ESSENTIAL NORMS OF A POTENTIAL THEORETIC BOUNDARY INTEGRAL OPERATOR IN L^1

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Abstract. Let $G \subset \mathbb{R}^m$ $(m \geqslant 2)$ be an open set with a compact boundary B and let $\sigma \geqslant 0$ be a finite measure on B. Consider the space $L^1(\sigma)$ of all σ-integrable functions on B and, for each $f \in L^1(\sigma)$, denote by $f\sigma$ the signed measure on B arising by multiplying σ by f in the usual way. $\mathcal{N}_\sigma f$ denotes the weak normal derivative (w.r. to G) of the Newtonian (in case $m > 2$) or the logarithmic (in case $n = 2$) potential of $f\sigma$, correspondingly. Sharp geometric estimates are obtained for the essential norms of the operator $\mathcal{N}_\sigma - \alpha I$ (here $\alpha \in \mathbb{R}$ and I stands for the identity operator on $L^1(\sigma)$) corresponding to various norms on $L^1(\sigma)$ inducing the topology of standard convergence in the mean w.r. to σ.

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1. Introduction.

In what follows $G \subset \mathbb{R}^m\,(m \geqslant 2)$ is an open set with a compact boundary $\partial G \equiv B$. $\mathcal{H}_k$ denotes the k-dimensional Hausdorff measure (with the usual normalization, so that $\mathcal{H}_m$ coincides with the Lebesgue measure in $\mathbb{R}^m$). We denote by

$$B_r(z) := \{x \in \mathbb{R}^m;\ |x - z| < r\}$$

the open ball of radius $r > 0$ centered at $z \in \mathbb{R}^m$ and put

$$S := \partial B_1(0), \quad A_m := \mathcal{H}_{m-1}(S) = \frac{2\pi^{\frac{1}{2}m}}{\Gamma(\frac{1}{2}m)}. \tag{1}$$

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We fix a Radon measure $\sigma \geqslant 0$ on $\mathbb{R}^m$ whose support coincides with B, spt $\sigma = B$, and denote by $L^1(\sigma)$ the Banach space of all (classes of) σ-integrable functions f on B with the usual norm

$$\|f\|_{L^1(\sigma)} := \int_B |f| \,\mathrm{d}\sigma. \tag{2}$$

The space of all signed Radon measures in $\mathbb{R}^m$ with support in B will be denoted by $\mathcal{C}'(B)$. Given $f \in L^1(\sigma)$ we denote by $\sigma f \in \mathcal{C}'(B)$ the signed measure which is absolutely continuous w.r. to σ and whose Radon-Nikodym derivative w.r. to σ coincides with f a.e.:

$$\frac{\mathrm{d}(\sigma f)}{\mathrm{d}\sigma} = f \quad \sigma\text{-a.e.}$$

In what follows h_z will stand for the fundamental harmonic function in $\mathbb{R}^m$ with a pole at $z \in \mathbb{R}^m$ whose value at $x \in \mathbb{R}^m \setminus \{z\}$ is given by

$$h_z(x) := \begin{cases} \frac{1}{(m-2)A_m}|x-z|^{2-m} & \text{if } m > 2, \\ \frac{1}{2\pi} \ln \frac{1}{|x-z|} & \text{if } m = 2; \end{cases}$$

we put $h_z(z) = +\infty$. For each $\mu \in C'(B)$ the potential

$$\mathcal{U}\mu(x) := \int_B h_z(x) \,\mathrm{d}\mu(z)$$

is well-defined for $x \in \mathbb{R}^m \setminus B$ and represents a harmonic function h on $G \subset \mathbb{R}^m$ whose first order partial derivatives $\partial_1 h, \dots, \partial_m h$ are Lebesgue integrable over each bounded Borel set contained in G. This makes it possible to consider the so-called weak normal derivative of h w.r. to G which is useful in connection with the Neumann boundary value problem (compare [9], [2], [7], [12]). This weak normal derivative $N^G h$ is a distribution defined over the space $\mathcal{D}$ of all infinitely differentiable functions φ with a compact support in $\mathbb{R}^m$ by

$$\langle N^G h, \varphi \rangle := \int_G \left(\sum_{j=1}^m \partial_j h \cdot \partial_j \varphi \right) \mathrm{d}\mathcal{H}_m, \quad \varphi \in \mathcal{D}.$$

The reason for this definition is motivated by the divergence theorem which permits, for smoothly bounded G and grad $h = [\partial_1 h, \dots, \partial_m h]$ continuously extendable from G to $G \cup B$, to transform $\langle N^G h, \varphi \rangle$ into

$$\int_B \varphi n \cdot \operatorname{grad} h \,\mathrm{d}\mathcal{H}_{m-1} = \int_B \varphi \frac{\partial h}{\partial n} \,\mathrm{d}\mathcal{H}_{m-1},$$

where $n : B \to S$ is the unit exterior normal to G (cf. [16]). It is easy to see that for each $\mu \in \mathcal{C}'(B)$ the distribution $N^G U\mu$ has its support contained in B (cf. [7], §1) and it is natural to inquire under which conditions on G it is possible to represent this weak normal derivative $N^G\mathcal{U}\mu$ by a signed measure $\nu_\mu \in \mathcal{C}'(B)$ in the sense that

$$\langle N^G\mathcal{U}\mu, \varphi\rangle = \int_B \varphi \,\mathrm{d}\nu_\mu, \quad \forall \varphi \in \mathcal{D};$$

