(\varphi,t) = R(\varphi,t)$).

For given $\varphi \in \mathcal{D}(\mathbb{R})$ and $\varepsilon > 0$, we define φ_ε by

$$(2.2)\qquad \varphi_\varepsilon(t) = \tfrac{1}{\varepsilon}\varphi\left(\tfrac{t}{\varepsilon}\right).$$

An element R of $\mathcal{E}[\mathbb{R}]$ is moderate if for every compact set K of $\mathbb{R}$ and every differential operator D_k there is $N \in \mathbb{N}$ such that the following condition holds: for every $\varphi \in \mathcal{A}_N$ there are $c > 0$ and $\varepsilon_0 > 0$ such that

$$(2.3)\qquad \sup_{t\in K} |D_kR(\varphi_\varepsilon,t)| \leqslant c\varepsilon^{-N} \qquad \text{for} \quad 0 < \varepsilon < \varepsilon_0.$$

We denote by $\mathcal{E}_M[\mathbb{R}]$ the set of all moderate elements of $\mathcal{E}[\mathbb{R}]$.

By Γ we denote the set of all increasing functions α from $\mathbb{N}$ into $\mathbb{R}^+$ such that $\alpha(q)$ tends to ∞ if $q \to \infty$.

We define an ideal $\mathcal{N}[\mathbb{R}]$ in $\mathcal{E}_M[\mathbb{R}]$ as follows: $R \in \mathcal{N}[\mathbb{R}]$ if for every compact set K of $\mathbb{R}$ and every differential operator D_k there are $N \in \mathbb{N}$ and $\alpha \in \Gamma$ such that the following condition holds: for every $q \geqslant N$ and $\varphi \in \mathcal{A}_q$ there are $c > 0$ and $\varepsilon_0 > 0$ such that

$$(2.4)\qquad \sup_{t\in K} |D_kR(\varphi_\varepsilon,t)| \leqslant c\varepsilon^{\alpha(q)-N} \qquad \text{if} \quad 0 < \varepsilon < \varepsilon_0.$$

The algebra $\mathcal{G}(\mathbb{R})$ (the Colombeau algebra) is defined as the quotient algebra of $\mathcal{E}_M[\mathbb{R}]$ with respect to $\mathcal{N}[\mathbb{R}]$ (see [1]).

We denote by $\mathcal{E}_0$ the set of all functions from $\mathcal{A}_1$ into $\mathbb{C}$. Next, we denote by $\mathcal{E}_M$ the set of all so-called moderate elements of $\mathcal{E}_0$ defined by

$$(2.5)\qquad \mathcal{E}_M = \{R \in \mathcal{E}_0\colon \text{ there is } N \in \mathbb{N} \text{ such that for every } \varphi \in \mathcal{A}_N \text{ there are } c > 0 \text{ and } \eta_0 \text{ such that } |R(\varphi_\varepsilon)| \leqslant c\varepsilon^{-N} \text{ if } 0 < \varepsilon < \eta_0\}$$

Further, we define an ideal $\mathcal{N}$ of $\mathcal{E}_M$ by

$$\text{(2.6)}\qquad \mathcal{N} = \{R \in \mathcal{E}_0\colon \text{ there are } N \in \mathbb{N} \text{ and } \alpha \in \Gamma \text{ such that for every } q \geqslant N \text{ and } \varphi \in \mathcal{A}_q \text{ there are } c > 0,\ \eta_0 > 0 \text{ such that } |R(\varphi_\varepsilon)| \leqslant c\varepsilon^{\alpha(q)-N} \text{ if } 0 < \varepsilon < \eta_0\}.$$

We define an algebra $\overline{\mathbb{C}}$ by setting $\overline{\mathbb{C}} = \frac{\mathcal{E}_M}{\mathcal{N}}$ (see [1]). It is known that $\overline{\mathbb{C}}$ is not a field. If $R \in \mathcal{E}_M[\mathbb{R}]$ is a representative of $G \in \mathcal{G}(\mathbb{R})$ then for a fixed t the map $Y\colon \varphi \to R(\varphi, t) \in \mathbb{C}$ is defined on $\mathcal{A}_1$ and $Y \in \mathcal{E}_M$. The class of Y in $\mathbb{C}$ depends only on G and t. This class is denoted by $G(t)$ and is called the value of the generalized function G at the point t (see [1]).

We say that $G \in \mathcal{G}(\mathbb{R})$ is a constant generalized function on $\mathbb{R}$ if it admits a representative $R(\varphi, t)$ which is independent of $t \in \mathbb{R}$. With any $Z \in \overline{\mathbb{C}}$ we associate a constant generalized function which admits $R(\varphi, t) = Z(\varphi)$ as its representative, provided we denote by Z a representative of Z (see [1]).

Throughout the paper K denotes a compact set in $\mathbb{R}$. We denote by $R_{A_{kj}}(\varphi, t)$, $R_{f_k}(\varphi, t), R_{x_{0j}}(\varphi), R_{x_j(t_0)}(\varphi), R_{x_j}(\varphi, t)$ and $R_{x'_j}(\varphi, t)$ representatives of elements $A_{kj}, f_k, x_{0j}, x_j(t_0), x_j$ and x'_j for $k, j = 1, \ldots, n$. Let $A(t) = (A_{kj}(t))$, $f(t) = (f_1(t), \ldots, f_n(t))^T$, $x(t) = (x_1(t), \ldots, x_n(t))^T$, $x'(t) = (x'_1(t), \ldots, x'_n(t))^T$, $x_0 = (x_{10}, \ldots, x_{n0})^T$, where T denotes the transpose. We put

