DISJOINT SEQUENCES IN BOOLEAN ALGEBRAS

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Abstract. We deal with the system Conv$B$ of all sequential convergences on a Boolean algebra B. We prove that if α is a sequential convergence on B which is generated by a set of disjoint sequences and if β is any element of Conv$B$, then the join $\alpha \vee \beta$ exists in the partially ordered set Conv$B$. Further we show that each interval of Conv$B$ is a Brouwerian lattice.

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1. Introduction

Some types of sequential convergences on Boolean algebras were investigated by Löwig [3], Novák and Novotný [4] and Papangelou [5].

This note is a continuation of [1]. Throughout the paper we assume that B is a Boolean algebra which has more than one element. $\operatorname{Conv} B$ is the system of all sequential convergences on B which are compatible with the structure of B. For the sake of completeness, the definition of $\operatorname{Conv} B$ as given in [1] is recalled in Section 2.

The system $\operatorname{Conv} B$ is partially ordered by the set-theoretical inclusion. It is a $\wedge$-semilattice with the least element (the discrete convergence on B). In general, $\operatorname{Conv} B$ fails to be a lattice; i.e., for α and β in $\operatorname{Conv} B$, the join $\alpha \vee \beta$ need not exist in the partially ordered set $\operatorname{Conv} B$.

A sufficient condition for $\operatorname{Conv} B$ to be a lattice was found in [2].

We denote by $D(B)$ the system of all sequences (x_n) in B such that

(i) $x_{n(1)} \wedge x_{n(2)} = 0$ whenever $n(1)$ and $n(2)$ are distinct positive integers;
(ii) $x_n > 0$ for each positive integer n.

The sequences belonging to $D(B)$ will be called disjoint.

We prove that for each subset A of $D(B)$ there exists a sequential convergence $\alpha \in \operatorname{Conv} B$ which is generated by A and that for any $\beta \in \operatorname{Conv} B$ the join $\alpha \vee \beta$ exists in the partially ordered set $\operatorname{Conv} B$.

Further we show that each interval of $\operatorname{Conv} B$ is a complete lattice satisfying the identity

$$\Big(\bigvee_{i\in I} \alpha_i\Big) \wedge \beta = \bigvee_{i\in I} (\alpha_i \wedge \beta).$$

This implies that each interval of $\operatorname{Conv} B$ is a Brouwerian lattice.

2. Preliminaries

We denote by S the system of all sequences in B. Let $\alpha \subseteq S \times B$. If $((x_n), x) \in \alpha$, then we denote this fact by writing $x_n \to_\alpha x$. For $a \in B$, $\operatorname{const} a$ denotes the sequence (x_n) such that $x_n = a$ for each $n \in \mathbb{N}$.

We recall the definitions of $\operatorname{Conv} B$ and $\operatorname{Conv}_0 B$ from [1].

2.1. Definition. A subset of $S \times B$ is said to be a convergence on B if the following conditions are satisfied:

(i) If $x_n \to_\alpha x$ and (y_n) is a subsequence of (x_n), then $y_n \to_\alpha x$.
(ii) If $(x_n) \in S$, $x \in B$ and if for each subsequence (y_n) of (x_n) there is a subsequence (z_n) of (y_n) such that $z_n \to_\alpha x$, then $x_n \to_\alpha x$.
(iii) If $a \in B$ and $(x_n) = \operatorname{const} a$, then $x_n \to_\alpha a$.
(iv) If $x_n \to_\alpha x$ and $x_n \to_\alpha y$, then $x = y$.
(v) If $x_n \to_\alpha x$ and $y_n \to_\alpha y$, then $x_n \vee y_n \to x \vee y$, $x_n \wedge y_n \to_\alpha x \wedge y$ and $x'_n \to_\alpha x'$.
(vi) If $x_n \leqslant y_n \leqslant z_n$ is valid for each $n \in \mathbb{N}$ and $x_n \to_\alpha x$, $z_n \to_\alpha x$, then $y_n \to_\alpha x$.

The system of all convergences on B is denoted by $\operatorname{Conv} B$.

