TOPOLOGICALLY MAXIMAL CONVERGENCES, ACCESSIBILITY, AND COVERING MAPS

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Abstract. Topologically maximal pretopologies, paratopologies and pseudotopologies are characterized in terms of various accessibility properties. Thanks to recent convergence-theoretic descriptions of miscellaneous quotient maps (in terms of topological, pretopological, paratopological and pseudotopological projections), the quotient characterizations of accessibility (in particular, those of G. T. Whyburn and F. Siwiec) are shown to be instances of a single general theorem. Convergence-theoretic characterizations of sequence-covering and compact-covering maps are used to refine various results on the relationship between covering and quotient maps (by A. V. Arhangel'skii, E. Michael, F. Siwiec and V. J. Mancuso) by deducing them from a single theorem.

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1. Introduction

Throughout this paper, the *topologically quotient map* means the classical quotient map.

Each closure operation in the sense of E. Čech[1] amounts to a pretopology. A pretopology τ is *coarser* than a pretopology ξ ($\tau \leqslant \xi$) whenever the closure corresponding to τ is larger than that corresponding to ξ. Topologies are those pretopologies for which the closure is idempotent. With every pretopology τ, we associate the *topology* $T\tau$ *of* τ, i. e., the finest among topologies that are coarser than τ. A pretopology τ is called *topologically maximal* if no pretopology $\pi > \tau$ fulfils $T\pi = T\tau$. A topological space is an *accessibility* space (G. T. Whyburn [19]) if for each x_0 and

[1] i. e., such that $A \subset \operatorname{cl} A$, $\operatorname{cl} \emptyset = \emptyset$ and $\operatorname{cl}(A \cup B) = \operatorname{cl} A \cup \operatorname{cl} B$.

every set H such that $x_0 \in \operatorname{cl}(H \setminus \{x_0\})$, there is a closed set F with $x_0 \in \operatorname{cl}(F \setminus \{x_0\})$ and $x_0 \notin \operatorname{cl}(F \setminus H \setminus \{x_0\})$. It is immediate that Fréchet topologies with unicity of sequential limits are accessibility topologies.

In [12], V. Kannan characterizes implicitly (i.e., without introducing the notion of maximality) topologies that are topologically maximal with respect to the class of pretopologies; one of these characterizations amounts to accessibility.

Theorem 1.1. ([12], Theorem 6.2.6) *A topology is topologically maximal (within the class of pretopologies) if and only if it is an accessibility topology.*

Unaware of [12] and of the notion of accessibility, S. Dolecki and G. Greco gave in [5] a characterization of topologically maximal pretopologies that extends the above theorem to pretopologies.

A map f from a pretopological space X to a pretopological space Y is *continuous* if for every subset A of Y, one has $\operatorname{cl} f^{-}(A) \subset f^{-}(\operatorname{cl} A)$. If for a given pretopology on X, the pretopology on Y is the finest pretopology for which f is continuous, then we say that f is a *pretopological quotient*. In particular, if X and Y are topological spaces, then f is a pretopological quotient if and only if it is *pseudo-open*, i.e., *hereditarily quotient*.

Theorem 1.2. (D. C. Kent [13]) *A topologically quotient map is pseudo-open if and only if it is pretopologically quotient.*

The maximality aspect of Theorem 1.1 enables one to easily deduce from Theorem 1.2 the following theorem of G. T. Whyburn ([20] under T_1) and V. Kannan [12].

Theorem 1.3. *A topology is an accessibility topology if and only if every topologically quotient map onto it is pseudo-open.*

This easy method of proving Theorem 1.3 hinges on the fact that if a topologically quotient map $f\colon X \to Y$ is not pseudo-open, then the finest pretopology on Y that makes the map continuous is not equal to the quotient topology (Theorem 1.2), while its topology is equal to the quotient topology; therefore the latter is not topologically maximal.

It turns out that this method admits natural extensions to general convergence spaces. By a convergence on X we understand a relation between filters $\mathscr{F}$ on X and points x of X, denoted $x \in \lim \mathscr{F}$ ($\mathscr{F}$ *converges* to x, or x is a *limit* of $\mathscr{F}$), such that $\mathscr{F} \subset \mathscr{G}$ implies $\lim \mathscr{F} \subset \lim \mathscr{G}$, the principal filter of x converges to x ($x \in \lim(x)$), and if $\bigcap_{i=1,\dots,n} \lim \mathscr{F}_i \subset \lim \bigwedge_{i=1,\dots,n} \mathscr{F}_i$ for every finite collection of filters $\mathscr{F}_1, \dots, \mathscr{F}_n$. In this paper we focus our attention on the following classes of convergences: topologies, pretopologies, paratopologies and pseudotopologies.

We provide a general unified characterization of topologically maximal pretopologies, paratopologies and pseudotopologies and generalize the results listed above. In particular, we recover Theorem 1.1 and deduce that a topology is a topologically maximal paratopology if and only if it is a strong accessibility topology (the latter notion is due to F. Siwiec [17]).

In this context we apply convergence-theoretic methods used in [4] to unify numerous facts concerning Fréchet, strongly Fréchet and bi-sequential spaces, sequence-covering maps and so on.

