DOMINATING FUNCTIONS OF GRAPHS WITH TWO VALUES

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Abstract. The Y-domination number of a graph for a given number set Y was introduced by D. W. Bange, A. E. Barkauskas, L. H. Host and P. J. Slater as a generalization of the domination number of a graph. It is defined using the concept of a Y-dominating function. In this paper the particular case where $Y = \{0, 1/k\}$ for a positive integer k is studied.

Keywords: Y-dominating function of a graph, Y-domination number of a graph

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This paper will concern a certain generalization of the domination number of a graph. All graphs considered will be finite undirected graphs without loops and multiple edges.

A subset D of the vertex set $V(G)$ of a graph G is called dominating in G, if for each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. The minimum number of vertices of a dominating set in G is called the domination number of G and denoted by $\gamma(G)$.

This well-known concept can be defined in another way, using domination functions. We will speak about functions f which map $V(G)$ into some set of numbers. If $S \subseteq V(G)$, then we denote $f(S) = \sum_{x \in S} f(x)$. If $x \in V(G)$, then by $N[x]$ we denote the closed neighbourhood of x in G, i.e. the set consisting of x and of all vertices which are adjacent to x in G. Besides, we will also consider the open neighbourhood $N(x) = N[x] - \{x\}$. Now we can formulate the alternative definition of the domination number.

A function $f\colon V(G) \to \{0, 1\}$ is called a dominating function of G, if $f(N[x]) \geqslant 1$ for each $x \in V(G)$. The minimum sum $f(V(G)) = \sum_{x \in V(G)} f(x)$ taken over all dominating functions f of G is called the domination number of G and denoted by $\gamma(G)$.

It is evident that these two definitions are equivalent. Namely, if D is a dominating set in G, then the function f defined so that $f(x)=1$ for $x\in D$ and $f(x)=0$ for $x\in V(G)-D$ is a dominating function of G. Conversely, if f is a dominating function of G, then the set $D=\{x\in V(G);\, f(x)=1\}$ is a dominating set in D.

The concept of a dominating function and obviously also the related concept of the domination number were generalized by some authors in such a way that the set of values $\{0,1\}$ was replaced by another number set. In [1] the signed dominating function and the signed domination number were defined by replacing the set $\{0,1\}$ by $\{-1,1\}$ and in [2] the minus dominating function and the minus domination number were defined by using the set $\{-1,0,1\}$. The fractional dominating function and the fractional domination number were introduced in [3] by using the set of real numbers. The most general case is the Y-dominating function and the Y-domination number, where a quite arbitrary set Y of values of f is used [4].

Therefore, following [4], a function $f\colon V(G)\to Y$, where Y is a given set of numbers, is called a Y-dominating function of G, if $f(N[x])\geqslant 1$ for each $x\in V(G)$. The minimum of $f(V(G))$ taken over all Y-dominating functions f of G is called the Y-dominating number of G and is denoted by $\gamma_Y(G)$.

We will not treat the domination is such a general way. We restrict our considerations to natural generalizations of the set $\{0,1\}$, namely to two-element number sets $\{0,t\}$, where t is a positive real number.

The following proposition is easy to prove.

Proposition 1. *Let $Y=\{0,t\}$, where t is a positive real number. Let G be a graph. The Y-domination number $\gamma_Y(G)$ of G is defined and at least one Y-dominating function of G exists if and only if $\delta(G)\geqslant 1/t-1$, where $\delta(G)$ denotes the minimum degree of a vertex of G.*

Let f be a function which maps $V(G)$ into the set of real numbers and let $x\in V(G)$. The vertex set x will be called a zero vertex of f, if $f(x)=0$.

The following theorem enables us to restrict our consideration to numbers t which are inverses of positive integers.

Theorem 1. *Let t be a positive real number, let G be a graph with $\delta(G)\geqslant 1/t-1$. Let $k=\lceil 1/t\rceil$ and $Y_1=\{0,t\}$, $Y_2=\{0,1/k\}$. Then $\gamma_{Y_1}(G)=kt\gamma_{Y_2}(G)$ and there exists a one-to-one correspondence between Y_1-dominating functions of G and Y_2-dominating functions of G such that the corresponding functions have the same set of zero vertices.*

