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REFLECTED DOUBLE LAYER POTENTIALS AND CAUCHY'S OPERATORS

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Abstract. Necessary and sufficient conditions are given for the reflected Cauchy's operator (the reflected double layer potential operator) to be continuous as an operator from the space of all continuous functions on the boundary of the investigated domain to the space of all holomorphic functions on this domain (to the space of all harmonic functions on this domain) equipped with the topology of locally uniform convergence.

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As usual, points $(a, b) \in \mathbb{R}^2$ in the Euclidean plane will be identified with the corresponding complex numbers $a + \mathrm{i}b \in \mathbb{C}$. If $A \subset \mathbb{R}^2$, then $\operatorname{cl} A, \partial A, A^0$ denote the closure, boundary and interior of A, respectively.

We denote by

$$B_r(z) := \{\eta \in \mathbb{C}; |\eta - z| < r\}$$

the disc of radius $r > 0$ centered at $z \in \mathbb{C}$; λ_2 denotes the Lebesgue measure in $\mathbb{R}^2 = \mathbb{C}$. In what follows we always assume that $A \subset \mathbb{C}$ is λ_2-measurable, ∂A is compact and

$$\lambda_2[A \cap B_r(z)] > 0$$

for each $z \in \partial A$ and $r > 0$. We denote by

$$\overline{d}(A, z) = \limsup_{r \to 0_+} \frac{\lambda_2[B_r(z) \cap A]}{\lambda_2[B_r(z)]}$$

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the upper density of A at $z \in \mathbb{C}$ and define the essential boundary $\partial_{\mathrm{es}} A$ of A by

$$\partial_{\mathrm{es}} A := \{z \in \mathbb{C}; \overline{d}(A,z) > 0, \overline{d}(\mathbb{C} \setminus A, z) > 0\}.$$

We denote by $\mathcal{C}_0^{(1)}$ the space of all real-valued continuously differentiable functions φ with a compact support in $\mathbb{C}$; $\mathcal{C}^{(1)}(\partial A)$ will stand for the space of all restrictions to ∂A of functions in $\mathcal{C}_0^{(1)}$, ∂_j will denote the partial derivative with respect to the j-th variable ($j = 1, 2$) and $\overline{\partial} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2)$. Given $z \in \mathbb{C} \setminus \partial A$ and $\varphi \in \mathcal{C}^{(1)}(\partial A)$ we choose a $\psi_\varphi \in \mathcal{C}_0^{(1)}$ vanishing in a neighbourhood of z such that $\psi_\varphi = \varphi$ on ∂A and define

$$\mathcal{K}^A \varphi(z) = \frac{2}{\pi \mathrm{i}} \int\limits_{\mathbb{C}-A} \frac{\overline{\partial}\psi_\varphi(\eta)}{\eta - z} \,\mathrm{d}\lambda_2(\eta).$$

The value $\mathcal{K}^A\varphi(z)$ is independent of the choice of ψ_φ with the properties specified above and the function

$$z \mapsto \mathcal{K}^A \varphi(z)$$

is holomorphic on $\mathbb{C} \setminus \partial A$.

Let now $D \subset \mathbb{C}$ be a bounded domain. A mapping $g\colon \mathcal{U} \to \mathbb{C}$ defined on a neighbourhood $\mathcal{U}$ of the boundary ∂D is called the reflection mapping corresponding to D if it satisfies the following conditions (i)–(iv):

(i) The complex conjugate $\overline{g}$ of g is 1-1 and holomorphic on $\mathcal{U}$.
(ii) $g(\eta) = \eta$ for any $\eta \in \partial D$.
(iii) $g(\mathcal{U} \cap D) = \mathcal{U} \setminus \mathrm{cl}\, D, g(\mathcal{U} \setminus \mathrm{cl}\, D) = \mathcal{U} \cap D$.
(iv) $g(g(z)) = z$ for any $z \in \mathcal{U}$.