2. Convergence classes determined with the aid of adherence operation

A convergence with unicity of limits is called *Hausdorff*. A convergence is a *pseudotopology* (G. Choquet [3]) if $x \in \lim \mathscr{F}$ whenever $x \in \lim \mathscr{U}$ for every ultrafilter $\mathscr{U}$ finer than $\mathscr{F}$. A convergence is a *pretopology* (G. Choquet [3]) if for every point x, its neighborhood filter $\mathscr{N}(x) = \bigcap_{x \in \lim \mathscr{F}} \mathscr{F}$ converges to x. A pretopology is a *topology* if each neighborhood filter admits a base of open sets (a set O in a convergence space is *open* if for every $x \in O$ and each filter $\mathscr{F}$ convergent to x, one has $O \in \mathscr{F}$). A convergence ξ is *finer* than a convergence τ ($\xi \geqslant \tau$) if $\lim_\xi \mathscr{F} \subset \lim_\tau \mathscr{F}$ for every filter $\mathscr{F}$.

The classes of pseudotopologies, pretopologies and topologies are closed for suprema. Therefore to every convergence ξ (on X), we assign the finest pseudotopology $S\xi$ (pretopology $P\xi$, topology $T\xi$) on X that is coarser than ξ. The maps S, P, T are isotone, contractive and idempotent on the class of convergences. We call such maps *projections*.[2]

The adherence adh_ξ associated with a convergence ξ is defined by

$$\operatorname{adh}_\xi \mathscr{F} = \bigcup_{\mathscr{H} \# \mathscr{F}} \lim_\xi \mathscr{H} = \bigcup_{\mathscr{G} \supset \mathscr{F}} \lim_\xi \mathscr{G},$$

where $\mathscr{H} \# \mathscr{F}$ means that $H \cap F \neq \emptyset$ for every $H \in \mathscr{H}$ and each $F \in \mathscr{F}$. In particular, the closure $\operatorname{cl}_\xi A$ is the adherence of the principal filter of A. Here $\mathscr{H}^{\#} = \{G : G \cap H \neq \emptyset \text{ for each } H \in \mathscr{H}\}$ is the *grill* of $\mathscr{H}$.

The projections S and P can be expressed in terms of adherence. Namely,

$$\lim_{J\tau} \mathscr{F} = \bigcap_{\mathscr{F} \# \mathscr{H} \in \mathfrak{J}(\tau)} \operatorname{adh}_\tau \mathscr{H}, \tag{2.1}$$

where $\mathfrak{J} = \mathfrak{J}(\tau)$ is equal, respectively, to the family of all filters (in the case of $J = S$) or of principal filters (when $J = P$).

[2] We do not use the category term *reflections*, since we make an abstraction of morphisms.

A convergence τ is a *paratopology* [4] if $J\tau = \tau$, where J is defined by (2.1) with $\mathfrak{J} = \mathfrak{J}(\tau)$, the family of countably based filters. It follows that the class of paratopologies is sup-closed; we denote the corresponding projection J by P_ω. The topological projection T also admits a characterization of the type (2.1) with $\mathfrak{J}(\tau)$ the set of all principal filters of τ-closed sets.

3. Topologically maximal convergences

Let J be a projection. We say that ξ is a *J-convergence* if $\xi = J\xi$. Of course, the class $\operatorname{fix} J$ (of J-convergences) is closed for suprema. Let J be a projection such that $T \leqslant J$, where T denotes the projection on the class of topologies. A J-convergence τ is *topologically maximal at* x_0 in $\operatorname{fix} J$ if $x_0 \in \lim_\tau \mathscr{F}$ implies $x_0 \in \lim_\xi \mathscr{F}$, for every J-convergence ξ such that $\xi \geqslant \tau$ and $T\xi = T\tau$. Let now J be a projection of the form (2.1); this is in particular the case with the projections S, P_ω, P on the classes of pseudotopologies, of paratopologies and of pretopologies.

We denote by $\mathscr{H} \setminus A$ the filter generated by $\{H \setminus A\colon H \in \mathscr{H}\}$ and abridge $\mathscr{H} \setminus x_0 = \mathscr{H} \setminus \{x_0\}$. We assume that $\mathscr{H} \in \mathfrak{J}$ implies $\mathscr{H} \setminus A \in \mathfrak{J}$ provided $\mathscr{H} \setminus A$ is nondegenerate.

Theorem 3.1. *Let the projection J be defined in* (2.1) *with the aid of the class $\mathfrak{J}$. A J-convergence τ is topologically maximal at x_0 in* $\operatorname{fix} J$ *if and only if for each $\mathscr{H} \in \mathfrak{J}$ with $x_0 \in \operatorname{adh}_\tau(\mathscr{H} \setminus x_0)$, there exists a τ-closed set F with $x_0 \in \operatorname{cl}_\tau(F \setminus \{x_0\})$ and such that*

$$\bigl(\forall\, H \in \mathscr{H}\bigr) \quad x_0 \notin \operatorname{cl}_\tau(F \setminus H \setminus \{x_0\}).$$

P r o o f. ($\Longrightarrow$) Let $\mathscr{H} \in \mathfrak{J}$ be such that $x_0 \in \operatorname{adh}_\tau(\mathscr{H} \setminus x_0)$ and such that for every τ-closed set F,

$$x_0 \in \operatorname{cl}_\tau(F \setminus x_0) \implies \bigl(\exists\, H \in \mathscr{H}\bigr)\ x_0 \in \operatorname{cl}_\tau(F \setminus H \setminus x_0). \tag{3.1}$$

If $\mathscr{H}_0 = \mathscr{H} \setminus x_0$, then $\mathscr{H}_0 \in \mathfrak{J}$ by our assumption. We define the following convergence ϑ:

$$\lim_\vartheta \mathscr{F} = \begin{cases} \lim_\tau \mathscr{F} \setminus \{x_0\}, & \text{if } \mathscr{H}_0 \# \mathscr{F}, \\ \lim_\tau \mathscr{F}, & \text{otherwise.} \end{cases} \tag{3.2}$$