$$\begin{aligned}
&R_A(\varphi, t) = (R_{A_{kj}}(\varphi, t)), \quad R_f(\varphi, t) = (R_{f_1}(\varphi, t), \ldots, R_{f_n}(\varphi, t))^T,\\
&R_x(\varphi, t) = (R_{x_1}(\varphi, t), \ldots, R_{x_n}(\varphi, t))^T, \quad R_{x'}(\varphi, t) = (R_{x'_1}(\varphi, t), \ldots, R_{x'_n}(\varphi, t))^T,\\
&R_{x_0}(\varphi) = (R_{x_{10}}(\varphi), \ldots, R_{x_{n0}}(\varphi))^T, \quad R_{x(t_0)}(\varphi) = (R_{x_1(t_0)}(\varphi), \ldots, R_{x_n(t_0)}(\varphi))^T,\\
&\int_{t_0}^{t} R_A(\varphi, s)\,\mathrm{d}s = \Big(\int_{t_0}^{t} R_{A_{kj}}(\varphi, s)\,\mathrm{d}s\Big),\\
&\int_{t_0}^{t} R_f(\varphi, s)\,\mathrm{d}s = \Big(\int_{t_0}^{t} R_{f_1}(\varphi, s)\,\mathrm{d}s, \ldots, \int_{t_0}^{t} R_{f_n}(\varphi, s)\,\mathrm{d}s\Big)^T,\\
&\|R_A(\varphi, t)\| = \Big(\sum_{k,j=1}^{n} |R_{A_{kj}}(\varphi, t)|^2\Big)^{1/2}, \quad \|R_f(\varphi, t)\| = \Big(\sum_{j=1}^{n} |R_{f_j}(\varphi, t)|^2\Big)^{1/2},\\
&\|R_A(\varphi, t)\|_K = \sup_{t \in K} \|R_A(\varphi, t)\|, \quad \|R_f(\varphi, t)\|_K = \sup_{t \in K} \|R_f(\varphi, t)\|.
\end{aligned}$$

We say that a generalized function G is real valued if it admits a real valued representative. Starting with those elements of $\mathcal{E}_0$ which are real valued we obtain in this way an algebra $\overline{\mathbb{R}}$ containing $\mathbb{R}$ as a subalgebra. Thus $\overline{\mathbb{C}} = \overline{\mathbb{R}} + \mathrm{i}\overline{\mathbb{R}}$, where $\mathrm{i}^2 = -1$ (see [1]).

If $A_{kj}, f_j \in \mathcal{G}(\mathbb{R})$, $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, $a_{kj}, b_j \in \mathcal{N}[\mathbb{R}]$, $m_{kj}, p_j \in \mathcal{N}$, $q_j \in \mathbb{C}$, $r_j \in \overline{\mathbb{C}}$ for $k, j = 1, \ldots, n$, then we write

$A = (A_{kj}) \in \mathcal{G}^{n\times n}(\mathbb{R})$, $f = (f_1, \ldots, f_n)^T \in \mathcal{G}^n(\mathbb{R})$,
$b = (b_1, \ldots, b_n)^T \in \mathcal{N}^n[\mathbb{R}]$, $a = (a_{kj}) \in \mathcal{N}^{n\times n}[\mathbb{R}]$,
$m = (m_{kj}) \in \mathcal{N}^{n\times n}$, $p = (p_1, \ldots, p_n)^T \in \mathcal{N}^n$,
$q = (q_1, \ldots, q_n)^T \in \mathbb{C}^n$, $r = (r_1, \ldots, r_n)^T \in \overline{\mathbb{C}}^n$,
$R_A(\varphi, t) \in \mathcal{E}^{n\times n}[\mathbb{R}]$, $R_x(\varphi, t) \in \mathcal{E}_M^n[\mathbb{R}]$ and $(u, v) = \sum\limits_{j=1}^{n} u_j v_j$.

We say that $x = (x_1, \ldots, x_n)^T \in \mathcal{G}^n(\mathbb{R})$ is a solution of the system (1.0) if x satisfies (1.0) in $\mathcal{G}^n(\mathbb{R})$; i.e. if $R_x(\varphi, t)$ is a representative of x, then there is $\eta \in \mathcal{N}^n[\mathbb{R}]$ such that

$$R_{x'}(\varphi, t) = R_A(\varphi, t) R_x(\varphi, t) + \eta(\varphi, t)$$

(for all $\varphi \in \mathcal{A}_1$ and $t \in \mathbb{R}$).

Let $U \in \mathcal{G}(\mathbb{R})$. U will be called locally of logarithmic growth with an exponent r, $r \geqslant 0$ if it has a representative $R_U(\varphi, t)$ with the property: for every compact subset $K \subset \mathbb{R}$ there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there exist constants $\varepsilon_0 > 0, c > 0$ such that

$$\|R_U(\varphi_\varepsilon, t)\|_K \leqslant \left(N \log \tfrac{c}{\varepsilon}\right)^r \qquad \text{for} \quad 0 < \varepsilon < \varepsilon_0.$$

If $r = 1$, then we say that U is locally of logarithmic growth (see [14]).

By $\mathcal{G}^2_{\frac{1}{2}}(\mathbb{R})$ we denote the set of all elements $y \in \mathcal{G}(\mathbb{R})$ such that y has locally logarithmic growth with the exponent $\frac{1}{2}$.

The symbol $\tilde{H}^s_{\frac{1}{2}}(\mathbb{R})$ denotes the set of all elements $y \in \mathcal{G}(\mathbb{R})$ such that $y^{(s)} \in \mathcal{G}^2_{\frac{1}{2}}(\mathbb{R})$. For $s = 0$ we put $\tilde{H}^o_{\frac{1}{2}}(\mathbb{R}) = \mathcal{G}^2_{\frac{1}{2}}(\mathbb{R})$.

By $g^{(-l)}$ we denote a generalized function whose l-th derivative is equal g (i.e. $(g^{(-l)})^{(l)} = g$. A function y belongs to $H^s(\mathbb{R})$ $(s \in \mathbb{N})$ if and only if $y^{(s)}$ (in the distributional sense) belongs to $L^2_{\mathrm{loc}}(\mathbb{R})$. For $s = 0$ we put $H^o(\mathbb{R}) = L^2_{\mathrm{loc}}(\mathbb{R})$.

A function y is said to belong to the class $\tilde{C}(\mathbb{R})$ if y is an absolutely continuous function in every compact interval $K \subset \mathbb{R}$. We put

$$\tilde{C}_n(\mathbb{R}) = \underbrace{\tilde{C}(\mathbb{R}) \times \ldots \times \tilde{C}(\mathbb{R})}_{n\text{-times}}, \qquad \tilde{C}(\mathbb{R}) = \tilde{C}_1(\mathbb{R}).$$

The symbol $L^{2(k)}_{\mathrm{loc}}(\mathbb{R})$ denotes the set of all k-th derivatives (in the distributional sense) of the functions of class $L^2_{\mathrm{loc}}(\mathbb{R})$.