if this is the case, then ν_μ is uniquely determined and will be identified with $N^G\mathcal{U}\mu \equiv \nu_\mu$. For this purpose it appears useful to consider the so-called essential boundary of G. Denoting by $\bar{d}(x, M)$ the upper density of $M \subset \mathbb{R}^m$ at $x \in \mathbb{R}^m$ defined by

$$\bar{d}(x, M) := \limsup_{r\downarrow 0} \frac{\mathcal{H}_m[B_r(x) \cap M]}{\mathcal{H}_m[B_r(x)]}$$

we introduce the essential boundary of G by

$$\partial_e G := \{x \in \mathbb{R}^m;\ \bar{d}(x, G) > 0,\ \bar{d}(x, \mathbb{R}^m \setminus G) > 0\}.$$

This essential boundary $\partial_e G \equiv B_e$ is a Borel subset of $\partial G \equiv B$. Given $z \in \mathbb{R}^m$ and $\theta \in S$, consider the intersection of the half-line issuing at z in the direction of θ with the essential boundary

$$B_e \cap \{z + t\theta;\ t > 0\}, \tag{3}$$

and denote by $n(z, \theta)$ the total number of points in (3) ($0 \leqslant n(z, \theta) \leqslant +\infty$). It appears that, for fixed $z \in \mathbb{R}^m$, the function

$$\theta \mapsto n(z, \theta)$$

is $\mathcal{H}_{m-1}$-measurable on S, so that it is possible to define

$$v(z) := \int_S n(z, \theta) \,\mathrm{d}\mathcal{H}_{m-1}(\theta).$$

It turns out that $v(z) < +\infty$ implies the existence at z of a well-defined density of G

$$d_G(z) := \lim_{r\downarrow 0} \frac{\mathcal{H}_m[B_r(z) \cap G]}{\mathcal{H}_m[B_r(z)]}. \tag{4}$$

Now the necessary and sufficient condition guaranteeing $N^G\mathcal{U}\mu \in \mathcal{C}'(B)$ whenever $\mu \in \mathcal{C}'(B)$ consists in

$$\sup_{z\in B} v(z) < +\infty. \tag{5}$$

This condition (5) is also necessary and sufficient for validity of the implication

$$f \in L^1(\sigma) \Rightarrow N^G\mathcal{U}(\sigma f) \in \mathcal{C}'(B)$$

(cf. [8]). If besides $N^G\mathcal{U}(\sigma f) \in \mathcal{C}'(B)$ we want this weak normal derivative to be absolutely continuous w.r. to σ for each $f \in L^1(\sigma)$ (and, consequently, to be representable by a $g_f \in L^1(\sigma)$ in the sense that

$$\langle N^G\mathcal{U}(\sigma f), \varphi\rangle = \int_B \varphi g_f \,\mathrm{d}\sigma \tag{6}$$

for each $\varphi \in \mathcal{D}$) then it is necessary and sufficient to require, besides (5), the validity of the implication

$$(M \subset B_e,\ \sigma(M) = 0) \Rightarrow \mathcal{H}_{m-1}(M) = 0 \tag{7}$$

for each Borel set M. Let us also recall that (5) implies

$$\sup_{z\in\mathbb{R}^m} v(z) < +\infty. \tag{8}$$

Assuming both the conditions (5) and (7) we can identify $N^G\mathcal{U}(\sigma f)$ with a certain $g_f \in L^1(\sigma)$ verifying (6) whenever $f \in L^1(\sigma)$; we thus arrive at a linear operator

$$\mathcal{N}_\sigma : f \mapsto g_f = \frac{\mathrm{d}N^G\mathcal{U}(\sigma f)}{\mathrm{d}\sigma}$$

which turns out to be bounded on $L^1(\sigma)$. Under the assumptions (5), (7) it is natural to interpret the weak Neumann problem for G with a boundary condition in $L^1(\sigma)$ as follows:

Given $g \in L^1(\sigma)$, determine an $f \in L^1(\sigma)$ such that $\mathcal{N}_\sigma f = g$. Denoting by I the identity operator on $L^1(\sigma)$ and defining the operator $\mathcal{T}$ on $L^1(\sigma)$ by

$$\tfrac{1}{2}(I + \mathcal{T}) = \mathcal{N}_\sigma$$

we may reduce the weak Neumann problem with a prescribed boundary condition $g \in L^1(\sigma)$ to the equation

$$(I + \mathcal{T})f = 2g \tag{9}$$

for an unknown $f \in L^1(\sigma)$. (For the case when $\sigma = \mathcal{H}_{m-1}|_{B_e}$ arises as the restriction of the Hausdorff measure $\mathcal{H}_{m-1}$ to the essential boundary of G this equation has been treated in [13], [14].) In connection with (9) the knowledge of the essential spectral radius of the operator $\mathcal{T}$ is important. According to [6] for its evaluation it is sufficient to determine, for each of the norms p on $L^1(\sigma)$ topologically equivalent to that given by (2), the corresponding p-essential norm $\omega_p(\mathcal{T})$ of $\mathcal{T}$ which is defined as the distance (measured w.r. to p) of $\mathcal{T}$ from the subspace $\mathcal{G}$ of all compact linear operators Q acting on $L^1(\sigma)$, i.e.