It follows that $x_0 \notin \operatorname{adh}_\vartheta \mathscr{H}_0$ and since $x_0 \in \operatorname{adh}_\tau \mathscr{H}_0$, the convergence ϑ is strictly finer than τ at x_0. We see that ϑ is a J-convergence. Indeed, as τ is a J-convergence, it is enough to show that $\lim_\vartheta \mathscr{F} = \lim_{J\vartheta} \mathscr{F}$ at the points where ϑ might differ from τ, i. e., at x_0. Therefore, consider a filter $\mathscr{F}$ for which $x_0 \in \lim_\tau \mathscr{F} \setminus \lim_\vartheta \mathscr{F}$. By (3.2), $\mathscr{H}_0 \# \mathscr{F}$ and because $\mathscr{H}_0 \in \mathfrak{J}$ and $x_0 \notin \operatorname{adh}_\vartheta \mathscr{H}_0$, we have $x_0 \notin \lim_{J\vartheta} \mathscr{F}$ by (2.1).

Let us show that $T\vartheta = T\tau$. If this were not the case, then there would be a set A such that $\mathrm{cl}_\vartheta A \subset A$ but $\mathrm{cl}_\tau A \setminus A \neq \emptyset$. As $\mathrm{cl}_\tau A \setminus \mathrm{cl}_\vartheta A \subset \{x_0\}$, we infer that $\mathrm{cl}_\tau A = A \cup \{x_0\}$ and $x_0 \notin A$. Because $\mathrm{cl}_\tau(A \cup \{x_0\}) = A \cup \{x_0\}$, the set $F = A \cup \{x_0\}$ is τ-closed and $x_0 \in \mathrm{cl}_\tau(F \setminus \{x_0\})$. By (3.1), there exists $H \in \mathscr{H}$ such that $x_0 \in \mathrm{cl}_\tau(A \setminus H)$ and thus there exists a filter $\mathscr{F}$ such that $x_0 \in \lim_\tau \mathscr{F}$ and $A \setminus H \in \mathscr{F}$, hence $H^c \in \mathscr{F}$ and thus $H \notin \mathscr{F}^\#$. Therefore $\mathscr{H}_0 \# \mathscr{F}$ does not hold so that by (3.2), $x_0 \in \lim_\vartheta \mathscr{F} \subset \mathrm{cl}_\vartheta A$, contrary to the assumption. We have proved that τ is not topologically maximal at x_0 in $\mathrm{fix}\, J$.

($\Longleftarrow$) If τ is not topologically maximal at x_0 in $\mathrm{fix}\, J$, then there exists a J-convergence ξ such that $\xi \geqslant \tau$ and $T\xi = T\tau$ and such that there are a filter $\mathscr{F}$ and $x_0 \in \lim_\tau \mathscr{F} \setminus \lim_\xi \mathscr{F}$. By (2.1), there exists $\mathscr{H}_0 \in \mathfrak{J}$ such that $x_0 \in \mathrm{adh}_\tau \mathscr{H}_0 \setminus \mathrm{adh}_\xi \mathscr{H}_0$. We define the convergence ϑ with the aid of (3.2). It has just been proved that ϑ is a J-convergence and $\xi \geqslant \vartheta > \tau$ at x_0, thus $T\vartheta = T\tau$.

Let us see that our $\mathscr{H}_0$ does not fulfil the condition of the theorem. Let F be an arbitrary τ-closed set with $x_0 \in \mathrm{cl}_\tau(F \setminus \{x_0\})$. Then $F \setminus \{x_0\}$ is not τ-closed, hence not ϑ-closed. Consequently, $x_0 \in \mathrm{cl}_\vartheta(F \setminus \{x_0\})$, that is, there is a filter $\mathscr{G}$ such that $x_0 \in \lim_\vartheta \mathscr{G}$ and $F \setminus \{x_0\} \in \mathscr{G}^\#$. By (3.2), the first condition implies the existence of $H \in \mathscr{H}$ with $H^c \in \mathscr{G}$ which, together with the second condition, yields $F \setminus H \setminus \{x_0\} \in \mathscr{G}^\#$ so that $x_0 \in \mathrm{cl}_\vartheta(F \setminus H \setminus \{x_0\}) \subset \mathrm{cl}_\tau(F \setminus H \setminus \{x_0\})$. $\square$

4. Accessibilities and types of quotient maps

We have already evoked the definition of accessibility. A topology τ is an *accessibility* topology at x_0 if for every set H with $x_0 \in \mathrm{cl}_\tau(H \setminus \{x_0\})$, there exists a τ-closed set F such that $x_0 \in \mathrm{cl}_\tau(F \setminus \{x_0\})$ and $x_0 \notin \mathrm{cl}_\tau(F \setminus H \setminus \{x_0\})$. Clearly, this definition makes sense also for pretopologies. Theorem 3.1 implies (for $J = P$ the projection on the class of pretopologies with $\mathfrak{J}$ the set of principal filters).