3. The main results

First we will introduce two hypotheses.

Hypothesis H_1.

$$A \in \mathcal{G}^{n\times n}(\mathbb{R}), \quad f \in \mathcal{G}^n(\mathbb{R}), \tag{3.0}$$

the matrix $A \in \mathcal{G}^{n\times n}(\mathbb{R})$ admits a representative $R_A(\varphi,t) = (R_{A_{kj}}(\varphi,t))$ having at least one of the following four properties:

$$R_{A_{kj}}(\varphi,t) \in \mathbb{R} \quad \text{for every} \quad \varphi \in \mathcal{A}_1 \text{ and } k,j = 1,\ldots,n, \tag{3.1}$$

$$R_A(\varphi,t) = (R_A(\varphi,t))^T \quad \text{for every} \quad \varphi \in \mathcal{A}_1, \tag{3.2}$$

$$R_A(\varphi,t) = -(R_A(\varphi,t))^T \quad \text{for every} \quad \varphi \in \mathcal{A}_1, \tag{3.3}$$

$$R_{A_{kj}}(\varphi,t) \ \text{ is locally of logarithmic growth for } \ k,j = 1,\ldots,n. \tag{3.4}$$

Hypothesis H_2.

The matrix A admits a representative $R_A(\varphi,t) = (R_{A_{kj}}(\varphi,t))$ with the following property: for every K there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $c > 0$ and $\varepsilon_0 > 0$ satisfying at least one of the following six conditions:

$$\Big\| \int_0^t |R_{A_{kj}}(\varphi_\varepsilon,s)|\,\mathrm{d}s \Big\|_K \leqslant c \quad \text{for } 0<\varepsilon<\varepsilon_0,\ k,j=1,\ldots,n; \tag{3.5}$$

$$\exp\Big(\Big\| \int_0^t |R_{A_{kj}}(\varphi_\varepsilon,s)|\,\mathrm{d}s \Big\|_K \Big) \leqslant \frac{c}{\varepsilon^N} \quad \text{for } 0<\varepsilon<\varepsilon_0,\ k,j=1,\ldots,n; \tag{3.6}$$

$$R_{A_{kj}}(\varphi_\varepsilon,t) = 0 \quad \text{for } t\in K,\ k<j,\ n>1 \text{ and} \tag{3.7}$$
$$\Big\| \int_0^t R_{A_{jj}}(\varphi_\varepsilon,s)\,\mathrm{d}s \Big\|_K \leqslant c \quad \text{if} \quad 0<\varepsilon<\varepsilon_0,\ j=1,\ldots,n;$$

$$R_{A_{kj}}(\varphi_\varepsilon,t) = 0 \quad \text{for } t\in K,\ k<j,\ n>1 \text{ and} \tag{3.8}$$
$$\exp\Big(\Big\| \int_0^t R_{A_{jj}}(\varphi_\varepsilon,s)\,\mathrm{d}s \Big\|_K \Big) \leqslant \frac{c}{\varepsilon^N} \quad \text{if } 0<\varepsilon<\varepsilon_0, \quad j=1,\ldots,n;$$

$$R_{A_{kj}}(\varphi_\varepsilon,t) = -R_{A_{jk}}(\varphi_\varepsilon,t) \quad \text{for } t\in K,\ j\neq k,\ n>1,\ j,k=2.\ldots,n \text{ and} \tag{3.9}$$
$$\exp\Big(\Big\| \int_0^t |R_{A_{jj}}(\varphi_\varepsilon,s)|\,\mathrm{d}s \Big\|_K \Big) \leqslant \frac{c}{\varepsilon^N} \quad \text{if} \quad 0<\varepsilon<\varepsilon_0, \quad j=1,\ldots,n;$$

(3.10) $A_0, A_1, \ldots, A_{n-k}, A_n \in \mathcal{G}^2_{\frac{1}{2}}(\mathbb{R}), \quad A_j^{(n-k-j)} \in \mathcal{G}^2_{\frac{1}{2}}(\mathbb{R}), \quad 2k \leqslant n, \quad k \geqslant 1,$
$j = n-k+1, \ldots, n-1.$

We shall consider the problems

$$x'(t) = \mathrm{i}A(t)x(t) + f(t), \quad x(t_0) = x_0, \quad t_0 \in \mathbb{R}, \quad x_0 \in \overline{\mathbb{C}}^n \tag{3.11$'$}$$

and

$$x'(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0, \quad t_0 \in \mathbb{R}, \quad x_0 \in \overline{\mathbb{C}}^n, \tag{3.11}$$

where $\mathrm{i}^2 = -1$.

Remark 3.1. If A and f have properties (3.0)–(3.2), then problem (3.11$'$) has exactly one solution $x \in \mathcal{G}^n(\mathbb{R})$ (see [13]). Besides, every solution x of system (3.11$'$) has a representation

$$x(t) = Z(t)c + Q(t), \tag{3.12}$$

where Z is a solution of the problem

$$Z'(t) = \mathrm{i}A(t)Z(t), \quad Z(t_0) = I, \tag{3.13}$$

I denotes the identity matrix, $c = (c_1, \ldots, c_n)^T$, c_j are generalized constant functions on $\mathbb{R}$ for $j = 1, \ldots, n$ and Q is a particular solution of system (3.11$'$). The solution x is the class of solutions of the problems

$$x'(t) = \mathrm{i}R_A(\varphi, t)x(t) + R_f(\varphi, t), \quad x(t_0) = R_{x_0}(\varphi), \quad \varphi \in \mathcal{A}_1 \quad \text{(see [13])}. \tag{3.14}$$

Remark 3.2. If $A \in \mathcal{G}^{n\times n}(\mathbb{R})$, $f \in \mathcal{G}^n(\mathbb{R})$ and at least one of conditions (3.3), (3.5), (3.7) is satisfied, then problem (3.11) has exactly one solution in $\mathcal{G}^n(\mathbb{R})$. Besides, every solution x of system (3.11) has a representation (3.12), where Z is a solution of the problem

$$Z'(t) = A(t)Z(t), \qquad Z(t_0) = I. \tag{3.15}$$

The solution x of problem (3.11) is the class of solutions of the problems

$$x'(t) = R_A(\varphi, t)x(t) + R_f(\varphi, t), \quad x(t_0) = R_{x_0}(\varphi), \quad \varphi \in \mathcal{A}_1, \tag{3.16}$$

(see [13]).