$$\omega_p(\mathcal{T}) := \inf\{p(\mathcal{T} - Q);\ Q \in \mathcal{G}\}. \tag{10}$$

It is the purpose of this paper to show that the essential norm (10) can be estimated and sometimes even precisely evaluated in geometric terms connected with G. For this purpose we denote by p' the norm on $L^\infty(\sigma)$ which is dual to p,

$$p'(u) := \sup\Big\{\int_B uf\,\mathrm{d}\sigma;\ f \in L^1(\sigma), p(f) \leqslant 1\Big\}, \quad u \in L^\infty(\sigma). \tag{11}$$

Let

$$L_1^\infty := \{u \in L^\infty(\sigma);\ p'(u) \leqslant 1\} \tag{12}$$

be the unit ball in $L^\infty(\sigma)$ corresponding to p'. Let us consider σ-essential majorants $q \in L^\infty(\sigma)$ of L_1^∞ enjoying the property

$$u \in L_1^\infty \Rightarrow u \leqslant q \quad \sigma\text{-a.e.}; \tag{13}$$

among them an important role is played by the σ-essential supremum of L_1^∞, to be denoted by $p^*(\in L^\infty(\sigma))$, which is the least σ-essential majorant of L_1^∞ characterized by the requirement

$$p^* \leqslant q \quad \sigma\text{-a.e.}$$

for each σ-essential majorant q fulfilling (13) (cf. [15], II.4.1). This supremum p^* is determined almost uniquely w.r. to σ and we may suppose that p^* is a non-negative bounded Baire function on B (this can be achieved by changing p^* eventually in a set of points of σ-measure zero).

Given a bounded Baire function $q \geqslant 0$ on B we introduce for $z \in \mathbb{R}^m$, $r > 0$, $\theta \in S$ the sum

$$n_r^q(z,\theta) := \sum_t q(z + t\theta), \quad 0 < t < r,\ z + t\theta \in B_e, \tag{14}$$

counting, with the corresponding weight given by q, all points in the intersection $B_e \cap \{z + t\theta; 0 < t < r\}$. For fixed $z \in \mathbb{R}^m$ and $r > 0$, the function

$$\theta \mapsto n_r^q(z,\theta) \tag{15}$$

is integrable on S w.r.t. to $\mathcal{H}_{m-1}$ so that we may define

$$v_r^q(z) = \frac{1}{A_m} \int_S n_r^q(z,\theta)\, \mathrm{d}\mathcal{H}_{m-1}(\theta). \tag{16}$$

(This quantity is not sensitive to changing q in a set of σ-measure zero. Note also that for $q \equiv 1$ and $r = +\infty$ this $v_\infty^1(z)$ reduces to $v(z)$ as defined above.) We are going to prove that the functions

$$v_r^{p^*}: y \mapsto v_r^{p^*}(y) \ (y \in B) \tag{17}$$

belong to $L^\infty(\sigma)$ and permit to obtain the estimate

$$\omega_p(\mathcal{T}) \leqslant 2 \inf_{r>0} p'(v_r^{p^*}); \tag{18}$$

besides that, the sign of equality holds in (18) for certain (e.g. weighted) norms p under suitable assumptions on the measure σ.

2. Notation. We denote by $\widehat{\partial} G \equiv \widehat{B}$ the so-called reduced boundary of G consisting of all the points $z \in \mathbb{R}^m$ for which there exists an $n \in S$ such that

$$\bar{d}(z, \{x \in \mathbb{R}^m; (x-z)\cdot n < 0\} \cap G) = 0 = \bar{d}(z, \{x \in \mathbb{R}^m; (x-z)\cdot n > 0\} \setminus G). \tag{19}$$

The corresponding vector $n \equiv n^G(z)$ is uniquely determined and is termed the interior normal of G at z in the sense of Federer; if there is no $n \in S$ satisfying (19) we agree to denote by $n^G(z) = 0$ ($\in \mathbb{R}^m$) the zero vector in $\mathbb{R}^m$. Then

$$z \mapsto n^G(z)$$

is a Borel measurable function on $\mathbb{R}^m$ (cf. [4]) so that, in particular, $\widehat{B}$ is a Borel set contained in B_e; besides that (cf. [5]),

$$\mathcal{H}_{m-1}(B_e) < \infty \Rightarrow \mathcal{H}_{m-1}(B_e \setminus \widehat{B}) = 0. \tag{20}$$

3. Lemma. *Assume* (5) *and consider a bounded Baire function* $q \geqslant 0$ *on* B. *Given* $z \in \mathbb{R}^m$, $r > 0$ *and* $\theta \in S$, *define* $n_r^q(z,\theta)$ *by* (14). *Then, for fixed* $z \in \mathbb{R}^m$ *and* $r > 0$, *the function* (15) *is integrable w.r.t. to* $\mathcal{H}_{m-1}$ *on* S *and defining* $v_r^q(z)$ *by* (16) *we have*

$$v_r^q(z) = \int_{B \cap B_r(z)} q(x) |n^G(x) \cdot \operatorname{grad} h_z(x)|\, \mathrm{d}\mathcal{H}_{m-1}(x). \tag{21}$$

For any fixed $r > 0$, the function

$$v_r^q : z \mapsto v_r^q(z) \tag{22}$$

is bounded and lower semicontinuous on $\mathbb{R}^m$.