Corollary 4.1. *A pretopology is topologically maximal at x_0 in* $\mathrm{fix}\, P$ *if and only if it is an accessibility pretopology at x_0.*

This corollary amounts to [5, Theorem 6.1] and extends Theorem 1.1. In [17], F. Siwiec defines strong accessibility. A topology τ is a *strong accessibility* topology at x_0 if for each decreasing sequence of sets $(H_n)_n$ such that $x_0 \in \mathrm{cl}_\tau(H_n \setminus \{x_0\})$ for each n, there exists a closed set F with $x_0 \in \mathrm{cl}_\tau(F \setminus \{x_0\})$ and such that $x_0 \notin \mathrm{cl}_\tau(F \setminus H_n \setminus \{x_0\})$ for each n.

This property can be generalized to paratopologies: a paratopology τ is a *strong accessibility* paratopology at x_0 if for each countably based filter $\mathscr{H}$ for which $x_0 \in \mathrm{adh}_\tau(\mathscr{H} \setminus x_0)$, there exists a τ-closed set F with $x_0 \in \mathrm{cl}_\tau(F \setminus \{x_0\})$ and such that

$$(\forall\, H \in \mathscr{H}) \quad x_0 \notin \mathrm{cl}_\tau(F \setminus H \setminus \{x_0\}). \tag{4.1}$$

If $J = P_\omega$ is the projection on the class of paratopologies (and $\mathfrak{J}$ is the set of countably based filters), Theorem 3.1 implies

Corollary 4.2. *A paratopology is topologically maximal at x_0 in $\operatorname{fix} P_\omega$ if and only if it is a strong accessibility paratopology at x_0.*

By analogy, we say that a pseudotopology τ is a *hyper-accessibility* pseudotopology at x_0 if for each filter $\mathscr{H}$ with $x_0 \in \operatorname{adh}_\tau(\mathscr{H} \setminus x_0)$, there exists a τ-closed set F such that $x_0 \in \operatorname{cl}_\tau(F \setminus \{x_0\})$ and (4.1) holds. Hyper-accessibility convergences form a very narrow class (see Proposition 5.9).

Theorem 3.1 yields

Corollary 4.3. *A pseudotopology is topologically maximal at x_0 in $\operatorname{fix} S$ if and only if it is a hyper-accessibility pseudotopology at x_0.*

A mapping $f\colon (X,\xi) \to (Y,\tau)$, abridged $f\colon \xi \to \tau$, is continuous if $x \in \lim_\xi \mathscr{F}$ implies $f(x) \in \lim_\tau f(\mathscr{F})$ for each filter $\mathscr{F}$ on X. The above somewhat abusive abbreviation should not be confounded with a mapping from ξ-open to τ-open sets.

If f is surjective and if ξ is a convergence on X, then $f\xi$ stands for the finest convergence on Y making f into a continuous mapping. Observe that $f\colon (X,\xi) \to (Y,\tau)$ is continuous if and only if $f\xi \geqslant \tau$. Of course, $T(f\xi)$ is the finest topology on Y for which f is continuous. In other words, f is a topologically quotient map if and only if $T(f\xi) = \tau$. Analogously, as pointed out in [13, 4], f is *pseudo-open* or *hereditarily quotient* if and only if $P(f\xi) = \tau$, *countably bi-quotient* if and only if $P_\omega(f\xi) = \tau$ and *bi-quotient* if and only if $S(f\xi) = \tau$. More generally, if J is a projection, then $f\colon \xi \to \tau$ is said to be a *J-quotient* if $\tau = J(f\xi)$. We consider also a broader notion of a *J-map*, i. e., such that

$$\tau \geqslant J(f\xi). \tag{4.2}$$

A J-map is a J-quotient map if and only if it is continuous. As observed in [4], the original definitions can be expressed with the aid of a single formula. Namely, a continuous map $f\colon \xi \to \tau$ fulfils

$$y_0 \in \operatorname{adh}_\tau \mathscr{F} \implies f^-(y_0) \cap \operatorname{adh}_\xi f^-(\mathscr{F}) \neq \emptyset \tag{4.3}$$

for every filter $\mathscr{F}$ on Y if and only if it is *bi-quotient at y_0* [11, 14], for every countably based filter $\mathscr{F}$ on Y if and only if it is *countably bi-quotient at y_0* [17, 18], for every principal filter $\mathscr{F}$ on Y if and only if it is *pseudo-open at y_0* [1], for every principal filter $\mathscr{F}$ of a $f\xi$-closed set if and only if it is *topologically quotient*. Of course, formula (4.3) makes sense for general convergences τ and ξ.

It was proved by G. Whyburn [19] and [20, Theorem 2] that a T_1 topology is an accessibility topology if and only if every topologically quotient map onto it is pseudo-open (see Theorem 1.3). V. Kannan, in [12], proved the same result without the condition T_1. On the other hand, F. Siwiec proves in [17, Theorem 4.3] that a topology is a strong accessibility topology if and only if every topologically quotient map onto it is countably bi-quotient. Both the results are special cases of the following characterization in which the projection J is equal either to P or to P_ω or else to S.

Theorem 4.4. *Let $J \geqslant T$ be a projection of the type* (2.1). *A topology τ is topologically maximal in* $\operatorname{fix} J$ *if and only if for every topologically quotient map f from (a topology) ξ to τ, one has $J(f\xi) = \tau$.*

Proof. Let f be a topologically quotient map from (X, ξ) onto (Y, τ) and let $J(f\xi) > \tau$. As $T\big(J(f\xi)\big) = T(f\xi) = \tau$, the topology τ is not topologically maximal in $\operatorname{fix} J$.