Remark 3.3. If δ denotes the generalized function (the delta Dirac's generalized function), which admits as the representative the function $R_\delta(\varphi, t) = \varphi(-t)$, then $R_\delta(\varphi, t)$ has property (3.5). It is not difficult to show that the problem

$$x'(t) = (\delta^2(t))'x(t), \qquad x(-1) = 1$$

has no solution in $\mathcal{G}(\mathbb{R})$ (see [13]).

R e m a r k 3.4. If the matrix A has property (3.4) or (3.5), then it has property (3.6). Note that properties (3.4) and (3.5) are independent. Indeed, take two generalized functions δ and d defined by

$$\delta = [\varphi(-t)], \quad d = [\ln \tfrac{1}{L(\varphi)}],$$

where $\varphi \in \mathcal{A}_1$ and $L(\varphi) = \sup\{|t| \colon \varphi(t) \neq 0\}$. It is not difficult to verify that δ satisfies (3.5) but d does not. Moreover, d has property (3.4), but δ has not this property.

If the matrix A has property (3.7), then it has property (3.8).

Theorem 3.1. *We assume conditions* (3.0) *and* (3.6). *Then problem* (3.11) *has exactly one solution* $x \in \mathcal{G}^n(\mathbb{R})$. *Besides, every solution* $x \in \mathcal{G}^n(\mathbb{R})$ *of equation* (3.11) *has properties* (3.12) *and it is the class of solutions of the problems* (3.16).

P r o o f. The proof of the theorem is similar to the proof of Theorem 4.2 in [13]. To this purpose we examine relation (3.16). Let $x(\varphi, t)$ be a solution of problem (3.16). Then

$$x(\varphi, t) = R_Z(\varphi, t) R_V(\varphi, t), \tag{3.17}$$

where

$$D_1 R_V(\varphi, t) = (R_Z(\varphi, t))^{-1} R_f(\varphi, t), \tag{3.18}$$

$$R_V(\varphi, t_0) = R_{x_0}(\varphi) \tag{3.19}$$

and $R_Z(\varphi, t)$ is a solution of problem (3.15). On the other hand,

$$D_1((R_Z(\varphi, t))^{-1}) = -((R_Z(\varphi, t))^{-1}) R_A(\varphi, t). \tag{3.20}$$

Using the Gronwall inequality and relations (3.6), (3.15) and (3.20) we obtain

$$\|R_U(\varphi_\varepsilon, t)\|_K \leqslant \sqrt{n} \exp\left(\left\| \int_{t_0}^{t} \|R_A(\varphi_\varepsilon, s)\| \,\mathrm{d}s \right\|_K\right) \leqslant \frac{\sqrt{n} c}{\varepsilon^N}, \tag{3.21}$$

where $0 < \varepsilon < \varepsilon_0$, $\varphi \in \mathcal{A}_N$ and $R_U(\varphi_\varepsilon, t) = R_Z(\varphi_\varepsilon, t)$ or $R_U(\varphi_\varepsilon, t) = (R_Z(\varphi_\varepsilon, t))^{-1}$.

By (3.20)–(3.21) there is $N_r \in \mathbb{N}$ such that for $\varphi \in \mathcal{A}_{N_r}$ and $0 < \varepsilon < \varepsilon_0$ we have

$$\|D_r R_U(\varphi_\varepsilon, t)\|_K \leqslant C_r \varepsilon^{-Nr}. \tag{3.22}$$

Hence

$$R_U(\varphi, t) \in \mathcal{E}_M^{n \times n}[\mathbb{R}]. \tag{3.23}$$

Relations (3.17), (3.18), (3.23) yield

$$x(\varphi,t)\in\mathcal{E}^n_M[\mathbb{R}]. \tag{3.24}$$

Denoting by x the class of $x(\varphi,t)$ in $\mathcal{G}^n(\mathbb{R})$, we get that x is a solution of problem (3.11). Let $y\in\mathcal{G}^n(\mathbb{R})$ be another solution of problem (3.11). Then

$$D_1R_y(\varphi,t)=R_A(\varphi,t)R_y(\varphi,t)+R_f(\varphi,t)+R_m(\varphi,t), \tag{3.25}$$

where

$$R_m(\varphi,t)\in\mathcal{N}^n[\mathbb{R}] \tag{3.26}$$

and

$$R_y(\varphi,t_0)-x(\varphi,t_0)\in\mathcal{N}^n. \tag{3.27}$$

In view of (3.16), (3.25)–(3.27) and the Gronwall inequality we conclude that (for $q\geqslant N',\ \varphi\in\mathcal{A}_q,\ 0<\varepsilon<\varepsilon_0'$)

$$\begin{aligned}\|x(\varphi_\varepsilon,t)-R_y(\varphi_\varepsilon,t)\|_K&\leqslant(\|x(\varphi_\varepsilon,t_0)-R_y(\varphi_\varepsilon,t_0)\|\\&\quad+\|R_m(\varphi_\varepsilon,t)\|_K)\exp\left(\left\|\int_{t_0}^{t}\|R_A(\varphi_\varepsilon,s)\|\,\mathrm{d}s\right\|_K\right)\leqslant c_0\varepsilon^{\alpha(q)-N'}.\end{aligned} \tag{3.28}$$

This yields

$$\|D_r(x(\varphi_\varepsilon,t)-R_y(\varphi_\varepsilon,t))\|_K\leqslant c_r\varepsilon^{\alpha(q)-N_r'} \tag{3.29}$$

for $\varphi\in\mathcal{A}_{q_r},\ \ q_r\geqslant N'r$ and $0<\varepsilon<\varepsilon_r'$ and consequently

$$x(\varphi,t)-R_y(\varphi,t)\in\mathcal{N}^n[\mathbb{R}]. \tag{3.30}$$

This proves the theorem. □

Theorem 3.2. *We assume conditions* (3.0) *and* (3.9). *Then problem* (3.11) *has exactly one solution* $x\in\mathcal{G}^n(\mathbb{R})$. *Besides, every solution* $x\in\mathcal{G}^n(\mathbb{R})$ *of equation* (3.11) *has properties* (3.12) *and it is the class of solutions of the problems* (3.16).