P r o o f. For any $z \in \mathbb{R}^m$ denote by $\mathcal{P}(z)$ the class of all non-negative Baire functions q on B for which the corresponding function $\theta \mapsto n_\infty^q(z,\theta)$ is $\mathcal{H}_{m-1}$-integrable on S and satisfies

$$\int_S n_\infty^q(z,\theta)\,\mathrm{d}\mathcal{H}_{m-1}(\theta) = A_m \int_B q(x)|n^G(x)\cdot \operatorname{grad} h_z(x)|\,\mathrm{d}\mathcal{H}_{m-1}(x). \tag{23}$$

As shown in Lemma 3 of [10] (p. 280), $\mathcal{P}(z)$ contains all positive bounded lower semicontinuous functions on B. In particular, the constant function equal to 1 on B belongs to $\mathcal{P}(z)$ so that

$$v(z) = \frac{1}{A_m}\int_S n_\infty^1(z,\theta)\,\mathrm{d}\mathcal{H}_{m-1}(\theta) = \int_B |n^G(x)\cdot \operatorname{grad} h_z(x)|\,\mathrm{d}\mathcal{H}_{m-1}(x),$$

which is a bounded function of the variable $z \in \mathbb{R}^m$, because our assumption (5) implies (8) (cf. [7]). Consequently, for any fixed z, the function

$$\theta \mapsto n_\infty^1(z,\theta)$$

is integrable ($\mathcal{H}_{m-1}$) on S. This permits us to conclude that $\mathcal{P}(z)$ contains the limit of any pointwise convergent uniformly bounded sequence of its elements. Indeed, given such a sequence $q_n \in \mathcal{P}(z)$, $|q_n| \leqslant c$ $(\in \mathbb{R})$, $q_n \to q$ pointwise on B, then all functions

$$\theta \mapsto n_\infty^{q_n}(z,\theta)$$

have

$$\theta \mapsto c n_\infty^1(z,\theta)$$

as a common $\mathcal{H}_{m-1}$-integrable majorant on S and converge to

$$\theta \mapsto n_\infty^q(z,\theta)$$

almost everywhere ($\mathcal{H}_{m-1}$) on S; passing to the limit under the integral sign we get (23) showing that $q \in \mathcal{P}(z)$, as asserted. These properties of $\mathcal{P}(z)$ guarantee that $\mathcal{P}(z)$ is rich enough to contain all bounded Baire functions $q \geqslant 0$ on B. Given such a q and denoting by $\chi_{B_r(z)}$ the characteristic function of $B_r(z)$ we may apply (23) with q replaced by $q \cdot \chi_{B_r(z)}$, which results in (21). It remains to verify that, for any fixed $r > 0$, the function (22) is lower semicontinuous. Consider an arbitrary convergent sequence of points $z_n \in \mathbb{R}^m$ tending to z as $n \to \infty$. For $x \in B \setminus \{z\}$ we have then

$$q(x)\chi_{B_r(z)}(x)|n^G(x) \cdot \operatorname{grad} h_z(x)| \leqslant \liminf_{n\to\infty} q(x)\chi_{B_r(z_n)}(x)|n^G(x) \cdot \operatorname{grad} h_{z_n}(x)|.$$

Integrating $\mathrm{d}\mathcal{H}_{m-1}(x)$ we get by Fatou's lemma $v_r^q(z) \leqslant \liminf_{n\to\infty} v_r^q(z_n)$, which completes the proof. □

4. Remark. The formula (21) shows that the quantity $v_r^q(z)$ is not influenced by changes of q in a set of points whose intersection with $\widehat{B}$ has vanishing $\mathcal{H}_{m-1}$-measure. The implication (7) guarantees that changing q in a set of points which meets $\widehat{B}$ in a set of vanishing σ-measure does not afflict $v_r^q(z)$, either. In what follows we always assume (5), which implies (8) and guarantees the existence of the density (4) at any $z \in \mathbb{R}^m$ (cf. [7] Theorem 2.16, Lemma 2.9). We also assume validity of the implication (7) for any Borel set M. We denote by $\widehat{\mathcal{H}}_{m-1}$ the restriction of the Hausdorff measure $\mathcal{H}_{m-1}$ to the reduced boundary $\widehat{B} \equiv \widehat{\partial} G$ which is defined on Borel sets M by

$$\widehat{\mathcal{H}}_{m-1}(M) = \mathcal{H}_{m-1}(M \cap \widehat{B}). \tag{24}$$

Since (8) implies finiteness of $\mathcal{H}_{m-1}(B_e)$ (cf. [7] Theorem 2.16, Theorem 2.12, [5] Theorem 4.5.6), in view of (20) replacing the reduced boundary $\widehat{B}$ by the essential boundary B_e in the definition (24) does not change the measure $\widehat{\mathcal{H}}_{m-1}$ which, as a consequence of the assumption (7), turns out to be absolutely continuous w.r. to σ. Accordingly, the Radon-Nikodym derivative

$$\widehat{h} := \frac{\mathrm{d}\widehat{\mathcal{H}}_{m-1}}{\mathrm{d}\sigma} \tag{25}$$

is meaningful; we may and will assume that $\widehat{h}$ is a Baire function defined and non-negative everywhere on $B = \partial G$ and vanishing on $B \setminus \widehat{B}$. It has been proved in [8] that, for $f \in L^1(\sigma)$ and σ-almost every $x \in B$, the integral

$$\int_{B\setminus\{x\}} \widehat{h}(x) n^G(x) \cdot \operatorname{grad} h_y(x) f(y) \,\mathrm{d}\sigma(y) \tag{26}$$

converges and represents a function which is σ-integrable w.r. to the variable $x \in B$; the operator $\mathcal{N}_\sigma$ is bounded on $L^1(\sigma)$ and transforms each $f \in L^1(\sigma)$ into a function which is given by the formula

$$\mathcal{N}_\sigma f(x) = d_G(x) f(x) - \int_{B\setminus\{x\}} \widehat{h}(x) n^G(x) \cdot \operatorname{grad} h_y(x) f(y) \,\mathrm{d}\sigma(y) \tag{27}$$

for σ-a.e. $x \in B$.