Conversely, if τ is not topologically maximal in $\operatorname{fix} J$, then there are y_0 and a filter $\mathscr{H}_0 \in \mathfrak{J}$ such that for every τ-closed set F with $y_0 \in \operatorname{cl}_\tau(F \setminus y_0)$, there exists $H \in \mathscr{H}$ such that $y_0 \in \operatorname{cl}_\tau(F \setminus H \setminus y_0)$. Let ϑ be the convergence as in (3.2). Since $y_0 \notin \bigcap\limits_{H \in \mathscr{H}_0} H$, there exists $H_0 \in \mathscr{H}_0$ such that $y_0 \in H^c$ for every $H \in \mathscr{H}_0$ with $H \subset H_0$. For every such H, let us consider the following topology τ_H on H^c: the neighborhood filter of y_0 is the trace on H^c of the neighborhood filter of y_0 in Y; all the other points are isolated. Let τ_0 be the topology on a copy Y_0 of Y for which y_0 is isolated and which coincides with τ for all the other points. The natural map f from the sum topology $\xi = \bigoplus\limits_{H_0 \supset H \in \mathscr{H}_0} \tau_H \oplus \tau_0$ on $\bigoplus\limits_{H_0 \supset H \in \mathscr{H}_0} H^c \oplus Y_0$ onto Y is topologically quotient: $T(f\xi) = \tau$. On the other hand, $f\xi = \vartheta$ and thus $J(f\xi) > T(f\xi)$. $\square$

5. Covering maps

The following properties are traditionally defined for topologies (the references we give below concern the topological case), but it is natural and essential for our approach to formulate them for general convergences. A convergence τ is

sequential [8] if each sequentially closed set is closed;

Fréchet at x_0 [1, 8, 9] if for every set A such that $x_0 \in \operatorname{cl}_\tau A$, there exists a sequence (x_n) in A convergent to x_0;

strongly Fréchet (or *countably bi-sequential*) *at* x_0 [17, 16] if for every countably based filter $\mathscr{F}$ adherent (in τ) to x_0, there is a countably based filter $\mathscr{G}$ convergent to x_0 (in τ) and such that $\mathscr{G} \# \mathscr{F}$; in the case of pretopologies τ, the above condition amounts to the following: if for every decreasing sequence of sets (A_n) with $x_0 \in \bigcap_n \mathrm{cl}_\tau A_n$, there is a sequence $x_n \in A_n$ that converges to x_0;

bi-sequential at x_0 [16] if for every filter $\mathscr{F}$ adherent (in τ) to x_0, there is a countably based filter $\mathscr{G}$ convergent to x_0 (in τ) and such that $\mathscr{G} \# \mathscr{F}$;

a sequence convergence at x_0 if for every filter $\mathscr{F}$ adherent (in τ) to x_0, there is a sequence filter $\mathscr{E}$ convergent to x_0 (in τ) and such that $\mathscr{E} \# \mathscr{F}$ (in view of [6, Theorem 3.5], this amounts to: if $x_0 \in \lim_\tau \mathscr{F}$, then there exists a sequence filter $\mathscr{E} \leqslant \mathscr{F}$ such that $x_0 \in \lim_\tau \mathscr{E}$).

In [17], F. Siwiec introduces the notion of sequence-covering maps: a continuous map $f\colon X \to Y$ is *sequence-covering* if for every sequence $(y_n)_n$ in Y convergent to y, there exists a sequence $(x_n)_n$ convergent to x so that $f(x_n) = y_n$ and $f(x) = y$. His Theorems 4.1, 4.2 and 4.4 are resumed in the following:

Theorem 5.1. *A topology τ is sequential (resp. Fréchet, strongly Fréchet) if and only if every sequence-covering map f from a topology ξ to τ is topologically quotient (resp. hereditarily quotient, countably bi-quotient).*

Before showing that these three theorems not only extend to arbitrary convergences, but are special cases of a single abstract result (Theorem 5.5), let us consider another group of results analogous to Theorem 5.1 in the sense that the role of convergent sequences is played by compact sets. We shall see that they also follow from Theorem 5.5. A mapping $f\colon \xi \to \tau$ is said to be *compact-covering* if for every τ-compact set K, there exists a ξ-compact set C such that $f(C) = K$ (a subset K of a convergence space is compact if $\mathrm{adh}\,\mathscr{H} \cap K \neq \emptyset$ for every filter $\mathscr{H}$ on K). In all the following definitions, ξ and τ are general convergences rather than topologies as is the case in classical definitions [7].

We say that a convergence τ is

locally compact if for every filter $\mathscr{F}$ that converges to x, there exists a compact set K such that $x \in K \in \mathscr{F}$;[3]

a *k-convergence* if a set is closed provided that its intersection with each compact set is closed;

a *k'-convergence* at x_0 if for every set A such that $x_0 \in \mathrm{cl}_\tau A$, there exists a compact set K such that $x_0 \in \mathrm{cl}_\tau(A \cap K)$;

a *strongly k'-convergence* at x_0 if for every countably based filter $\mathscr{F}$ adherent (in τ) to x_0, there exists a compact set K such that $x_0 \in \mathrm{adh}_\tau(\mathscr{F} \vee K)$, where $\mathscr{F} \vee K$ stands for the supremum of $\mathscr{F}$ and the principal filter of K.

[3] In [10] a convergence is called locally compact if every filter contains a compact set; the two defintions coincide for Hausdorff convergences.

The following theorem collects and generalizes the results of F. Siwiec and V. J. Mancuso [18] (for locally compact topologies and k'-topologies), A. V. Arhangel'skii [2] and E. Michael [15, Lemma 11.2] (for k-topologies) and F. Siwiec [17] (for strongly k'-topologies).