First we shall prove two lemmas.

Lemma 3.1. *We assume conditions* (3.0) *and* (3.9). *Then the problem*

$$x'(t)=A(t)x(t),\quad x(t_0)=x_0,\ x_0\in\overline{\mathbb{C}}^n,\ t_0\in\mathbb{R} \tag{3.31}$$

has exactly one solution $x\in\mathcal{G}^n(\mathbb{R})$.

P r o o f of Lemma 3.1. For a fixed $\varphi \in \mathcal{A}_1$ the problem

$$x'(t) = R_A(\varphi, t)x(t), \quad x(t_0) = R_{x_0}(\varphi) \tag{3.32}$$

has exactly one solution $x(\varphi, t)$ on $\mathbb{R}$. By (3.9) we get

$$\begin{aligned} &\int_{t_0}^{t} \big((x(\varphi_\varepsilon, s))^T, x'(\varphi_\varepsilon, s)\big)\,\mathrm{d}s = \int_{t_0}^{t} ((x(\varphi_\varepsilon, s))^T, R_A(\varphi_\varepsilon, s)x(\varphi_\varepsilon, s))\,\mathrm{d}s \\ &= \int_{t_0}^{t} \Big(\sum_{j=1}^{n} R_{A_{jj}}(\varphi_\varepsilon, s)x_j^2(\varphi_\varepsilon, s)\Big)\,\mathrm{d}s = \tfrac{1}{2}\Big(\sum_{j=1}^{n} x_j^2(\varphi_\varepsilon, t) - \sum_{j=1}^{n} x_j^2(\varphi_\varepsilon, t_0)\Big). \end{aligned} \tag{3.33}$$

Using the Gronwall inequality we have

$$\|(\|x(\varphi_\varepsilon, t)\|^2)\|_K \leqslant \ \exp\bigg(\bigg\|\int_{t_0}^{t} 2T_rR_{|A|}(\varphi_\varepsilon, s)\,\mathrm{d}s\bigg\|_K\bigg)\|x(\varphi_\varepsilon, t_0)\|^2 \leqslant \frac{c_0}{\varepsilon^N} \tag{3.34}$$

for $0 < \varepsilon < \varepsilon_0$ and $\varphi \in \mathcal{A}_N$, where

$$T_rR_{|A|}(\varphi_\varepsilon, s) = \sum_{j=1}^{n} |R_{A_{jj}}(\varphi_\varepsilon, s)|.$$

Hence

$$\|D_r x(\varphi_\varepsilon, t)\|_K \leqslant Cr\varepsilon^{-N'r} \tag{3.35}$$

for $0 < \varepsilon < \varepsilon'_r$ and $\varphi \in \mathcal{A}_{N'_r}$.

So $x(\varphi, t) \in \mathcal{E}_M^n[\mathbb{R}]$. We denote by x the class of $x(\varphi, t)$ in $\mathcal{G}^n(\mathbb{R})$. Therefore x is a solution of problem (3.31). If $y \in \mathcal{G}^n(\mathbb{R})$ is another solution of problem (3.31), then

$$D_1R_y(\varphi, t) = R_A(\varphi, t)R_y(\varphi, t) + \eta(\varphi, t), \tag{3.36}$$

where

$$\eta(\varphi, t) \in \mathcal{N}^n[\mathbb{R}] \tag{3.37}$$

and

$$R_y(\varphi, t_0) - x(\varphi, t_0) \in \mathcal{N}^n. \tag{3.38}$$

In view of (3.9), (3.32) and (3.36) we deduce that

$$\tag{3.39} \begin{aligned} &\int\limits_{t_0}^{t} \big(((x(\varphi_\varepsilon, s))^T, x'(\varphi_\varepsilon)) - ((R_y(\varphi_\varepsilon, s))^T, R_y'(\varphi_\varepsilon, s))\big)\,\mathrm{d}s \\ &= \frac{1}{2}\bigg(\sum_{j=1}^{n}(x_j^2(\varphi_\varepsilon, t) - R_{yj}^2(\varphi_\varepsilon, t))\bigg) - \frac{1}{2}\bigg(\sum_{j=1}^{n}(x_j^2(\varphi_\varepsilon, t_0) - R_{yj}^2(\varphi_\varepsilon, t_0))\bigg) \\ &= \int\limits_{t_0}^{t}\bigg(\sum_{j=1}^{n} R_{A_{jj}}(\varphi_\varepsilon, s)(x_j^2(\varphi_\varepsilon, s) - R_{yj}^2(\varphi_\varepsilon, s))\bigg)\,\mathrm{d}s \\ &\quad - \int\limits_{t_0}^{t}(R_y^T(\varphi_\varepsilon, s)), R_\eta(\varphi_\varepsilon, s))\,\mathrm{d}s. \end{aligned}$$

Evidently

$$\tag{3.40} \int\limits_{t_0}^{t}(R_y^T(\varphi, s)), R_\eta(\varphi, s))\,\mathrm{d}s = \tilde{\eta}(\varphi, t) \in \mathcal{N}[\mathbb{R}].$$

Taking into account (3.39)–(3.40) and the Gronwall inequality we infer that

$$\tag{3.41} \begin{aligned} &\|(\|x(\varphi_\varepsilon, t) - R_y(\varphi_\varepsilon, t)\|^2)\|_K \leqslant (\|x(\varphi_\varepsilon, t_0) - R_y(\varphi_\varepsilon, t_0)\|^2 \\ &\quad + 2\|\tilde{\eta}(\varphi_\varepsilon, t)\|_K) \cdot \exp\Bigg(\bigg\|\int\limits_{t_0}^{t} 2TrR_{|A|}(\varphi_\varepsilon, s)\,\mathrm{d}s\bigg\|_K\Bigg) \leqslant c_0'\varepsilon^{\alpha(q)-N_0'} \end{aligned}$$

for $0 < \varepsilon < \varepsilon_0'$, $\varphi \in \mathcal{A}_q$ and $q \geqslant N_0'$.