5. Proposition. *Let p be a norm on $L^1(\sigma)$ which is topologically equivalent to that given by* (2) *and suppose that the norm p' on $L^\infty(\sigma)$ which is dual to p (cf.* (11)*) has the property*

$$(u,\ v \in L^\infty(\sigma),\ |u| \leqslant v) \Rightarrow p'(u) \leqslant p'(v). \tag{28}$$

(Note that this is true if p satisfies the requirement $p(|f|) \leqslant p(f)$, $f \in L^1(\sigma)$.) Denote, as above, by p^* the σ-essential supremum of L_1^∞ (cf. (12)*) and consider, for each $r > 0$, the corresponding function* (17) *(which is known from Lemma* 3 *to be bounded and lower semicontinuous on B). Let I be the identity operator on $L^1(\sigma)$. Then for $\alpha \in \mathbb{R}$*

$$\begin{aligned}\omega_p(\mathcal{N}_\sigma - \alpha I) &\leqslant \inf_{r>0} p'[y \mapsto |d_G(y) - \alpha|p^*(y) + v_r^{p^*}(y)] \\ &\leqslant p'[y \mapsto |d_G(y) - \alpha|p^*(y)] + \inf_{r>0} p'(v_r^{p^*}).\end{aligned} \tag{29}$$

If, in addition

$$\sigma(\{y \in B;\ d_G(y) \neq \tfrac{1}{2}\}) = 0, \tag{30}$$

then

$$\omega_p(\mathcal{N}_\sigma - \alpha I) \leqslant |\tfrac{1}{2} - \alpha| p'(p^*) + \inf_{r>0} p'(v_r^{p^*}). \tag{31}$$

P r o o f. If $p(|f|) \leqslant p(f)$ whenever $f \in L^1(\sigma)$ and if $u, v \in L^\infty(\sigma)$ satisfy $|u| \leqslant v$, then by (11)

$$\begin{aligned}p'(u) &\leqslant \sup\Big\{\int_B |u|\cdot|f|\,\mathrm{d}\sigma;\ f \in L^1(\sigma),\ p(f) \leqslant 1\Big\} \\ &\leqslant \sup\Big\{\int_B vg\,\mathrm{d}\sigma;\ g \in L^1(\sigma),\ p(g) \leqslant 1\Big\} = p'(v)\end{aligned}$$

and (28) is verified. In what follows we assume validity of (28). Fix $r > 0$ and choose an infinitely differentiable function γ_r on $\mathbb{R}^m$ such that

$$0 \leqslant \gamma_r \leqslant 1,\ \gamma_r(B_{\frac{1}{2}r}(0)) = \{0\},\ \gamma_r(\mathbb{R}^m \setminus B_r(0)) = \{1\}.$$

It has been proved in [8] (cf. Corollaire, pp. 153–154) that

$$[x, y] \mapsto n^G(x) \cdot \operatorname{grad} h_y(x) \widehat{h}(x)$$

represents a function of Baire on $B \times B \setminus \Delta$ where $\Delta = \{[x,x];\ x \in B\}$ and that, for each $f \in L^1(\sigma)$, the integral

$$\int\int_{B \times B \setminus \Delta} |n^G(x) \cdot \operatorname{grad} h_y(x)| \cdot |f(y)| \widehat{h}(x) \, \mathrm{d}\sigma(x) \, \mathrm{d}\sigma(y)$$

is convergent. Consequently, also the function

$$[x,y] \mapsto \gamma_r(x-y)\ n^G(x) \cdot \operatorname{grad} h_y(x) \widehat{h}(x)$$

which we extend by 0 to Δ represents a function of Baire on $B \times B$ and, for any $f \in L^1(\sigma)$, the functions

$$\begin{aligned} T_r f(x) &= -\int_B \widehat{h}(x)\gamma_r(x-y) n^G(x) \cdot \operatorname{grad} h_y(x) f(y) \, \mathrm{d}\sigma(y), \\ V_r f(x) &= -\int_B \widehat{h}(x)[1-\gamma_r(x-y)] n^G(x) \cdot \operatorname{grad} h_y(x) f(y) \, \mathrm{d}\sigma(y) \end{aligned}$$

are defined for σ-a.e. $x \in B$ and are integrable (σ). In view of (27) we have

$$(\mathcal{N}_\sigma - \alpha I) f(x) = [d_G(x) - \alpha] f(x) + T_r f(x) + V_r f(x) \tag{32}$$

for σ-a.e. $x \in B$. Using the properties of γ_r it is easy to verify the estimates (where $x, y, y_j \in B,\ j = 1,2$)

$$\begin{gathered} \gamma_r(x-y) |n^G(x) \cdot \operatorname{grad} h_y(x)| \leqslant A_m^{-1} (\tfrac{1}{2} r)^{1-m}, \\ |\gamma_r(x-y_1) - \gamma_r(x-y_2)| \leqslant |y_1 - y_2| \max\{|\operatorname{grad} \gamma_r(z)|; z \in R^m\}, \\ \gamma_r(x-y_j)| \operatorname{grad} h_{y_1}(x) - \operatorname{grad} h_{y_2}(x)| \leqslant (m+1) A_m^{-1} |y_1 - y_2| (\tfrac{1}{4} r)^{-m} \text{ for } |y_1 - y_2| \leqslant \tfrac{1}{4} r. \end{gathered} \tag{33}$$