For each $\alpha \in \operatorname{Conv} B$ we put

$$\alpha_0 = \{(x_n) \in S \colon x_n \to_\alpha 0\}.$$

Further we define

$$\operatorname{Conv}_0 B = \{\alpha_0 \colon \alpha \in \operatorname{Conv} B\}.$$

Both the systems $\operatorname{Conv} B$ and $\operatorname{Conv}_0 B$ are partially ordered by the set-theoretical inclusion; the suprema and infima (if they exist) in $\operatorname{Conv} B$ or in $\operatorname{Conv}_0 B$ are denoted by the symbol $\vee$ or $\wedge$, respectively.

Next, we denote by d the system of all $((x_n), x) \in S \times B$ such that the set $\{n \in \mathbb{N}\colon x_n \neq x\}$ is finite. Then d is the least element of $\operatorname{Conv} B$.

For each $\alpha \in \operatorname{Conv} B$ we put $f(\alpha) = \alpha_0$.

2.2. Lemma. *The mapping f is an isomorphism of the partially ordered set $\operatorname{Conv} B$ onto the partially ordered set $\operatorname{Conv}_0 B$.*

P r o o f. We have $f(\operatorname{Conv} B) = \operatorname{Conv}_0 B$. In view of 1.4 in [1], f is a monomorphism.

Let $\alpha, \beta \in \operatorname{Conv} B$, $\alpha \leqslant \beta$. Further let $(x_n) \in \alpha_0$. Hence $((x_n), 0) \in \alpha$, thus $((x_n), 0) \in \beta$ and then $(x_n) \in \beta_0$. Thus $\alpha_0 \leqslant \beta_0$.

Now let $\alpha, \beta \in \operatorname{Conv} B$, $\alpha_0 \leqslant \beta_0$. Assume that $((x_n), x) \in \alpha$. In view of 1.3 in [1] we have

$$x_n \wedge x' \to_\alpha 0, \quad x'_n \wedge x \to_\alpha 0.$$

Thus from the relation $\alpha_0 \leqslant \beta_0$ we obtain

$$x_n \wedge x' \to_\beta 0, \quad x'_n \wedge x \to_\beta 0.$$

Then by applying 1.3 in [1] again we get $x_n \to_\beta x$. Hence $\alpha \leqslant \beta$. □

As a consequence we obtain that d_0 is the least element of $\operatorname{Conv}_0 B$.

2.3. Lemma. (Cf. [1].) (i) *$\operatorname{Conv}_0 B$ is a $\wedge$-semilattice and each interval of $\operatorname{Conv}_0 B$ is a complete lattice.*

(ii) *If $\emptyset \neq \{\alpha_i^0\}_{i \in I} \subseteq \operatorname{Conv}_0 B$, then*

$$\bigwedge_{i \in I} \alpha_i^0 = \bigcap_{i \in I} \alpha_i^0.$$

(iii) *There exists a Boolean algebra B_1 such that $\operatorname{Conv}_0 B_1$ fails to be a lattice.*

From 2.2 and 2.3 we infer

2.4. Proposition. *$\operatorname{Conv} B$ is a $\wedge$-semilattice and each interval of $\operatorname{Conv} B$ is a complete lattice. There exists a Boolean algebra B_1 such that $\operatorname{Conv} B_1$ is not a lattice.*

3. On the set $D(B)$

We apply the notation as in the previous sections. A subset T of S is called regular if there exists $\alpha_0 \in \operatorname{Conv}_0 B$ such that $T \subseteq \alpha_0$.

Let T be a regular subset of S and let α_0 be as above. Then in view of 2.3 there exists an element $\alpha^0(T)$ of $\operatorname{Conv}_0 B$ such that $\alpha^0(T)$ is the least element of $\operatorname{Conv}_0 B$ having T as a subset. We say that $\alpha^0(T)$ is the element of $\operatorname{Conv}_0 B$ which is generated by T. We also say that T generates the convergence α, where $\alpha_0 = \alpha^0(T)$.

If T is regular, then clearly each subset of T is regular.