P r o o f. Let $f\colon V(G)\to Y_1$, $g\colon V(G)\to Y_2$ and suppose that f,g have the same set of zero vertices. Then $f(x)=ktg(x)$ and also $f(N[x])=ktg(N[x])$ for each $x \in V(G)$. Suppose that g is a Y_2-dominating function of G: then $g(N[x]) \geqslant 1$ for each $x \in V(G)$. Evidently $kt \geqslant 1$ and thus $f(N[x]) \geqslant g(N[x]) \geqslant 1$ for each $x \in V(G)$ and f is a Y_1-dominating function of G. Now suppose that g is not a Y_2-dominating function of G. There exists $x \in V(G)$ such that $g(N[x]) < 1$. If $k = 1$, then $g(N[x])$ must be a non-negative integer and therefore $g(N[x]) = 0$. This is possible only if $g(y) = 0$ for each $y \in N[x]$. But then also $f(y) = 0$ for each $y \in N[x]$ and $f(N[x]) = 0$; the function f is not a Y_1-dominating function of G. If $k \geqslant 2$, then the number of vertices of $N[x]$ which are not zero vertices of g is at most $k-1$. But these vertices are exactly those vertices which are not zero vertices of f. We have $f(N[x]) \leqslant (k-1)t$. Evidently $1/t > k-1$ and thus $f(N[x]) \leqslant (k-1)t < 1$; the function f is not a Y_1-dominating function of G. If g_0 is a minimal (i.e. with the minimum sum on $V(G)$) Y_2-dominating function, then the corresponding function f_0 is a minimal Y_1-dominating function. We have $\gamma_{Y_1}(G) = \sum\limits_{x \in V(G)} f_0(x) = \sum\limits_{x \in V(G)} ktg_0(x) = kt \sum\limits_{x \in V(G)} g_0(x) = kt\gamma_{Y_2}(G)$. □

For each positive integer k we denote $Y(k) = \{0, 1/k\}$ and $\gamma(k, G) = \gamma_{Y(k)}G$. From Proposition 1 we have the following corollary.

Corollary 1. *Let k be a positive integer, let G be a graph. The $Y(k)$-domination number $\gamma(k, G)$ is defined and at least one $Y(k)$-dominating function of G exists if and only if $\delta(G) \geqslant k-1$.*

Note that $\gamma(1, G) = \gamma(G)$, the usual domination number of G.

If we speak about a function $f\colon V(G) \to Y(k)$, we will use the notation $V^0 = \{x \in V(G);\, f(x) = 0\}$, $V^+ = \{x \in V(G);\, f(x) = 1/k\}$.

Theorem 2. *Let G be a regular graph of degree $k-1$ with n vertices. Then $\gamma(k, G) = n/k$.*

P r o o f. The neighbourhood $N[x]$ for each $x \in V(G)$ has exactly k vertices. If f is a $Y(k)$-dominating function, then f must assign the value $1/k$ to all vertices of $N[x]$. As x was chosen arbitrarily, it assigns $1/k$ to all vertices of G, which implies the assertion. □

By G^2 we denote the square of the graph G, i.e. the graph such that $V(G^2) = V(G)$ and two vertices are adjacent in G^2 if and only if their distance in G is at most 2. The symbol $\alpha_0(G)$ denotes the independence number of G, i.e. the maximum number of pairwise non-adjacent vertices in G.

Theorem 3. *Let G be a regular graph of degree k with n vertices. Then $\gamma(k, G) = (n - \alpha_0(G^2))/k$.*

P r o o f. For each vertex x of G the set $N[x]$ has $k+1$ vertices. If f is a $Y(k)$-dominating function of G, then $N[x]$ contains at most one zero vertex of f. The distance between two zero vertices of f cannot be 1; then the closed neighbourhood of either of them would contain them both. This distance cannot be 2; then there would exist a vertex adjacent to both of them and its closed neighbourhood would contain them both. Therefore the distance between two zero vertices of f in G is at least 3 and in G^2 at least 2; they form an independent set in G^2. Therefore there are at most $\alpha_0(G^2)$ zero vertices of f and at least $n-\alpha_0(G^2)$ vertices x such that $f(x)=1/k$. This implies the assertion. □

Corollary 2. *Let C_n be the circuit of length n. Then $\gamma(3,C_n)=n/3$ and $\gamma(2,C_n)=n/3$ for $n\equiv 0(\operatorname{mod} 3)$, $\gamma(2,C_n)=n/3-1/6$ for $n\equiv 1(\operatorname{mod} 3)$, $\gamma(2,C_n)=n/3+1/3$ for $n\equiv 2(\operatorname{mod} 3)$.*

A path is a similar case. If f is a $Y(2)$-dominating function of a path P_n of length n, then again the distance between any two zero vertices of f is at least 3 and moreover neither the vertices of degree 1, not the vertices adjacent to them may be zero vertices of f. This yields the result.