This yields

$$\tag{3.42} \|D_r(x(\varphi_\varepsilon, t) - R_y(\varphi_\varepsilon, t)\|_K \leqslant C_r\varepsilon^{\alpha(q)-N_r'}$$

(for $\varphi \in \mathcal{A}_{q_r}$, $q_r \geqslant N_r'$ and $0 < \varepsilon < \varepsilon_r'$).

Thus, by (3.42) we obtain

$$x(\varphi, t) - R_y(\varphi, t) \in \mathcal{N}^n[\mathbb{R}].$$

This proves Lemma 3.1. □

From Lemma 3.1 we get

Lemma 3.2. *We assume conditions* (3.0), (3.9). *Then problem* (3.15) *has exactly one solution* $Z \in \mathcal{G}^{n\times n}(\mathbb{R})$.

P r o o f of Theorem 3.2. Proof of Theorem 3.2 is similar to the proof of Theorem 3.1. We start from relations (3.17)–(3.19), where $x(\varphi,t)$ is a solution of problem (3.16). Taking into account (3.34)–(3.35) we show relation (3.23). So, by virtue of (3.42) and (3.17) we obtain (3.24). The uniqueness of a solution of problem (3.11) follows from Lemma 3.1. $\square$

Theorem 3.3. *We assume conditions* (3.0), (3.8). *Then problem* (3.11) *has exactly one solution* $x \in \mathcal{G}^n(\mathbb{R})$. *Besides, every solution* $x \in \mathcal{G}^n(\mathbb{R})$ *of equation* (3.11) *has properties* (3.12) *and it is the class of solutions of the problems* (3.16).

P r o o f. Proof of Theorem 3.3 is similar to the proof of Theorem 4.2 in [13]. First we examine the problem

$$(3.43)_{1k} \qquad z_{1k}'(t) = R_{A_{11}}(\varphi,t)z_{1k}(t)$$

$$\dots$$

$$(3.43)_{nk} \qquad z_{nk}'(t) = R_{A_{n1}}(\varphi,t)z_{1k}(t) + \dots + R_{A_{nn}}(\varphi,t)z_{nk}(t),$$

$$(3.44) \qquad z_{kj}(t_0) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \quad \text{where} \quad k,j = \dots, n.$$

From $(3.43)_{1k}$–$(3.43)_{nk}$, (3.44) we infer that

$$(3.45) \qquad \|z_{1k}(\varphi_\varepsilon,t)\|_K \leqslant \exp\left(\left\|\int_{t_0}^{t} R_{A_{11}}(\varphi_\varepsilon,s)\,\mathrm{d}s\right\|_K\right)$$

and

$$(3.46) \qquad \|D_r z_{1k}(\varphi_\varepsilon,t)\|_K \leqslant c_1\varepsilon^{-N_1}$$

(for $\varphi \in \mathcal{A}_{N_1}$, $0 < \varepsilon < \varepsilon_1$ and $k = 1,\dots,n$), where $(z_{1k}(\varphi,t),\dots,z_{nk}(\varphi,t))^T$ is a solution of problem $(3.43)_{1k}$–(3.44). If $n > 1$, then $(3.43)_{nk}$, (3.45) and (3.8) imply

$$(3.47) \qquad \|D_r(z_{mk}(\varphi_\varepsilon,t)\|_K \leqslant c_m\varepsilon^{-N_m}$$

for $0 < \varepsilon < \varepsilon_m,\ \varphi \in \mathcal{A}_{N_m}$.

Let $x(\varphi,t)$ be a solution of problem (3.16). Then $x(\varphi,t)$ has the properties (3.17)–(3.20), (3.23), (3.25)–(3.27), (3.29), which completes the proof of the theorem. $\square$

Corollary 3.1. *We assume that*

$$(3.48) \qquad a_j, f \in \mathcal{G}(\mathbb{R}) \quad \text{for} \quad j = 1,\dots,n;$$

(3.49) *all elements* a_j *(for* $j = 1,\dots,n$*) have property* (3.6).

Then the problem

$$y^{(n)}(t)+\sum_{j=0}^{n-1}a_j(t)y^{(j)}(t)+f(t)=0, \tag{3.50}$$

$$y^{(i)}(t_0)=y_{0i},\quad y_{0i}\in\overline{\mathbb{C}},\quad t_0\in\mathbb{R},\quad i=0,1,\ldots,n-1 \tag{3.51}$$

has exactly one solution $y\in\mathcal{G}(\mathbb{R})$.

Now we shall consider linear differential equations of order n of the form

$$B_{nk}(y)+A_n^{(k)}(t)=0, \tag{3.52}$$

where

$$B_{nk}(y)=y^{(n)}(t)+\sum_{i=0}^{n-2k}A_i^{(k)}(t)y^{(i)}(t)+\sum_{i=n-2k+1}^{n-1}A_i^{(n-k-i)}(t)y^{(i)}(t), \tag{3.53}$$

$k\geqslant 1,\ 2k\leqslant n$ and A_j have property (3.10) (for $j=0,1,\ldots,n-1$).

Remark 3.5. It is worth noting that if $y\in\tilde{H}_r^s(\mathbb{R})$, then y does not belong to $\mathcal{G}_r^2(\mathbb{R})$ in general. In fact, let $y(t)=1+\frac{1}{[L(\varphi)]}$, where $L(\varphi)$ is defined by Remark 3.4. Then $y\in\tilde{H}_1^1(\mathbb{R})$ and $y\notin\mathcal{G}_{\frac{1}{2}}^2(\mathbb{R})$. If $G_1,G_2\in\mathcal{G}_{\frac{1}{2}}^2(\mathbb{R})$, then for every K there is $N\in\mathbb{N}$ such that for every $\varphi\in\mathcal{A}_N$ there exist $\varepsilon_0>0$ and $c>0$ such that

$$\|R_{G_1}(\varphi_\varepsilon,t)R_{G_2}(\varphi_\varepsilon,t)\|_K\leqslant N\log(\tfrac{c}{\varepsilon})\quad\text{and}\quad \tilde{y}=\Bigl[\int_0^t R_{G_1}(\varphi,s)\,\mathrm{d}s\Bigr]\in\mathcal{G}_{\frac{1}{2}}^2(\mathbb{R}) \tag{3.54}$$

for $0<\varepsilon<\varepsilon_0$.

Remark 3.6. To the linear differential equation (3.52) we shall apply a modification of the transformation introduced by R. Pfaff in [17].