We say that a convergence τ is *topologically Hausdorff* if $T\tau$ is a Hausdorff topology.

Theorem 5.2. *Let τ be either a topologically Hausdorff pseudotopology or an arbitrary topology. Then τ is a k-convergence (k', strongly k', locally compact) if and only if each compact-covering map f from a topology ξ to τ is topologically quotient (resp. pseudo-open, countably bi-quotient, bi-quotient).*

Observe that the above properties of convergences are, of topological, pretopological, paratopological, pseudotopological and general nature in the sense that τ has a property if and only if, respectively, $T\tau$, $P\tau$, $P_\omega\tau$, $S\tau$ (and τ) does.

In order to put the listed concepts into a unified framework, let τ be a convergence and consider $\mathrm{First}\,\tau$, the least first-countable convergence finer than τ,[4] the least sequence convergence $\mathrm{Seq}\,\tau$, finer than τ,[5] and $K\tau$ the least locally compact convergence finer than τ.[6] The mappings Seq, First and K are *co-projections*, that is, isotone expansive idempotent mappings.

All classes of convergences that have been described in this section admit the common characterization [4]

$$(5.1)\qquad \big(\forall\, \mathscr{F} \in \mathfrak{J}(\tau)\big)\big(x_0 \in \mathrm{adh}_\tau \mathscr{F} \implies x_0 \in \mathrm{adh}_{E\tau} \mathscr{F}\big),$$

where E is equal either to First or K, and $\mathfrak{J}(\tau)$ is the family of principal filters of $(E\tau)$-closed sets, of principal filters, of countably based filters and of arbitrary filters. Sequential, Fréchet and strongly Fréchet convergences, but not bi-sequential convergences, can be also characterized by (5.1) with $E = \mathrm{Seq}$.

It was observed in [4] that (5.1) amounts to

$$(5.2)\qquad \tau \geqslant JE\tau,$$

where the projection J corresponds via (2.1) to the family of filters $\mathfrak{J}(\tau)$. Namely, the topologization T corresponds to the class of the principal filters of closed sets, P to the class of principal filters, P_ω to the class of countably based filters, and S to the class of all filters. The following table recapitulates the corresponding properties of the type (5.2).

[4] $x \in \lim_{\mathrm{First}\,\tau} \mathscr{F}$ if there exists a countably based filter $\mathscr{G}$ such that $x \in \lim_\tau \mathscr{G}$ and $\mathscr{G} \subset \mathscr{F}$.

[5] $x \in \lim_{\mathrm{Seq}\,\tau} \mathscr{F}$ if there exists a sequence filter $\mathscr{E}$ such that $x \in \lim_\tau \mathscr{E}$ and $\mathscr{E} \subset \mathscr{F}$.

[6] $x \in \lim_{K\tau} \mathscr{F}$ if $x \in \lim_\tau \mathscr{F}$ and if there exists a τ-compact set $C \in \mathscr{F}$.

J \ E	Seq	First	K
T	sequential	sequential	k-convergence
P	Fréchet	Fréchet	k'-convergence
P_ω	strongly Fréchet	strongly Fréchet	strongly k'-convergence
S	sequence convergence	bi-sequential	locally compact

Table. Convergences of the type $\tau \geqslant JE\tau$.

Of course, a mapping f from ξ to τ is continuous if and only if[7] $f\xi \geqslant \tau$. On the other hand, f is sequence-covering if and only if $\operatorname{Seq}\tau \geqslant f(\operatorname{Seq}\xi)$, and if f is compact-covering, then $K\tau \geqslant Sf(K\xi)$.

Let E be a co-projection. A map $f\colon \xi \to \tau$ is an *E-relatively J-map* [4], if $f\colon E\xi \to E\tau$ is a J-map, that is, if

$$E\tau \geqslant Jf(E\xi). \tag{5.3}$$

Therefore, sequence-covering maps are exactly Seq-relatively I-maps, where I stands for the identity map. As for compact-covering maps, we have

Proposition 5.3. *Let $f\colon \xi \to \tau$ be a continuous map. If $K\tau$ is Hausdorff, then each property below implies its successor:*

1. *f is a K-relatively I-map;*
2. *f is compact-covering;*
3. *f is a K-relatively S-map.*

Proof. (1) $\Longrightarrow$ (2) If f is not compact-covering, then there exists a τ-compact set K such that $K \not\subset f(C)$ for every ξ-compact set C, that is, $K \cap f(C)^c \neq \emptyset$. The family $\mathscr{F} = \{f(C)^c\colon C \in \mathscr{K}(\xi)\}$, where $\mathscr{K}(\xi)$ stands for the family of ξ-compact sets, is a filter base and $K \in \mathscr{F}^{\#}$. If $\mathscr{U}$ is an ultrafilter finer than $\mathscr{F} \vee K$, then $\lim_{K\tau} \mathscr{U} \neq \emptyset$. If $\mathscr{G}$ is a filter of X such that $f(\mathscr{G}) = \mathscr{U}$, then for each $C \in \mathscr{K}(\xi)$ there exists $G \in \mathscr{G}$ such that $f(G) \subset f(C)^c$, hence $f(G) \cap f(C) = \emptyset$ and thus $G \cap C = \emptyset$, that is, $G \subset C^c$. Therefore $C^c \in \mathscr{G}$ for every ξ-compact set C so that $C \notin \mathscr{G}$ and hence $\mathscr{G}$ is not $K\xi$-convergent. Consequently, $\mathscr{U}$ is not $f(K\xi)$ convergent and $K\tau \not\geqslant fK\xi$.