Proposition 2. *Let P_n be a path of length n. Then $\gamma(2,P_n)=n/3+1$ for $n\equiv 0(\operatorname{mod} 3)$, $\gamma(2,P_n)=n/3+2/3$ for $n\equiv 1(\operatorname{mod} 3)$, $\gamma(2,P_n)=n/3+5/6$ for $n\equiv 2(\operatorname{mod} 3)$.*

Now we turn to complete graphs and complete bipartite graphs.

Theorem 4. *Let k,n be positive integers, $k\leqslant n$. Then $\gamma(k,K_n)=1$.*

P r o o f. In the complete graph K_n we have $N[x]=V(K_n)$ for each vertex x. If f is a $Y(k)$-dominating function, then $f(V(K_n)=f(N[x])\geqslant 1$. Moreover, there exists a function f which assigns the value $1/k$ to k vertices and the value 0 to the remaining $n-k$ vertices: then $f(V(K_n))=1$. □

Theorem 5. *Let k,m,n be positive integers, $k-1\leqslant m\leqslant n$. If $k<m$, then $\gamma(k,K_{m,n})=2$. If $m=k-1$, then $\gamma(k,K_{m,n})=(m+n)/k=(k+n-1)/k$. If $m=k$, then $\gamma(k,K_{m,n})=2-1/k$.*

P r o o f. Let $k<m$. Let A,B be the bipartition classes of K, $|A|=m$, $|B|=n$. For each vertex $x\in A$, its open neighbourhood satisfies $N(x)\subseteq B$. As $N[x]=\{x\}\cup N(x)$ and $f(N[x])\geqslant 1$ for a $Y(k)$-dominating function f, there are at least $k-1$ vertices $y\in N(x)\subseteq A$ which are in V^+. If moreover $f(x)=0$, then there are at least k such vertices. Therefore either $f(x)=1/k$ for all $x\in A$ and

$f(y) = 1/k$ for at least $k-1$ vertices of B, or $f(y) = 1/k$ for at least k vertices of B. In the former case $f(V(K_{m,n})) \geqslant (m+k-1)/k \geqslant 2$. In the latter case analogously either $f(x) = 1/k$ for all $x \in B$ and $f(y) = 1/k$ for at least $k-1$ vertices of A, or $f(y) = 1/k$ for at least k vertices of A. In both these cases again $f(V(K_{m,n})) \geqslant 2$. A function f which assigns $1/k$ to exactly k vertices of A and to exactly k vertices of B has $f(V(K_{m,n})) = 2$.

Now suppose $m = k-1$. Then $|A| = k-1$. Let $x \in B$ and again let f be a $Y(k)$-dominating function of $K_{m,n}$. The set $N[x]$ has exactly k vertices and thus $f(x) = 1/k$ for each $y \in N[x]$. This means that $f(y) = 1/k$ for each $y \in A$ and also $f(x) = 1/k$. As x is an arbitrary vertex of B, we have $f(x) = 1/k$ for all $x \in V(K_{m,n})$ and $f(V(K_{m,n})) = (k-1+n)/k$. Another $Y(k)$-dominating function does not exist and thus $\gamma(k, K_{m,n}) = (k-1+n)/k$.

Finally, let $k = m$. If f is a $Y(k)$-dominating function, then either $f(x) = 1/k$ for each $x \in A$ and for at least $k-1$ vertices x of B, or $f(x) = 1/k$ for exactly $k-1$ vertices of A and all vertices $x \in B$. In the former case $f(V(K_{m,n})) \geqslant (2k-1)/k = 2-1/k$, in the latter case $f(V(K_{m,n})) \geqslant (k-1+n)/k \geqslant (2k-1)/k = 2-1/k$. If f assigns the value $1/k$ to all vertices of A and to exactly $k-1$ vertices of B, then $f(V(K_{m,n})) = 2-1/k$, therefore $\gamma(k, K_{m,n}) = 2-1/k$. □

By the symbol $G \oplus H$ we denote the Zykov sum of graphs G and H, i.e. the graph obtained from vertex-disjoint graphs G and H by joining all vertices of G with all vertices of H by edges.