Denoting by T_r' the dual operator to T_r we have for $u \in L^\infty(\sigma)$ and σ-a.e. $y \in B$

$$\begin{aligned} T_r' u(y) &= -\int_B \widehat{h}(x)\gamma_r(x-y) n^G(x) \cdot \operatorname{grad} h_y(x) u(x) \, \mathrm{d}\sigma(x) \\ &= -\int_B \gamma_r(x-y) n^G(x) \cdot \operatorname{grad} h_y(x) u(x) \, \mathrm{d}\mathcal{H}_{m-1}(x). \end{aligned}$$

Hence we conclude by virtue of (33) that T_r' maps the unit ball in $L^\infty(\sigma)$ into a family of uniformly bounded functions satisfying the Lipschitz condition with the same coefficient on B. By Arzela's theorem, this family is relatively compact in $L^\infty(\sigma)$. We have thus verified that

$$T_r \colon f \mapsto T_r f$$

is a compact operator on $L^1(\sigma)$. Defining

$$U_r f(x) = [d_G(x) - \alpha] f(x) + V_r f(x)$$

we may rewrite (32) in the form

$$\mathcal{N}_\sigma - \alpha I = U_r + T_r.$$

Since T_r is compact, we have

$$\omega_p(\mathcal{N}_\sigma - \alpha I) \leqslant p(U_r) = p'(U_r'),$$

where U_r' denotes the dual operator to U_r sending any $u \in L^\infty(\sigma)$ into a function determined for σ-a.e. $y \in B$ by

$$\begin{aligned} U_r' u(y) &= [d_G(y) - \alpha] u(y) - \int_{B \setminus \{y\}} u(x)[1 - \gamma_r(x-y)] n^G(x) \cdot \operatorname{grad} h_y(x) \widehat{h}(x) \, \mathrm{d}\sigma(x) \\ &= [d_G(y) - \alpha] u(y) - \int_{B} u(x)[1 - \gamma_r(x-y)] n^G(x) \cdot \operatorname{grad} h_y(x) \, \mathrm{d}\mathcal{H}_{m-1}(x). \end{aligned}$$

If $u \in L_1^\infty$ then

$$|u| \leqslant p^*$$

σ-a.e. on B and, in view of (7), the same inequality holds $\mathcal{H}_{m-1}$-a.e. on $\widehat{B}$. Taking into account that

$$1 - \gamma_r(x-y) = 0 \quad \text{for } x \in \mathbb{R}^m \setminus B_r(y)$$

we obtain from Lemma 3 for $u \in L_1^\infty$ and σ-a.e. $y \in B$ that

$$\begin{aligned} |U_r' u(y)| &\leqslant |d_G(y) - \alpha| p^*(y) + \int_{B \cap B_r(y)} p^*(x) |n^G(x) \cdot \operatorname{grad} h_y(x)| \, \mathrm{d}\mathcal{H}_{m-1}(x) \\ &= |d_G(y) - \alpha| p^*(y) + v_r^{p^*}(y), \end{aligned}$$

whence using (28) we get

$$\begin{aligned} p'(U_r') = \sup_{u \in L_1^\infty} p'(U_r' u) &\leqslant p'[y \mapsto |d_G(y) - \alpha| p^*(y) + v_r^{p^*}(y)] \\ &\leqslant p'[y \mapsto |d_G(y) - \alpha| p^*(y)] + p'(v_r^{p^*}) \end{aligned}$$

for any $r > 0$, which implies (29). Assuming (30) we obtain

$$p'[y \mapsto |d_G(y) - \alpha| p^*(y)] = |\tfrac{1}{2} - \alpha| p'(p^*),$$

which completes the proof. $\square$

6. Notation. If w is a function on $M \subset B$ then its σ-essential supremum on M is defined as

$$\inf \big\{\lambda \in \mathbb{R};\ \sigma(\{x \in M; w(x) > \lambda\}) = 0\big\};$$

it will be denoted by the symbols

$$\sigma\text{-}\sup_M w \equiv \sigma\text{-}\sup_{x \in M} w(x).$$

7. Corollary. *Let q be a function of Baire on B satisfying σ-a.e. on B the inequalities*

$$c_1 \leqslant q \leqslant c_2 \tag{34}$$

for suitable constants $0 < c_1 \leqslant c_2 < +\infty$, and define a norm p on $L^1(\sigma)$ by

$$p(f) = \int_B q|f| \, \mathrm{d}\sigma,\ \ f \in L^1(\sigma). \tag{35}$$

Then for any $\alpha \in \mathbb{R}$

$$\omega_p(\mathcal{N}_\sigma - \alpha I) \leqslant \inf_{r>0} \sigma\text{-}\sup_{x \in B} \Big[|d_G(x) - \alpha| + \frac{v_r^q(x)}{q(x)}\Big] \leqslant \sigma\text{-}\sup_{x \in B} |d_G(x) - \alpha| + \inf_{r>0} \sigma\text{-}\sup_{x \in B} \frac{v_r^q(x)}{q(x)}.$$