Given such D and g we now assume that $A \subset D$ is compact, $D \setminus \mathcal{U} \subset A^0$ and define

$$G = (\mathcal{U} \setminus A) \cap g(\mathcal{U} \setminus A),$$

which is an open set containing $(D \setminus A) \cup \partial D$.

To each $\varphi \in \mathcal{C}^{(1)}(\partial A)$ we assign a function $\mathcal{J}^A\varphi(z)$ defined on G by

$$\mathcal{J}^A\varphi(z) = \mathcal{K}^A\varphi(z) - \overline{\mathcal{K}^A\varphi(g(z))}, \quad z \in G,$$

where the bar denotes the complex conjugate. The function

$$\mathcal{J}^A\varphi\colon z \mapsto \mathcal{J}^A\varphi(z)$$

is holomorphic on G. Now $\mathcal{A}(G)$ will denote the space of all holomorphic functions on G, $\mathcal{H}(G)$ will stand for the space of all real-valued harmonic functions on G. The operators

(1) $$\mathcal{J}^A\colon \varphi \mapsto \mathcal{J}^A\varphi$$

(the reflected Cauchy's operator) and

$$\operatorname{Im}\mathcal{J}^A\colon \varphi \mapsto \operatorname{Im}\mathcal{J}^A\varphi \tag{2}$$

(the reflected double layer potential operator) acting from $\mathcal{C}^{(1)}(\partial A)$ into $\mathcal{A}(G)$ and $\mathcal{H}(G)$, respectively, proved to be useful in treating some boundary value problems (cf. [1]). We equip $\mathcal{C}^{(1)}(\partial A)$ with the topology of uniform convergence on ∂A and consider the topology of locally uniform convergence in $\mathcal{A}(G)$ and $\mathcal{H}(G)$. In connection with these topologies the question of continuity of the operators (1), (2) naturally arises. We are going to prove the following result characterizing this continuity in geometric terms connected with ∂A; λ_1 will denote the 1-dimensional Hausdorff measure (length) as introduced in [4], chap. II, §8.

Theorem. *The following conditions* (a)–(c) *are equivalent:*
(a) $\lambda_1(\partial_{\mathrm{es}} A) < \infty$.
(b) *The operator* (1) *[acting from* $\mathcal{C}^{(1)}(\partial A)$ *into* $\mathcal{A}(G)$*] is continuous.*
(c) *The operator* (2) *[acting from* $\mathcal{C}^{(1)}(\partial A)$ *into* $\mathcal{H}(G)$*] is continuous.*

Proof. Since the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) were proved in [1], it remains to verify the implication (c) $\Rightarrow$ (a).

Fix $x \in \partial A$. First we prove that there are $u, v \in G$ such that the vectors

$$\frac{x-u}{|x-u|^2} + \frac{x-g(u)}{|x-g(u)|^2}, \quad \frac{x-v}{|x-v|^2} + \frac{x-g(v)}{|x-g(v)|^2}$$

are linearly independent. Suppose the opposite. Then there is a unit vector θ such that

$$\theta \cdot \left[\frac{x-z}{|x-z|^2} + \frac{x-g(z)}{|x-g(z)|^2}\right] = 0 \tag{3}$$

for each $z \in G$. For $z \in D \cap \mathcal{U}$ put

$$f(z) = \theta \cdot \left[x - z + |x-z|^2 \frac{x-g(z)}{|x-g(z)|^2}\right].$$

Since $g(z) \notin D, x \in D \cap \mathcal{U}$ the function f is infinitely differentiable on $D \cap \mathcal{U}$. Since $f(x) = 0, \frac{\partial f}{\partial \theta}(x) = -1$, the implicit function theorem yields that there is a neighbourhood V of the point x such that $V \cap \{y; f(y) = 0\}$ is the graph of an infinitely differentiable function in a suitable Cartesian coordinate system and thus we obtain $\lambda_2(V \cap \{y; f(y) = 0\}) = 0$. The assumptions yield that $x \in \operatorname{cl} G$. Since $V \cap G$ is an nonempty open set, we have $\lambda_2(V \cap G) > 0$. Since $V \cap G \subset V \cap \{y; f(y) = 0\}$ by (3) we obtain $\lambda_2(V \cap \{y; f(y) = 0\}) > 0$, which is a contradiction.