(2) $\Longrightarrow$ (3) Suppose that there exists an ultrafilter $\mathscr{U}$ such that $y \in \lim_{K\tau} \mathscr{U}$. Hence $y \in \lim_\tau \mathscr{U}$ and there exists a τ-compact set K in $\mathscr{U}$. Since f is compact-covering, there exists a ξ-compact set C such that $f(C) = K$. Therefore, C is

[7] Recall that $f\xi$ stands for the finest convergence on Y making f into a continuous mapping.

in $f^-(\mathscr{U})^\#$. Consider an ultrafilter $\mathscr{W}$ of $f^-(\mathscr{U}) \vee C$. Then $f(\mathscr{W}) = \mathscr{U}$, and $\mathscr{W}$ converges for $K\xi$ to an element x of C and consequently $f(x) \in \lim_{f(K\xi)} \mathscr{U}$. Since, by continuity, $f(K\xi)$ is finer than $K\tau$, $f(x) \in \lim_{K\tau} \mathscr{U}$ and, by unicity of limits, y is equal to $f(x)$. □

Generalizing the classical Theorems 5.1 and 5.2, we prove in Theorem 5.4 that a convergence τ fulfils (5.3) if and only if every E-relatively J-map is a J-map. Actually we prove more, namely that a necessary condition for (5.3) to hold is that every weakly E-relatively J-map is a J-map. A map $f\colon \xi \to \tau$ is a *weakly E-relatively J-map* if $f\colon \xi \to E\tau$ is a J-map, that is, if

$$E\tau \geqslant J(f\xi). \tag{5.4}$$

The latter class is essentially broader than the former; Example 5.6 shows that Theorem 5.4 improves the quoted classical theorems even in the context of mere topologies.

For example, a map $f\colon X \to Y$ is weakly first-countable-relatively I-map if for every $y \in Y$ and each countably based filter $\mathscr{F}$ such that $y \in \lim \mathscr{F}$, there exists an arbitrary (!) filter $\mathscr{G}$ such that $\mathscr{F} = f(\mathscr{G})$ and $\lim \mathscr{G} \cap f^-(y) \neq \emptyset$; it is weakly locally-compact-relatively S-map if for every $y \in Y$ and each filter $\mathscr{F}$ containing a compact set and such that $y \in \lim \mathscr{F}$, there exists an arbitrary (!) filter $\mathscr{G}$ such that $\mathscr{F} = f(\mathscr{G})$ and $\lim \mathscr{G} \cap f^-(y) \neq \emptyset$.

Taking into account Formula (4.2) that characterizes various quotient maps, we are in a position to state the following general theorem:

Theorem 5.4. *Let E be a co-projection and $D \geqslant J$ be projections. Then the following properties are equivalent:*

1. $\tau \geqslant JE\tau$*;*
2. *every E-relatively D-map onto τ is a J-map;*
3. *every weakly E-relatively D-map onto τ is a J-map.*

Proof. (1) $\Longrightarrow$ (2) Let $\tau \geqslant JE\tau$ and let $f\colon \xi \to \tau$ be a weakly E-relatively D-map. Using (5.3), we have

$$\tau \geqslant JE\tau \geqslant JDf(E\xi) \geqslant J(f\xi). \tag{5.5}$$

(2) $\Longrightarrow$ (3) Because every E-relatively D-map is a weakly E-relatively D-map.

(3) $\Longrightarrow$ (1) If $\tau \ngeqslant JE\tau$, then the identity map $i\colon \xi = E\tau \to \tau$, which is always a weakly E-relatively D-map, is not a J-map. □

Theorem 5.4 specializes for other important co-projections. We say that a mapping $f\colon \xi \to \tau$ is *first-countable-covering* if for every countably based filter $\mathscr{F}$ that τ-converges to y, there exists $x \in f^{-}(y)$ and a countably based filter $\mathscr{G}$ that ξ-converges to x and satisfies $f(\mathscr{G}) = \mathscr{F}$; this amounts to the inequality $\mathrm{First}\,\tau \geqslant f(\mathrm{First}\,\xi)$.

We could now apply Theorem 5.4 and obtain analogues of Theorem 5.1. Instead we are going to improve Theorem 5.4 in the case where E is one of the three co-projections Seq, First, K. The improvement consists in characterizations in terms of E-relatively J-maps or weakly E-relatively J-maps with topologies (rather than general convergences) as domains.

Theorem 5.5. *Let $J \leqslant S$ be a projection. Let E be a co-projection equal to* Seq *or* First, *or let τ be a topologically Hausdorff pseudotopology and $E = K$. Then for every projection $D \geqslant J$, the following properties are equivalent:*

1. $\tau \geqslant JE\tau$;
2. *each E-relatively D-map (or S-map) from a topology onto τ is a J-map;*
3. *each weakly E-relatively D-map (or weakly E-relatively S-map) from a topology onto τ is a J-map.*

Moreover, if τ is a topology, then the Hausdorff condition can be dropped.

Proof. By Theorem 5.4, (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

(3) $\Longrightarrow$ (1) Suppose that $\tau \ngeqslant JE\tau$. Then $i\colon E\tau \to \tau$ is a weakly E-relatively D-map for every projection D, but not a J-map.

In the case of $E = \mathrm{Seq}$, let ξ be the sum topology of all convergent τ-sequences with their limits and let $h\colon \xi \to \mathrm{Seq}\,\tau$ be the canonical (convergence) quotient map.