If (30) *holds, then*

$$\omega_p(\mathcal{N}_\sigma - \alpha I) \leqslant |\alpha - \tfrac{1}{2}| + \inf_{r>0} \sigma\text{-}\sup_{x \in B} \frac{v_r^q(x)}{q(x)}.$$

Proof. If p is defined by (35) then the dual norm of any $u \in L^\infty(\sigma)$ is given by

$$p'(u) = \sigma\text{-}\sup_B \frac{|u|}{q} \tag{36}$$

(cf. (11)). We see that $q \in L_1^\infty$ so that, denoting by p^* the σ-essential supremum of the family L_1^∞, we get

$$q \leqslant p^* \quad \sigma\text{-a.e.}$$

On the other hand, in view of (13) we obtain from the σ-essential minimality of p^* the inequality

$$p^* \leqslant q \quad \sigma\text{-a.e.},$$

so that

$$p^* = q \quad \sigma\text{-a.e.}$$

We may thus replace p^* by q in Proposition 5 and (36) yields

$$\begin{aligned}\omega_p(\mathcal{N}_\sigma - \alpha I) &\leqslant \inf_{r>0} \sigma\text{-}\sup_{x\in B} \Big[|d_G(x) - \alpha| + \frac{v_r^q(x)}{q(x)}\Big] \\ &\leqslant \sigma\text{-}\sup_{x\in B} |d_G(x) - \alpha| + \inf_{r>0} \sigma\text{-}\sup \frac{v_r^q}{q}.\end{aligned}$$

If (30) holds, then (31) combined with (36) and $p'(p^*) \leqslant 1$ yield

$$\omega_p(\mathcal{N}_\sigma - \alpha I) \leqslant |\tfrac{1}{2} - \alpha| + \inf_{r>0} \sigma\text{-}\sup \frac{v_r^q}{q},$$

which completes the proof. □

The following simple lemma will be useful in the course of the proof of our main theorem.

8. Lemma. *Let q be a finite function of Baire on B and let $\widehat{q}_\sigma$ associate with each $x \in B$ the σ-essential limes inferior of q at x which is defined as the supremum of all $\lambda \in \mathbb{R}$, for which there exists an $r > 0$ such that*

$$\sigma(\{y \in B_r(x) \cap B;\ q(y) < \lambda\}) = 0. \tag{37}$$

Then $\widehat{q}_\sigma$ is a lower semicontinuous function on B and

$$\sigma(\{x \in B;\ q(x) < \widehat{q}_\sigma(x)\}) = 0. \tag{38}$$

P r o o f. Let $x \in B$ and $\lambda_0 < \widehat{q}_\sigma(x)$. Then there are $\lambda > \lambda_0$ and $r > 0$ satisfying (37). Put $\varrho = \frac{1}{2}r$ and consider an arbitrary $x_0 \in B_\varrho(x) \cap B$. Since $B_\varrho(x_0) \cap B \subset B_r(x) \cap B$ we have

$$\sigma(\{y \in B_\varrho(x_0) \cap B,\ q(y) < \lambda\}) = 0,$$

whence

$$\widehat{q}_\sigma(x_0) \geqslant \lambda > \lambda_0.$$

We have thus shown that for each $\lambda_0 < \widehat{q}_0(x)$ there is a $\varrho > 0$ such that

$$x_0 \in B_\varrho(x) \cap B \Rightarrow \widehat{q}_\sigma(x_0) > \lambda_0,$$

which proves the lower semicontinuity of $\widehat{q}_\sigma$ at x.

Since both q and $\widehat{q}_\sigma$ are functions of Baire we see that

$$\{x \in B;\ q(x) < \widehat{q}_\sigma(x)\}$$

is a Borel set. Admitting that its σ-measure is positive we obtain from Luzin's theorem the existence of a compact

$$K \subset \{x \in B;\ q(x) < \widehat{q}_\sigma(x)\}$$

with $\sigma(K) > 0$ such that the restriction of q to K is continuous. The set consisting of all $x \in B$ for which $\sigma(B_r(x) \cap K) = 0$ for suitable $r = r(x) > 0$ has vanishing σ-measure. Consequently, there is an $x_0 \in K$ such that

$$\sigma(B_\varrho(x_0) \cap K) > 0 \tag{39}$$

for each $\varrho > 0$. In view of $q(x_0) < \widehat{q}_\sigma(x_0)$ there are $\lambda > q(x_0)$ and $r > 0$ such that

$$\sigma(\{y \in B_r(x_0) \cap B;\ q(y) < \lambda\}) = 0. \tag{40}$$

Since the restriction of q to K is continuous we can choose $\varrho \in (0, r)$ small enough to have

$$y \in B_\varrho(x_0) \cap K \Rightarrow \lambda > q(y),$$

which together with (40) violates (39). Thus (38) is established. $\square$

9. Theorem. *Let q be a function of Baire on B satisfying σ-a.e. on B the inequalities* (34) *where $0 < c_1 \leqslant c_2 < +\infty$ are constants, and define a norm p on $L^1(\sigma)$ by* (35). *Assume that σ satisfies* (30) *and does not charge singletons:*

$$\sigma(\{y\}) = 0 \quad \textit{for each} \quad y \in B. \tag{41}$$

Then

$$\omega_p(\mathcal{N}_\sigma - \tfrac{1}{2}I) = \inf_{r>0} \sigma\text{-}\sup_B \frac{v_r^q}{q}. \tag{42}$$

