DIGRAPHS CONTRACTIBLE ONTO $^*K_3$.

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Abstract. We show that any digraph on $n \geqslant 3$ vertices and with not less than $3n-3$ arcs is contractible onto $^*K_3$

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Introduction

The notation of contraction is well known for non-oriented graphs (cf. [4]). In this paper, Hadwiger gave the following conjecture: If $\chi(G) = p$ then G is contractible onto the complete graph on p vertices K_p. Here $\chi(G)$ denotes the chromatic number of G. Dirac [2] showed this conjecture to be true for $p \leqslant 4$. Wagner [11] showed that the four color theorem implies the case $p = 5$. Robertson, Seymour and Thomas [9] proved the case $p = 6$. For a good survey on the relationship between the minor's existence in G and the generalization of the coloring notion to the digraphs we refer the reader to [5], where an oriented version of Hadwiger's conjecture is given, too. Recently, Jagger [6] has shown that if p is large enough, then any digraph on n vertices having at least $10^5 p\sqrt{\log_2 p} \cdot n$ arcs is contractible onto $^*K_p$. Nevertheless, this nice asymptotical result does not give a right information about the "little" cases. In this direction, Duchet and Kaneti [3] proved that any digraph on n vertices with not less than $5n-8$ arcs is contractible onto $^*K_4$. We give a short proof of the following result discovered by Meyniel [8].

Theorem. *Any digraph on $n \geqslant 3$ vertices and with not less than $3n-3$ arcs is contractible onto $^*K_3$, and this bound is attained for any n.*

We consider only finite digraphs without loops and parallel arcs. An arc of a digraph $G = (V(G), A(G))$ from x to y is the couple (x, y). We say that (x, y) is *incident* to x and y. The couple of arcs (x, y) and (y, x) is called a *symmetrical* arc and is denoted by xy. We will say *edge* instead of arc whenever the orientation is insignificant. The set of *out-neighbours (in-neighbours)* of x is $A^+(x) = \{y \in V(G)\colon (x, y) \in A(G)\}$ $(A^-(x) = \{y \in V(G)\colon (y, x) \in A(G)\})$. $A(x) = A^+(x) \cup A^-(x)$ is the set of neighbours of x. We denote by $d^+(x)(d^-(x))$ the *in-degree (out-degree)* and by $d(x) = d^+(x) + d^-(x)$ the *degree* of x. By contracting one arc we mean identifying its extremities and omitting the loop(s) created. We say that the digraph G is *contractible* onto G' (or G' is a *minor* of G) and we denote $G \geqslant G'$ if G' can be obtained from G by a sequence (possibly empty) of contractions of arcs or removing of arcs or removing of vertices. Clearly, this relation is transitive. The digraph $^*K_p$ contains p vertices and a symmetrical arc between any pair of vertices. The digraph $^*K_3$ is given in Fig. 1:

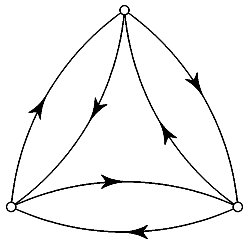

Fig. 1. The digraph $^*K_3$.

Proof and remarks

P r o o f. Let $G = (V(G), A(G))$ be a digraph with $n = |V(G)|$ and $m = |A(G)|$. The proof is done by induction on $n + m$. The result is clearly true for $n = 3$, so we suppose that G has at least 4 vertices. If G contains a vertex x with $d(x) \leqslant 3$, then it is easy to see that $G' = G - x$ verifies the induction hypothesis. This means that $G' \geqslant {}^*K_3$ and by the transitivity of "$\geqslant$", we have $G \geqslant {}^*K_3$. We can assume, in the following, that $d(x) \geqslant 4$ for every vertex x of G. If all vertices of G have a degree $\geqslant 6$, then G has at least $3n$ arcs and the induction hypothesis applies to the graph G' obtained from G by removing one arc. So, we can suppose that G contains at least one vertex u such that $d(u) \in \{4, 5\}$. We can also assume that the following condition is verified:

$(*)$ If G' is obtained from G by contraction of one arc with both its end-vertices in $A(u)$, then $|A(G')| \leqslant |A(G)| - 4$.