We denote

$$\begin{aligned} &N_{j,u}=(-1)^j\binom{u}{j}\quad\text{for}\quad j,u\in\mathbb{N}_0,\\ &m_{ijl}=n-2k+l-i-j,\quad n_{ij}=n-k-i+j,\\ &p_{ijl}=i+l-n-j,\quad v=n-k,\quad m=n-2k, \end{aligned} \tag{3.55}$$

and for $l = 1, \dots, k-1$

$$
\begin{aligned}
z_0 &= y,\\
z_1 &= y',\\
&\dots\\
z_{v-1} &= y^{(v-1)},\\
z_v &= \sum_{i=0}^{m} Ay^{(i)} + y^{(v)} + A_n, \qquad (3.56)\\
z_{v+l} &= \sum_{i=0}^{m}\sum_{j=0}^{l} N_{j,k}(A_i y^{(i+j)})^{(l-j)} + \sum_{l=m+1}^{m+l}\sum_{j=0}^{m+l-i} N_{j,v-i}(A_i y^{(i+j)})^{(m_{ijl})}\\
&\quad + \sum_{l=n-l}^{n-1}\sum_{j=0}^{i+l-n} N_{j,i-v}(y^{(v)}A_i^{(n_{ij})})^{(P_{ijl})} + y^{(v+l)} + A_n^{(l)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
z_0' &= z_1,\\
z_1' &= z_2,\\
&\dots\\
z_{v-2}' &= z_{v-1},\\
z_{v-1}' &= \sum_{i=0}^{m} A_i z_i + z_v - A_n,\\
z_v' &= A_0 A_{n-1}^{(-k+1)} z_0 + \sum_{i=1}^{m}(kA_{i-1} + A_i A_{n-1}^{(-k+1)})z_i\\
&\quad + (kA_m - A_{m+1})z_{m+1} - A_{n-1}^{(-k+1)}(z_v - A_n) + z_{v+1},\\
z_{v+l-1}' &= \sum_{i=0}^{m}\left(\sum_{j=1}^{l} N_{l-j,k-j}A_{n-j}^{(l-k)}\right)A_i z_i \qquad (3.57)\\
&\quad - \sum_{i=l}^{m+l} N_{l,k}A_{i-l}z_i - \left(\sum_{j=1}^{l} N_{l-j,k-j}A_{m+j}\right)z_{m+l}\\
&\quad - \left(\sum_{j=1}^{l} N_{l-j,k-j}A_{n-j}^{(l-k)}\right)(z_v - A_n) + z_{v+l},\\
z_{n-1}' &= \sum_{i=0}^{m}\left(\sum_{j=m}^{n-1}(-1)^{v-j}A_j\right)A_i z_i\\
&\quad - \left(\sum_{i=k}^{v-1}(-1)^k A_{i-k} z_i\right) - \left(\sum_{j=m}^{n-1}(-1)^{v-j}A_j\right)(z_v - A_n),
\end{aligned}
$$

where $l = 2, \dots, k-1$ (see [17], [12]).

Obviously $y \in \mathcal{G}(\mathbb{R})$ is a solution of equation (3.52) if and only if $z = (z_0, \dots, z_{n-1})^T \in \mathcal{G}^n(\mathbb{R})$ is a solution of system (3.57). System (3.57) will be written in the simplified form:

$$z'(t) = A(t)z(t) + b(t). \tag{3.58}$$

By virtue of (3.53) we infer that system (3.58) is a system of linear differential equations with coefficients locally of logarithmic growth. We can consider system (3.58) with the conditions

$$z_i(t_0) = z_{i0}, \quad z_{i0} \in \overline{\mathbb{C}},\ t_0 \in \mathbb{R}, \quad i = 0, 1, \dots, n-1. \tag{3.59}$$

Applying Theorem 3.1 we can show that problem (3.58)–(3.59) has exactly one solution $z \in \mathcal{G}^n(\mathbb{R})$. Hence, by (3.56)–(3.59), we get that problem (3.51), (3.52) has exactly one solution $y \in \mathcal{G}(\mathbb{R})$.

R e m a r k 3.7. We assume that $A_0, A_1, \dots, A_{n-k},\ \ A_n \in L^2_{\mathrm{loc}}(\mathbb{R}),\ A_j^{(n-k-j)} \in H^{j-n+k}_{\mathrm{loc}}(\mathbb{R}),\ j = n-k+1, \dots, n+1,\ k \geqslant 1,\ 2k \leqslant n,\ z_{r_0} \in \mathbb{R}$ for $r = 0, 1, \dots, n-1$, derivative is understood in the distributional sense, product of distributions $f \in L^{2(p)}_{\mathrm{loc}}(\mathbb{R}),\ G \in H^p_{\mathrm{loc}}(\mathbb{R})$ is defined by

$$fG = Gf = \sum_{i=0}^{p} (-1)^i \binom{p}{i} (FG^{(i)})^{(p-i)},$$

where $F^{(p)} = f$. Then problem (3.58)–(3.59) has exactly one solution $z \in C_n(\mathbb{R})$ (in the distributional sense) and $y \in H^{n-k}_{\mathrm{loc}}(\mathbb{R})$ (see [12], [17]).

4. Final remarks

R e m a r k 4.1. Let all elements of a matrix A and vector f be $C^\infty(\mathbb{R})$ on $\mathbb{R}$. Moreover, let $x_0 \in \mathbb{C}$. Then the classical and the generalized solutions (i.e. solutions in the Colombeau algebra) of problem (1.0)–(1.1) give rise to the same element of $\mathcal{G}^n(\mathbb{R})$. If $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, we define

$$R_f(\varphi, t) = \int_{-\infty}^{\infty} f(t+u)\varphi(u)\,\mathrm{d}u, \quad \varphi \in \mathcal{A}_1. \tag{4.1}$$

Obviously $R_f(\varphi, t) \in \mathcal{E}_M[\mathbb{R}]$ and $R_f(\varphi, t)$ has properties (3.4)–(3.5).

It is known that every distribution is moderate (see [1]). If f_1 and f_2 are not elements of the space $C^\infty(\mathbb{R})$, then the product $f_1 \circ f_2$ in the Colombeau algebra and the classical product $f_1 \cdot f_2$ do not give rise to the same element of $\mathcal{G}(\mathbb{R})$ in general. Hence we observe that the classical solutions (in the Caratheodory sense) and the generalized solutions (in the Colombeau sense) are different in general. The solutions are equal in the weaker sense.