Now we shall prove that there are positive constants $r(x), M(x)$ such that for each $\varphi \in \mathcal{C}_0^{(1)}, |\varphi| \leqslant 1, \operatorname{spt}\varphi \subset B_{r(x)}(x)$ and $i = 1, 2$

$$\int\limits_{\mathbb{R}^2\setminus A} \partial_i \varphi \,\mathrm{d}\lambda_2 \leqslant M(x). \tag{4}$$

Choose points z_1, z_2 in G such that

$$\frac{x - z_1}{|x - z_1|^2} + \frac{x - g(z_1)}{|x - g(z_1)|^2}, \quad \frac{x - z_2}{|x - z_2|^2} + \frac{x - g(z_2)}{|x - g(z_2)|^2}$$

are linearly independent vectors. Then there is a positive constant r such that $B_{2r}(x) \cap \{z_1, z_2, g(z_1), g(z_2)\} = \emptyset$ and

$$\frac{y - z_1}{|y - z_1|^2} + \frac{y - g(z_1)}{|y - g(z_1)|^2}, \quad \frac{y - z_2}{|y - z_2|^2} + \frac{y - g(z_2)}{|y - g(z_2)|^2}$$

are linearly independent vectors for each $y \in B_{2r}(x)$. Fix $\theta \in \partial B_1(0)$. Then there are α_1, α_2, infinitely differentiable functions in $B_{2r}(x)$ such that

$$\theta = \sum_{j=1}^{2} \alpha_j(y) \left[\frac{y - z_j}{|y - z_j|^2} + \frac{y - g(z_j)}{|y - g(z_j)|^2} \right]$$

on $B_{2r}(x)$. If $\varphi \in \mathcal{C}_0^{(1)}, |\varphi| \leqslant 1, \operatorname{spt}\varphi \subset B_r(x)$ we define $\alpha_j\varphi = 0$ on $\mathbb{R}^2 \setminus B_r(x)$. Then

$$\begin{aligned}
\int\limits_{\mathbb{R}^2\setminus A} \frac{\partial \varphi}{\partial \theta} \,\mathrm{d}\lambda_2 &= \int\limits_{B_r(x)\setminus A} \sum_{j=1}^{2} \alpha_j(y) \nabla\varphi(y) \cdot \left[\frac{y - z_j}{|y - z_j|^2} + \frac{y - g(z_j)}{|y - g(z_j)|^2} \right] \mathrm{d}\lambda_2(y) \\
&= \sum_{j=1}^{2} \Big\{ \int\limits_{\mathbb{R}^2\setminus A} \nabla(\alpha_j(y)\varphi(y)) \cdot \left[\frac{y - z_j}{|y - z_j|^2} + \frac{y - g(z_j)}{|y - g(z_j)|^2} \right] \mathrm{d}\lambda_2(y) \\
&\qquad - \int\limits_{B_r(x)\setminus A} \varphi(y) \nabla\alpha_j(y) \cdot \left[\frac{y - z_j}{|y - z_j|^2} + \frac{y - g(z_j)}{|y - g(z_j)|^2} \right] \mathrm{d}\lambda_2(y) \Big\}.
\end{aligned}$$