Otherwise, the induction hypothesis applies to G'.

Let us suppose that $d^+(u) \geqslant d^-(u)$. We can ensure this to be always the case by changing the orientation of all arcs.

Now, let $d(u) = 4$. If $|A(u)| = 2$, let x and y be the neighbours of u. G contains the symmetrical arcs ux and uy because $d(u) = 4$. The condition $(*)$ implies the existence of the symmetrical arc xy.

If $|A(u)| = 3$, let $A(u) = \{x, y, z\}$ and suppose that there is a symmetrical arc ux. By condition $(*)$, there must be at least one symmetrical arc either between x and y or x and z (see Fig. 2).

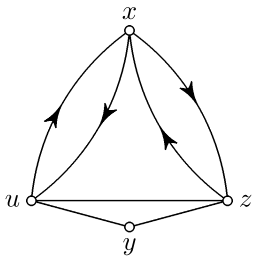

Fig. 2.

If this is not the case, then it is easy to see that for any orientation of the arcs between vertices of $A(u)$, the contraction of any arc incident to u decreases the number of arcs by at most 3. So, let xz be a symmetrical arc. By the same argument, we can see that there is at least one arc between y and z (drawn as a segment because we don't know its orientation). The graph obtained by the contraction of the edge uz must verify $(*)$.

This implies that either the two arcs incident to y go to y, or they come out from y. But then, for any orientation of uz, by contracting either the edge uy or the edge yz, we obtain the desired ${}^*K_3$.

Let $|A(u)| = 4$. The condition $(*)$ implies that there is at least one arc between any pair of vertices of $A(u)$. If $A(u) = A^+(u)$, then it is clear by $(*)$ that the graph induced by $A(u)$ is ${}^*K_4$. If $A^+(u) = \{x, y\}$ and $A^-(u) = \{z, v\}$ then we obtain a ${}^*K_3$ by identifying x with z and y with v. If $A^+(u) = \{x, y, z\}$ and $A^-(u) = \{v\}$, then the graph induced by $A^+(u)$ is a ${}^*K_3$ and this completes the case $d(u) = 4$.

Suppose now that u has the degree $d(u) = 5$. Let $|A(u)| = 3$ and let x, y and z be the neighbours of u. Suppose (u, x) is not a symmetrical arc. The condition $(*)$ implies that there is at least one symmetrical arc either between x and y or between x and z, for example between x and y, and at least one of the edges xz or xy, say xz. Then we obtain a ${}^*K_3$ by contracting xz.

If $|A(u)| = 4$ and $A^+(u) = \{x, y, z, v\}$ and $A^-(u) = \{x\}$ then the graph G_1 induced by $u \cup A(u)$ contains at least 14 arcs and if $n = 5$ we have $14 \geqslant 3n - 3$. So, we can remove one of these arcs and apply the induction hypothesis to G_1.

Suppose now that $A^+(u) = \{x, y, z\}$ and $A^-(u) = \{x, v\}$. If there is at least one symmetrical arc between x and one of y, z or v, say between x and y, then $(*)$ implies the existence of at least one of edges vy or vz and at least one edge yz. We obtain a $^*K_3$ by contracting all thee edges. If there is only one edge between x and all the other vertices of $A(u)$, then there is (by $(*)$) a symmetrical arc yz and the edge zv. We obtain a $^*K_3$ by contracting xy and zv.