An element U of $\mathcal{G}(\mathbb{R})$ is said to admit a number $W \in \mathcal{D}'(\mathbb{R})$ as the associated distribution if it has a representative $R_U(\varphi, t)$ with the following property: for every $\psi \in \mathcal{D}(\mathbb{R})$ there is $N \in \mathbb{N}$ such the for every $\varphi \in \mathcal{A}_N$ we have

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} R_U(\varphi_\varepsilon, t)\psi(t)\,\mathrm{d}t = W(\psi)$$

(see [1]). If $x = (x_1, \ldots, x_n)^T$ is a solution of problem (1.0)–(1.1) in the Caratheodory sense ($x_{0k} \in \mathbb{R}$, $A_{kj}, f_k \in L^1_{\mathrm{loc}}(\mathbb{R})$ for $k, j = 1, \ldots, n$) and $\overline{x} = (\overline{x_1}, \ldots, \overline{x_n})^T \in \mathcal{G}^n(\mathbb{R})$ is a solution of the problem

$$x'_k(t) = \sum_{j=1}^{n} A_{kj}(t) \circ x_j(t) + f_k(t), \quad x_k(t_0) = x_{0k}, \ k = 1, \ldots, n,$$

then $\overline{x_i}$ admits an associated distribution which equls x_i for $i = 1, \ldots, n$ (see [13]).

Remark 4.2. L. Schwartz proved in [20] that there exists no algebra $\tilde{A}$ such that: the algebra of continuous functions on $\mathbb{R}$ is a subalgebra of $\tilde{A}$, the function 1 is the unit element of $\tilde{A}$, elements of $\tilde{A}$ are C^∞ with respect to a derivation which coincides with the usual one in $C^1(\mathbb{R})$, the usual formula for the derivation of a product holds, the algebra $\tilde{A}$ contains the Dirac delta distribution.

Remark 4.3. It is worth noting that if A is a matrix such that $A = (A_{ij}) \in \mathcal{G}^{n\times n}(\mathbb{R})$, $A_{jj} = dB'_{jj}, B_{jj}$ are continuous functions, $j = 1, \ldots, n$, the derivative is meant in the Colombeau sense, $A_{kj} = 0$ for $k < j$ and $n > 1$, then the matrix A has property (3.8) (d is defined in Remark 3.4).

Remark 4.4. The definition of generalized functions on an open interval $(a, b) \subset \mathbb{R}$ is almost the same as the definition in the whole $\mathbb{R}$ (see [1]). It is not difficult to observe that the above proved theorems are also true in the case when generalized functions A_{kj}, f_k and x_k are considered on an interval (a, b) for $k, j = 1, \ldots, n$. For this purpose it is necessary to formulate properties (3.0–(3.10) on the interval (a, b).

Remark 4.5. Generalized solutions of ordinary differential equations can be considered in other ways, too (for example: [1]–[12], [15]–[18], [21]).

References

[1] *J. F. Colombeau*: Elementary Introduction to New Generalized Functions. North Holland, Amsterdam, 1985.

[2] *J. F. Colombeau*: Multiplication of Distributions. Lecture Notes in Math. 1532, Springer, Berlin, 1992.

[3] *S. G. Deo, S. G. Pandit*: Differential Systems Involving Impulses. Lecture Notes in Math. 954, Springer, Berlin, 1982.

[4] *V. Doležal*: Dynamics of Linear Systems. Academia, Praha, 1967.

[5] *Y. Egorov*: A theory of generalized functions. Uspehi Math. Nauk *455* (1990), 3–40. (In Russian.)

[6] *A. F. Filippov*: Differential Equations with Discontinuous Right Part. Nauka, Moscow, 1985. (In Russian.)

[7] *I. M. Gel'fand, G. E. Shilov*: Generalized Functions I. Academic Press, New York, 1964.

[8] *T. H. Hildebrandt*: On systems of linear differential Stieltjes integral equations. Illinois J. Math. *3* (1959), 352–373.

[9] *J. Kurzweil*: Generalized ordinary differential equations and continuous dependence on a parameter. Czechoslovak Math. J. *7* (1957), 418–449.

[10] *J. Kurzweil*: Linear differential equations with distributions coefficients. Bull. Acad. Polon. Sci. Ser. Math. Phys. *7* (1959), 557–560.

[11] *J. Ligęza*: On distributional solutions of some systems of linear differential equations. Časopis Pěst. Mat. *102* (1977), 37–41.

[12] *J. Ligęza*: Weak Solutions of Ordinary Differential Equations. Prace Nauk. Uniw. Śląsk. Katowic. 842, 1986.

[13] *J. Ligęza*: Generalized solutions of ordinary linear differential equations in the Colombeau algebra. Math. Bohem. *118* (1993), 123–146.

[14] *M. Oberguggenberger*: Hyperbolic systems with discontinuous coefficients: Generalized solutions and a transmission problem in acoustics. J. Math. Anal. Appl. *142* (1989), 452–467.

[15] *M. Pelant, M. Tvrdý*: Linear distributional differential equations in the space of regulated functions. Math. Bohem. *118* (1993), 379–400.

[16] *J. Persson*: The Cauchy system for linear distribution differential equations. Functial Ekvac. *30* (1987), 162–168.

[17] *R. Pfaff*: Gewöhnliche lineare Differentialgleichungen n-ter Order mit Distributionskoefizienten. Proc. Roy. Soc. Edingburgh, Sect. A. *85* (1980), 291–298.

[18] *Š. Schwabik, M. Tvrdý, O. Vejvoda*: Differential and Integral Equations. Academia, Praha, 1979.

[19] *E. E. Rosinger*: Nonlinear Partial Differential Equations, Sequential and Weak Solutions. Math. Studies 44, North-Holland, 1980.

[20] *L. Schwartz*: Sur l'impossibilité de la multiplication des distributions. C. R. Acad. Sci. Paris Sér. I Math. *239* (1954), 847–848.

[21] *Z. Wyderka*: Some problems of optimal control for linear systems with measures as coefficients. Systems Sci. *5* (1979), 425–431.

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