For $(x_n), (y_n) \in S$ we put $(x_n) \leqslant (y_n)$ if $x_n \leqslant y_n$ for each $n \in \mathbb{N}$. Then S turns out to be a Boolean algebra. Let A be a nonempty subset of S. We denote by

A^*—the set of all $(x_n) \in S$ such that for each subsequence (y_n) of (x_n) there exists a subsequence (z_n) of (y_n) which belongs to A;

$[A]$—the ideal of the Boolean algebra generated by the set A;

δA—the set of all subsequences of sequences belonging to A.

The following assertion is easy to verify.

3.1. Lemma. *Let A be a nonempty subset of S. Then $[A]$ is the set of all sequences $(z_n) \in S$ such that there exist $k \in \mathbb{N}$ and $(w_n^1), (w_n^2), \ldots, (w_n^k) \in A$ having the property that the relation*

$$z_n \leqslant w_n^1 \vee w_n^2 \vee \ldots \vee w_n^k$$

is valid for each $n \in \mathbb{N}$.

3.2. Lemma. (Cf. [1], 2.9.) *Let $\emptyset \neq A \subseteq S$. Then the following conditions are equivalent:*

(i) *A is regular.*

(ii) *If $(y_n^1), (y_n^2), \ldots, (y_n^k)$ are elements of δA and if b is an element of B such that $b \leqslant y_n^1 \vee y_n^2 \vee \ldots \vee y_n^k$ is valid for each $n \in \mathbb{N}$, then $b = 0$.*

From the definition of $\operatorname{Conv}_0 B$ and from [1], 2.5 we conclude

3.3. Lemma. *Let $A \neq \emptyset$ be a regular subset of S. Then $[\delta A]^*$ is an element of $\operatorname{Conv}_0 B$ which is generated by the set A.*

3.4. Lemma. (Cf. [1], 5.2.) *Let $(x_n) \in D(B)$. Then the set $\{(x_n)\}$ is regular.*

3.5. Lemma. *Let $(x_n) \in D(B)$ and suppose that $(y_n^1), (y_n^2), \ldots, (y_n^k)$ are subsequences of (x_n). Put $(z_n) = y_n^1 \vee y_n^2 \vee \ldots \vee y_n^k$ for each $n \in \mathbb{N}$. Then there exists a subsequence (t_n) of (z_n) such that $(t_n) \in D(B)$.*

P r o o f. For each $i \in \{1,2,\dots,k\}$ and each $n \in \mathbb{N}$ there is a positive integer $j(i,n)$ such that

$$y_n^i = x_{j(i,n)}.$$

Thus for each $i \in \{1,2,\dots,k\}$ we have

$$j(i,n) \to \infty \quad \text{as} \quad n \to \infty. \tag{1}$$

We define the sequence (t_n) by induction as follows. We put $t_1 = z_1$. Suppose that $n > 1$ and that $t_1, t_2, \dots, t_{n-1}$ are defined. Hence there are $\ell(1), \ell(2), \dots, \ell(n-1) \in \mathbb{N}$ with

$$t_s = z_{\ell(s)} \quad \text{for } s = 1,2,\dots,n-1.$$

In view of (1) there exists the least positive integer p having the property that for each $s \in \{1,2,\dots,n-1\}$ and each $i(1), i(2) \in \{1,2,\dots,k\}$ the relation

$$j(i(1),s) < j(i(2),p)$$

is valid. Then we put $t_n = z_p$.

Hence $t_n \wedge t_s = 0$ for $s = 1,2,\dots,n-1$. Thus $(z_n) \in D(B)$. $\square$

3.6. Lemma. *Let $\emptyset \neq A_1$ be a regular subset of S and let $(x_n) \in D(B)$. Then the set $A_1 \cup \{(x_n)\}$ is regular.*

P r o o f. We denote by α_0 the element of $\operatorname{Conv}_0 B$ which is generated by the set A_1. Put $A = A_1 \cup \{(x_n)\}$. By way of contradiction, suppose that A fails to be regular. Then in view of 3.2 there are $(y_n^1), (y_n^2), \dots, (y_n^m) \in \delta A$ and $0 < b \in B$ such that the relation

$$0 < b \leqslant y_n^1 \vee y_n^2 \vee \ldots \vee y_n^m$$

is valid for each $n \in \mathbb{N}$. Put

$$M_1 = \{i \in \{1,2,\dots,m\} \colon (y_n^i) \in A_1\},$$
$$M_2 = \{1,2,\dots,m\} \setminus M_1.$$

Since the set A_1 is regular, in view of 3.2 the relation $M_2 = \emptyset$ cannot hold. Further, according to 3.4 and 3.2, the set M_1 cannot be empty. Denote

$$z_n^1 = \bigvee y_n^i \ \ (i \in M_1), \quad z_n^2 = \bigvee y_n^i \ \ (i \in M_2).$$

Then $(z_n^1) \in \alpha_0$.