Since

$$\operatorname{Im} \mathcal{J}^A(\alpha_j\varphi)(z_j) = \operatorname{Im} \Big[\frac{2}{\pi \mathrm{i}} \int\limits_{\mathbb{C}\setminus A} \frac{\overline{\partial}(\alpha_j\varphi)(y)}{y - z_j} \,\mathrm{d}\lambda_2(y) + \frac{2}{\pi \mathrm{i}} \int\limits_{\mathbb{C}\setminus A} \frac{\overline{\partial}(\alpha_j\varphi)(y)}{y - g(z_j)} \,\mathrm{d}\lambda_2(y) \Big]$$

$$= -\frac{1}{\pi} \Big[\int\limits_{\mathbb{R}^2\setminus A} \frac{y - z_j}{|y - z_j|^2} \cdot \nabla(\alpha_j\varphi)(y) \,\mathrm{d}\lambda_2 + \int\limits_{\mathbb{R}^2\setminus A} \frac{y - g(z_j)}{|y - g(z_j)|^2} \cdot \nabla(\alpha_j\varphi)(y) \,\mathrm{d}\lambda_2(y) \Big]$$

we obtain

$$\int\limits_{\mathbb{R}^2\setminus A} \frac{\partial\varphi}{\partial\theta}\,\mathrm{d}\lambda_2 \leqslant \sum_{j=1}^{2}(-\pi)\operatorname{Im}\mathcal{J}^A(\alpha_j\varphi)(z_j) + \sum_{j=1}^{2}\lambda_2(B_r(x))\sup_{B_r(x)}|\nabla\alpha_j|\frac{2}{r},$$

because $z_j, g(z_j) \notin B_{2r}(x)$. Since

$$|\alpha_j\varphi| \leqslant \sup_{B_r(x)}|\alpha_j|$$

the continuity of the operator (2) yields the estimate (4).

Since ∂A is compact there is a finite set $x^1,\dots,x^k$ of points in ∂A such that

$$\partial A \subset \bigcup_{j=1}^{k} B_{r(x^j)}(x^j).$$

Further, there are $\alpha_1,\dots,\alpha_k \in \mathcal{C}_0^{(1)}$ such that $0 \leqslant \alpha_j \leqslant 1$,

$$\operatorname{spt}\alpha_j \subset B_{r(x^j)}(x^j), \quad \sum_{j=1}^{k}\alpha_j = 1 \text{ on a neighbourhood of } \partial A.$$

If $\varphi \in \mathcal{C}_0^{(1)}, |\varphi| \leqslant 1$ then

$$\begin{aligned}
\int\limits_A \partial_j\varphi\,\mathrm{d}\lambda_2 &= \int\limits_{\mathbb{R}^2}\partial_j\varphi\,\mathrm{d}\lambda_2 - \int\limits_{\mathbb{R}^2\setminus A}\partial_j\varphi\,\mathrm{d}\lambda_2\\
&= \int\limits_{\mathbb{R}^2\setminus A}\sum_{n=1}^{k}\partial_j(-\alpha_n\varphi)\,\mathrm{d}\lambda_2 + \int\limits_{\mathbb{R}^2\setminus A}\partial_j\Big[\Big(\sum_{n=1}^{k}\alpha_n - 1\Big)\varphi\Big]\,\mathrm{d}\lambda_2\\
&\leqslant \sum_{n=1}^{k}M(x^n) + \int\limits_{\mathbb{R}^2}\partial_j\Big[\Big(\sum_{n=1}^{k}\alpha_n - 1\Big)\varphi\chi_{(\mathbb{R}^2\setminus A)}\Big]\,\mathrm{d}\lambda_2\\
&= \sum_{n=1}^{k}M(x^n),
\end{aligned}$$

where χ_C denotes the characteristic function of the set C. Since the so called perimeter of A

$$P(A) = \sup\Big\{\int\limits_A \operatorname{div} w\,\mathrm{d}\lambda_2; w = (w_1,w_2), w_j \in \mathcal{C}_0^{(1)}, w_1^2 + w_2^2 \leqslant 1\Big\}$$

is finite, we have $\lambda_1(\partial_{\mathrm{es}}A) < \infty$ by [F], Theorem 4.5.11. $\square$

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