In the case of $E = \mathrm{First}$, let ξ be the topological sum of the form $\bigoplus_{\mathscr{G} \in \mathfrak{G}} X_{\mathscr{G}}$, where $\mathfrak{G}$ is the collection of all τ-convergent countably based filters the elements of which contain the limit; here $\mathscr{G}$ is the neighborhood filter of $\lim_\tau \mathscr{G}$ in $X_{\mathscr{G}}$ while the other points of $X_{\mathscr{G}}$ are isolated. Let $h\colon \xi \to E\tau$ be the canonical (convergence) quotient map.

Consider now the case $E = K$. Recall that $\mathscr{K}(\tau)$ denotes the family of all τ-compact sets. For every compact set K, the restriction $\tau\!\restriction_K$ of the pseudotopology τ to K is a topology, because $T\tau$ is Hausdorff [10]. If τ is already a topology, $\tau\!\restriction_K$ is a topology without the Hausdorff assumption. Let $\xi = \bigoplus_{K \in \mathscr{K}(\tau)} \tau\!\restriction_K$. Then $f = i \circ h$ fulfils (5.4), but $J(f\xi) = J\big(i \circ h(\xi)\big) = JE\tau$ and $\tau \ngeqslant JE\tau$. $\square$

The class of weakly E-relatively J-maps is essentially broader than the class of E-relatively S-maps.

Example 5.6. *(a weakly* Seq*-relatively I-map non* Seq*-relatively I-map)* Let $X = \{x_\infty\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{(n,k)} : n, k \in \mathbb{N}\}$ be the domain of a bi-sequence that converges to x_∞. Denote by ξ the subspace topology of $Y = \{x_\infty\} \cup \{x_{(n,k)} : n, k \in \mathbb{N}\}$

and by τ the subspace topology of Z $= \{x_\infty\} \cup \{x_n : n \in \mathbb{N}\}$. Let $f\colon Y \to Z$ be defined by $f(x_\infty) = x_\infty$ and $f(x_{(n,k)}) = x_n$ for each k. Now, τ is sequential and locally compact (i. e., $\tau = \operatorname{Seq}\tau = K\tau$), and the map f is open, hence almost-open (i. e., $\tau \geqslant f\xi$). Hence f is a weakly Seq-relatively I-map and a weakly K-relatively I-map (i. e., $K\tau = \operatorname{Seq}\tau \geqslant f\xi$). On the other hand, $\iota = K\xi = \operatorname{Seq}\xi$ is the discrete topology ι, and thus $f\iota$ is also the discrete topology; therefore f is neither a Seq-relatively I-map nor a K-relatively I-map.

In view of the table on page 380 and because bi-quotient maps are precisely continuous S-maps, we have the following

Corollary 5.7. *A pseudotopology τ is a sequence convergence if and only if each sequence-covering map f from a topology ξ to τ is bi-quotient.*

On the other hand, we have

Corollary 5.8. *A pseudotopology τ is a bi-sequential convergence if and only if each first-countable-covering map f from a topology ξ to τ is bi-quotient.*

As mentioned above, each Fréchet pretopology with unicity of sequence limits is an accessibility pretopology and each strongly Fréchet paratopology with unicity of sequence limits is a strong accessibility paratopology.

Not every bi-sequential Hausdorff pseudotopology is a hyper-accessibility pseudotopology. In fact, in view of the following proposition the natural topology of the unit interval is an example.

Proposition 5.9. *If a sequential Hausdorff pretopology is a hyper-accessibility pretopology, then it is a sequence convergence.*

Proof. Let τ be a Hausdorff pretopology which is sequential ($T\operatorname{Seq}\tau = \tau$) and which is not a sequence convergence ($\operatorname{Seq}\tau > \tau$). By [6, Theorem 6.3, Corollary 7.4], $\operatorname{Seq}\tau = \operatorname{Seq}T\tau$ and by [6, Theorem 5.4], $S\operatorname{Seq}\tau = \operatorname{Seq}\tau$ so that $\operatorname{Seq}\tau$ is a pseudotopology strictly finer than τ and with the same topological projection. □

F. Siwiec [17] mentioned the converse of one of the preceding remarks, namely that each Hausdorff sequential strong accessibility topology is strongly Fréchet. If we set $D = P_\omega$ and $E = \mathrm{First}$ or $E = \mathrm{Seq}$, then we see that the observation of Siwiec is a special case of the following general fact:

Theorem 5.10. *Let $D \geqslant T$ be a projection and E a co-projection. If $TE\tau = \tau$ is topologically maximal in* fix D, *then $DE\tau = \tau$.*

Proof. Let $TE\tau = \tau$. As $DE\tau \geqslant TE\tau$, by maximality, $DE\tau = \tau$. □

On the other hand, we have

E x a m p l e 5.11. *(Hausdorff, non sequential, topologically maximal topology in* $\operatorname{fix} S$*)* Let $\mathscr{U}$ be a free ultrafilter on X. Let τ be the topology on $X \cup \{\infty\}$ for which $\mathscr{U}$ is the trace of $\mathscr{N}_\tau(\infty)$ on X and all the other points are isolated. This Hausdorff topology is not sequential (more precisely, $T\,\mathrm{Seq}\,\tau$ is the discrete topology), but it is topologically maximal in $\operatorname{fix} S$. In fact, the only pseudotopology that is strictly finer than τ is the discrete topology ι.

Incidentally, the identity $i\colon \iota \to \tau$ is obviously sequence-covering but not topologically quotient.

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