According to 3.5 there exists a mapping $\varphi\colon \mathbb{N} \to \mathbb{N}$ such that φ is increasing and the sequence $(z^2_{\varphi(n)})$ belongs to $D(B)$. We have

$$0 < b \leqslant z^1_{\varphi(n)} \vee z^2_{\varphi(n)} \quad \text{for each } n \in \mathbb{N}.$$

Put

$$b \wedge z^1_{\varphi(n)} = q^1_n, \quad b \wedge z^2_{\varphi(n)} = q^2_n.$$

Then

$$b = q^1_n \vee q^2_n$$

for each $n \in \mathbb{N}$. We have $(q^1_n) \in \alpha_0$ and $(q^2_n) \in D(B)$.

Since $b = q^1_{n+1} \vee q^2_{n+1}$ we get

$$q^2_n = q^2_n \wedge b = q^2_n \wedge (q^1_{n+1} \vee q^2_{n+1}) = (q^2_n \wedge q^1_{n+1}) \vee (q^2_n \wedge q^2_{n+1}) = q^2_n \wedge q^1_{n+1}$$

and clearly $(q^2_n \wedge q^1_{n+1}) \in \alpha_0$. Therefore $(q^1_n \vee q^2_n) \in \alpha_0$ yielding that $\operatorname{const} b \in \alpha_0$, which is impossible. □

By the obvious induction, from 3.6 we obtain

3.7. Lemma. *Let $\emptyset \neq A_1$ be a regular subset of S, $m \in \mathbb{N}$, $(x^1_n), (x^2_n), \ldots, (x^m_n) \in D(B)$. Then the set $A_1 \cup \{(x^1_n), (x^2_n), \ldots, (x^m_n)\}$ is regular.*

Since the system of sequences which is dealt with in the condition (ii) of 3.2 is finite, from 3.7 we conclude

3.8. Proposition. *Let $\emptyset \neq A_1$ be a regular subset of S. Then the set $A_1 \cup D(B)$ is regular.*

It is obvious that if $\emptyset \neq A_2 \subseteq S$, then A_2 is regular if and only if the set $\{\operatorname{const} 0\} \cup A_2$ is regular. Hence by putting $A_1 = \{\operatorname{const} 0\}$, from 3.8 we obtain

3.9. Proposition. *The set $D(B)$ is regular.*

In view of 3.9, there exists $\gamma \in \operatorname{Conv} B$ which is generated by the set $D(B)$.

Let $\alpha_0 \in \operatorname{Conv}_0 B$. According to 3.8, the set $\alpha_0 \cup D(B)$ is regular. Hence there exists $\beta_0 \in \operatorname{Conv}_0 B$ such that β_0 is generated by the set $\alpha_0 \cup D(B)$.

In view of 3.3, we have $\alpha_0 \leqslant \beta_0$ and $\gamma_0 \leqslant \beta_0$. Let $\beta_1 \in \operatorname{Conv}_0 B$, $\beta_1 \geqslant \alpha_0$, $\beta_1 \geqslant \gamma_0$. Thus $D(B) \subseteq \beta_1$ and hence $\alpha_0 \cup D(B) \subseteq \beta_1$. By using 3.3 again we get $\beta_0 \leqslant \beta_1$. Therefore $\beta_0 = \alpha_0 \vee \gamma_0$. We obtain

3.10. Proposition. *Let $\alpha_0 \in \operatorname{Conv}_0 B$. Then the join $\alpha_0 \vee \gamma_0$ exists in the partially ordered set $\operatorname{Conv}_0 B$.*

In view of 2.2 we conclude

3.11. Corollary. *Let $\alpha \in \operatorname{Conv} B$. Then the join $\alpha \vee \gamma$ exists in the partially ordered set* $\operatorname{Conv} B$.