If $|A(u)| = 5$ and $|A^+(u)| = 4$ or 5 then the graph G_1 induced by $u \cup A(u)$ contains at least 16 arcs and for $n = 6$ we have $16 > 3n - 3$, so the induction hypothesis applies to G_1. Suppose that $A^+(u) = \{x, y, z\}$ and $A^-(u) = \{v, w\}$. If there is one arc (a, b) for any $a \in A^-(u)$ and for any $b \in A^+(u)$ then, by contracting (w, x) and (v, y), we obtain a $^*K_3$. On the contrary, if (w, x) is not an arc of G then G contains (by $(*)$) the arcs $(x, y), (x, z), (y, x), (w, y), (w, z)$ and (v, w). We obtain a $^*K_3$ by identifying the vertices of the sets $\{y, w\}$ and $\{x, z, v\}$ and this completes the proof of the inequality in the theorem. The graph G drawn in Fig. 3 is an example showing that the bound of the theorem is attained for any n.

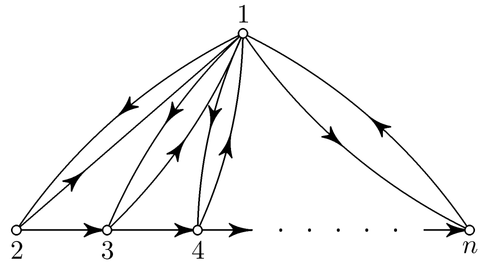

Fig. 3. The bound is attained for any n.

In this graph there is a symmetrical arc between 1 and i for $i = 2, \ldots, n$ and an arc $(i, i + 1)$ for $i = 2, \ldots, n - 1$. Thus, G has $3n - 4$ arcs and it is not contractible onto $^*K_3$. This completes the proof of the theorem. □

We conclude this paper with some remarks. First, by our Theorem and the result of Duchet and Kaneti [3] the following intuitive conjecture is suggested:

C o n j e c t u r e . If a digraph G on p vertices has at least $(2h - 3)p - h(h - 2)$ arcs, then G is contractible onto $^*K_h$ for any integer $h \geqslant 3$.

We remark that this conjecture is not true for "great" valued of p. This fact is a consequence of a result from Bollobas, Catlin and Erdös [1], by taking a "great" a non-oriented graph and by replacing any edge by a symmetrical arc.

Concerning the contraction of non-oriented graphs onto cliques, Kostochka and then Thomason [10] have shown that if there is a constant c_p that any graph on n vertices having $c_p n$ edges is contractible onto K_p then $c_p = \mathrm{o}(p\sqrt{\log p})$. Unfortunately, this bound does not apply to the digraphs.

References

[1] *Bollobas B., Catlin P. A., Erdös P.*: Hadwiger's conjecture is true for almost every graph. European J. Combin. *1* (1980), 195–199.

[2] *Dirac G. A.*: A property of 4-chromatic graphs and some remarks on critical graphs. J. London Math. Soc. *27* (1952), 85–92.

[3] *Duchet P., Kaneti V.*: Sur la contractibilité d'un graphe orienté en $^*K_4$. Discrete Math. *130* (1994), 57–68.

[4] *Hadwiger H.*: Über eine Klassifikation der Streckenkcomplexe. Vierteljschr. Naturforsch. Ges. Zürich *88* (1943), 133–142.

[5] *Jacob H., Meyniel H.*: Extensions of Turan's and Brooks theorem and new notions of stability and colouring in digraphs. Ann. Discrete Math. *17* (1983), 365–370.

[6] *Jagger C.*: An extremal function for digraph subcontraction. J. Graph. Theory *21* (1996), 343–350.

[7] *Kostochka A. V.*: Lower bound on the Hadwiger number of graph by their average degree. Combinatorica *4* (1984), 307–316.

[8] *Meyniel H.*: Contractibilité de graphes orienté on $^*K_3$. Private communication.

[9] *Robertson N., Seymour P., Thomas R.*: Hadviger's conjecture for K_6-free graphs. Combinatorica *13* (1993), 279–361.

[10] *Thomason A.*: An extremal function for contractions of graphs. Math. Proc. Combridge Philos. Soc. *95* (1984), 261–265.

[11] *Wagner K.*: Bemerkungen zu Hadwigers Vermutung. Math. Ann. *141* (1960), 433–451.

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