Theorem 6. *Let k, q be positive integers, let G, H be two graphs such that $\gamma(k, G)$, $\gamma(k, H)$ are defined and $q \leqslant 1+\min(\gamma(k, G), \gamma(k, H))$. Then $\gamma(kq, G \oplus H) \leqslant (\gamma(k, G) + \gamma(k, H))/q$.*

Proof. Let g and h be minimal $Y(k)$-dominating functions of G and H, respectively. Let $f\colon V(G) \cup V(H) \to Y(kq)$ be defined so that $f(x) = g(x)/q$ for $x \in V(G)$ and $f(x) = h(x)/q$ for $x \in V(H)$. Consider $x \in V(G)$. The closed neighbourhood of x in $G \oplus H$ is the disjoint union of the closed neighbourhood of x in G and of $V(H)$. The sum of values of f over the closed neighbourhood of x in G is at least $1/q$, its sum over $V(H)$ is at least $\gamma(k, H)/q$. It follows from the assumption that $1/q + \gamma(k, H)/q \geqslant 1$. For $x \in V(H)$ this may be proved quite analogously. Therefore f is a $Y(kq)$-dominating function of $G \oplus H$. This implies the assertions. □

For the particular case $k = 1$ we have a corollary.

Corollary 3. *Let q be a positive integer, let G, H be two graphs such that $q \leqslant 1 + \min(\gamma(G), \gamma(H))$. Then $\gamma(q, G \oplus H) \leqslant (\gamma(G) + \gamma(H))/q$.*

A similar assertion holds for $G \oplus K_1$, i.e. the graph which is obtained from G by adding a new vertex and joining it with all vertices of G by edges.

Theorem 7. *Let k be a positive integer, let G be a graph for which $\gamma(k,G)$ is defined. Then*

$$\gamma(k+1, G \oplus K_1) = \gamma(k,G) \cdot \tfrac{k}{k+1} + \tfrac{1}{k+1}.$$

Proof. Let f be a minimal $Y(k)$-dominating function of G. Let w be the added vertex. Let $g\colon V(G) \cup \{w\} \to Y(k+1)$ be defined so that $g(x) = kf(x)/(k+1)$ for $x \in V(G)$ and $g(w) = 1/(k+1)$. Then the sum of $g(x)$ over the closed neighbourhood of x in $G \oplus K_1$ is equal to the sum of g over the closed neighbourhood of x in G plus $g(w)$. The sum of g over the closed neighbourhood of x in G is at least $k/(k+1)$ and $g(w) = 1/(k+1)$, therefore the sum of g over the closed neighbourhood of x in $G \oplus K_1$ is at least 1. The closed neighbourhood of w in $G \oplus K_1$ is $V(G) \cup \{w\}$ and the sum of g over it is greater than or equal to this sum over the closed neighbourhood of any other vertex, therefore it is also at least 1 and

$$\sum_{x \in V(G) \cup \{w\}} g(x) = \tfrac{k}{k+1} \sum_{x \in V(G)} f(x) + g(w) = \tfrac{k}{k+1}\gamma(k,G) + \tfrac{1}{k+1}.$$

Hence $\gamma(k+1, G \oplus K_1) \leqslant \frac{k}{k+1}\gamma(k,G) + \frac{1}{k+1}$. On the other hand, let g_0 be a minimal $Y(k+1)$-dominating function of $G \oplus K_1$ and let $f_0\colon V(G) \to Y(k)$ be defined so that $f_0(x) = (k+1)g_0(x)/k$ for each $x \in V(G)$. The sum of values of g over the closed neighbourhood of any vertex $x \in V(G)$ in G is at least $1 - 1/(k+1)$ and thus such a sum of f_0 is at least 1. We have $\sum\limits_{x \in V(G)} f_0(x) = \sum\limits_{x \in V(G)} (k+1)g_0(x)/k = \frac{k+1}{k} \sum\limits_{x \in V(G)} g_0(x) = \frac{k+1}{k}\gamma(k+1, G \oplus K_1) - g_0(w) = \frac{k+1}{k}\gamma(k+1, G \oplus K_1) - \frac{1}{k}$ and thus $\gamma(k,G) \leqslant \frac{k+1}{k}\gamma(k+1, G \oplus K_1) - \frac{1}{k}$, which yields $\gamma(k+1, G \oplus K_1) \geqslant \frac{k}{k+1}\gamma(k,G) + \frac{1}{k+1}$. Hence we have the equality $\gamma(k+1, G \oplus K_1) = \frac{k}{k+1}\gamma(k,G) + \frac{1}{k+1}$. □

In the end we will consider the number $\gamma(k,G)$ for different numbers k and for the same graph G.