If A_0 is a nonempty subset of $D(B)$, then it is regular and thus there exists $\gamma_1 \in \operatorname{Conv} B$ which is generated by A_0. Clearly $\gamma_1 \leqslant \gamma$; from 3.11 and 2.4 we obtain

3.12. Corollary. *Under the notation as above, for each $\alpha \in \operatorname{Conv} B$ there exists $\alpha \vee \gamma_1$ in* $\operatorname{Conv} B$.

4. A distributive identity

Suppose that μ_1 and μ_2 are elements of $\operatorname{Conv}_0 B$ such that $\mu_1 \leqslant \mu_2$. Consider the interval $[\mu_1, \mu_2]$ of the partially ordered set $\operatorname{Conv}_0 B$. In view of 2.3, this interval is a complete lattice.

Let $\emptyset \neq \{\alpha_i\}_{i\in I} \subseteq [\mu_1, \mu_2]$ and $\beta \in [\mu_1, \mu_2]$. Then the elements

$$\nu_1 = \Big(\bigvee_{i\in I} \alpha_i\Big) \wedge \beta, \quad \nu_2 = \bigvee_{i\in I} (\alpha_i \wedge \beta)$$

exist in $[\mu_1, \mu_2]$ and $\nu_1 \geqslant \nu_2$. Put

$$A_1 = \bigcup_{i\in I} \alpha_i, \quad A_2 = \bigcup_{i\in I} (\alpha_i \cap \beta).$$

Suppose that $(v_n) \in \nu_1$. Hence according to 2.3 we have

$$(v_n) \in \beta \quad \text{and } (v_n) \in \bigvee_{i\in I} \alpha_i.$$

From the second relation and from Lemma 3.3 in [1] we obtain

$$(v_n) \in [A_1]^*.$$

Hence for each subsequence (t_n^1) of (v_n) there is a subsequence (t_n^2) of (t_n^1) such that $(t_n^2) \in [A_1]$.

Let (t_n^1) and (t_n^2) have the mentioned properties. Therefore in view of 3.1 there are $(w_n^1), (w_n^2), \ldots, (w_n^k)$ in A such that the relation

$$t_n^2 \leqslant w_n^1 \vee w_n^2 \vee \ldots \vee w_n^k$$

is valid for each $n \in \mathbb{N}$. Put

$$q_n^j = t_n^2 \wedge w_n^j$$

for each $n \in \mathbb{N}$ and each $j \in \{1, 2, \ldots, k\}$. Thus

$$t_n^2 = q_n^1 \vee q_n^2 \vee \ldots \vee q_n^k \quad \text{for each } n \in \mathbb{N},$$

and $(q_n^1), (q_n^2), \ldots, (q_n^k) \in A_1$. At the same time we have $(q_n^1), (q_n^2), \ldots, (q_n^k) \in \beta$. Hence for each $j \in \{1, 2, \ldots, k\}$ there is $i(j) \in I$ such that

$$(q_n^j) \in \alpha_{i(j)} \cap \beta.$$

In view of 3.1, this yields that (t_n^2) belongs to $[A_2]$. Therefore $(v_n) \in [A_2]^*$. Thus by applying Lemma 3.3 in [1] we get $(v_n) \in \nu_2$.

Summarizing, we have

4.1. Proposition. *Let* $[\mu_1, \mu_2]$ *be an interval of* $\operatorname{Conv}_0 B$, $\beta \in [\mu_1, \mu_2]$, $\emptyset \neq \{\alpha_i\}_{i \in I} \subseteq [\mu_1, \mu_2]$. *Then*

$$\left(\bigvee_{i \in I} \alpha_i\right) \wedge \beta = \bigvee_{i \in I} (\alpha_i \wedge \beta). \tag{1}$$

4.2. Corollary. *Each interval of* $\operatorname{Conv}_0 B$ *is Brouwerian.*

From 4.1 and 2.2 we obtain

4.3. Corollary. *Each interval of* $\operatorname{Conv} B$ *satisfies the identity* (1).

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