Theorem 8. *Let k, q be positive integers. Then there exists a graph G such that $\gamma(k+1, G) - \gamma(k,G) = q$.*

Proof. Denote $p = kq + q + 1$ and let G be the Zykov sum $K_k \oplus \overline{K}_p$, where $\overline{K}_p$ denotes the complement of the complete graph $\overline{K}_p$, i.e. the graph consisting of p isolated vertices. If f is a function such that $f(x) = 0$ for $x \in V(\overline{K}_p)$ and $f(x) = 1/k$ for $x \in V(K_k)$, then f is a $Y(k)$-dominating function of G; namely, we have $V(K_k) \subseteq N[x]$ for each $x \in V(G)$ and $f(V(K_k)) = 1$. We have $\gamma(k,G) = 1$. Each vertex of

$\overline{K}_p$ has degree k in G and therefore for each $Y(k+1)$-dominating function g we have $g(y) = 1/(k+1)$ for each $y \in V(G)$ and $\gamma(k+1, G) = (p+k)/(k+1) = q+1$. $\square$

The next theorem is not expressed for k in general, but only for $\gamma(1, G)$ and $\gamma(2, G)$.

Theorem 9. *Let q be a positive integer. Then there exists a graph G such that $\gamma(1, G) - \gamma(2, G) = q$.*

P r o o f. Let H be a graph obtained from the circuit of length 4 by adding a new vertex u and joining it to a vertex v of the circuit by an edge. Take $2q$ pairwise vertex-disjoint copies $H_1, \ldots, H_{2q}$ of H. Take a vertex w and join it by edges with the vertex corresponding to u in each of the graphs $H_1, \ldots, H_{2q}$. Finally, take a new vertex x and join it with w by an edge. The resulting graph will be G. For $q = 4$ this graph is shown Fig. 1. The number $\gamma(1, G)$ is the usual domination number $\gamma(G)$ of G,

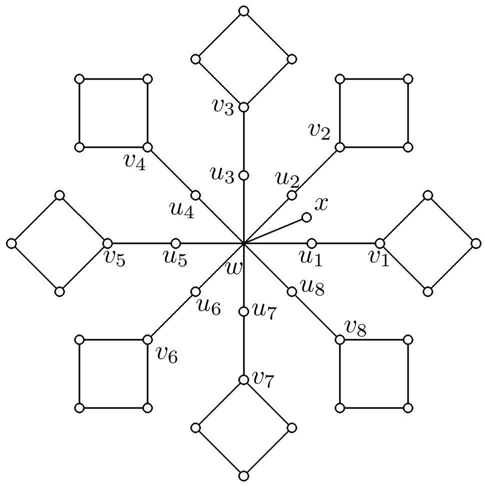

Fig. 1

i.e. the minimum number of vertices of a dominating set D in G. Evidently such a dominating set must contain at least one of the vertices w, x and at least two vertices from each H for $i = 1, \ldots, 2q$: hence $\gamma(G) \geqslant 4q+1$. If D consists of w, of the vertices corresponding to v in H and of one other vertex of the circuit in H for $i = 1, \ldots, 2q$, then D is dominating in G and $|D| = 4q+1$, which implies $\gamma(G) = 4q+1$. Now let V^+ be the set consisting of all vertices of D and, moreover, of x and of one more vertex of the circuit in each H for $i = 1, \ldots, 2q$. We have $|V^+| = 6q+2$. If $f(x) = \frac{1}{2}$ for $x \in V^+$ and $f(x) = 0$ for $x \in V(G) - V^+$, then f is a $V(2)$-dominating function of G and is evidently minimal. We have $\gamma(2, G) = f(V(G)) = \frac{1}{2}|V^+| = 3q+1$. Hence $\gamma(1, G) - \gamma(2, G) = q$. $\square$

P r o b l e m. Can Theorem 10 be generalized to a theorem analogous to Theorem 9?

A final remark. The $Y(k)$-domination number of a graph can be defined in another way, without using the concept of a $Y(k)$-dominating function:

A subset D of $V(G)$ is called k-tuply dominating in G, if for each $x \in V(G) - D$ there exist k vertices $y_1, \ldots, y_k$ od D adjacent to x and for each $y \in D$ there exist $k-1$ vertices $z_1, \ldots, z_{k-1}$ adjacent to y. The minimum number of vertices of a k-tuply dominating set in G is called the $Y(k)$-domination number of G.

A k-tuply dominating set was defined and used also in [5], but in a weaker form: the requirement of existence of $z_1, \ldots, z_{k-1}$ for $y \in D$ was not used there.

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