P r o o f. As we have seen in the course of the proof of Corollary 7 the dual norm $p'(u)$ of any $u \in L^\infty(\sigma)$ is given by (36) and q coincides σ-a.e. on B with the σ-essential supremum p^* of the family L_1^∞. We have to verify the inequality

$$\omega_p(\mathcal{N}_\sigma - \tfrac{1}{2}I) \geqslant \inf_{r>0} \sigma\text{-}\sup_B \frac{v_r^q}{q}; \tag{43}$$

the rest will follow from Corollary 7.

According to (27), (30) we have for $f \in L^1(\sigma)$ and σ-a.e. $x \in B$

$$(44) \qquad (\mathcal{N}_\sigma - \tfrac{1}{2}I)f(x) = -\int_{B\setminus\{x\}} \widehat{h}(x) n^G(x) \cdot \operatorname{grad} h_y(x) f(y) \,\mathrm{d}\sigma(y).$$

Fix an arbitrary $\varepsilon > 0$. According to Theorem 10 and Corollary 11 in Chap. VI, §8 in [3] there are mutually disjoint Borel sets $M_1, \dots, M_n \subset B$ and functions $g_1, \dots, g_n \in L^1(\sigma)$ such that the finite dimensional operator

$$(45) \qquad T : f \mapsto \sum_{j=1}^{n} g_j \int_{M_j} f \,\mathrm{d}\sigma$$

acting on $L^1(\sigma)$ satisfies

$$(46) \qquad p(\mathcal{N}_\sigma - \tfrac{1}{2}I - T) < \varepsilon + \omega_p(\mathcal{N}_\sigma - \tfrac{1}{2}I).$$

We infer from (44) that the operator $(\mathcal{N}_\sigma - \frac{1}{2}I)'$ which is dual to $(\mathcal{N}_\sigma - \frac{1}{2}I)$ sends any $u \in L^\infty(\sigma)$ into a function in $L^\infty(\sigma)$ whose values for σ-a.e. $y \in B$ are given by

$$(\mathcal{N}_\sigma - \tfrac{1}{2}I)'u(y) = -\int_B u(x) n^G(x) \cdot \operatorname{grad} h_y(x) \,\mathrm{d}\mathcal{H}_{m-1}(x).$$

Denoting by m_j the characteristic function of M_j on B we obtain from (45) that the operator T' dual to T has the form

$$(47) \qquad T' : u \mapsto T'u = \sum_{j=1}^{n} m_j \int_B u g_j \,\mathrm{d}\sigma, \;\; u \in L^\infty(\sigma).$$

In view of the equality

$$(48) \qquad p(\mathcal{N}_\sigma - \tfrac{1}{2}I - T) = p'(\mathcal{N}_\sigma - \tfrac{1}{2}I - T)'$$

it will suffice to derive a lower estimate for $p'(\mathcal{N}_\sigma - \frac{1}{2}I - T)'$. Choose $c > 0$ small enough to have $c < q$ σ-a.e. on B and fix a $\delta > 0$ such that for any Borel set $M \subset B$,

$$(49) \qquad \sigma(M) < \delta \Rightarrow \int_M q|g_j| \,\mathrm{d}\sigma < \varepsilon c, \;\; j = 1, \dots, n.$$

According to our assumption (41) we can fix $r > 0$ small enough to guarantee that

$$(50) \qquad y \in B \Rightarrow \sigma(B \cap B_r(y)) < \delta.$$

Observe that any $u \in L^\infty(\sigma)$ with $p'(u) \leqslant 1$ vanishing outside the ball $B_r(y)$ centered at an $y \in B$ satisfies

$$|(T'u)(x)| \leqslant \sum_{j=1}^{n} m_j(x) \int_{B\cap B_r(y)} q|g_j| \, \mathrm{d}\sigma < \varepsilon c$$

for σ-a.e. $x \in B$, so that

$$p'(T'u) \leqslant \varepsilon. \tag{51}$$

Put $H_1 := \{x \in B;\ q(x) < \widehat{q}_\sigma(x)\}$ and recall that $\sigma(H_1) = 0$ by (38). Given $y \in B \setminus H_1$ and $k > q(y)$ we thus have

$$\sigma(\{x \in B_\tau(y) \cap B;\ q(x) < k\}) > 0,\ \tau > 0. \tag{52}$$

Putting $H_2 := \{x \in B;\ d_G(x) \neq \frac{1}{2}\}$, $H_0 := H_1 \cup H_2$ we conclude from (30) that

$$\sigma(H_0) = 0.$$

Fix now an arbitrary $y \in B \setminus H_0$ and $k > q(y)$. We are looking for a $u \in L^\infty(\sigma)$ with

$$p'(u) \leqslant 1,\ u(B \setminus B_r(y)) = \{0\} \tag{53}$$

such that

$$p'((\mathcal{N}_\sigma - \tfrac{1}{2}I)'u) \geqslant \frac{v_r^q(y)}{k} - \varepsilon.$$

According to (21) we can fix $\varrho \in (0, r)$ small enough to have

$$\int_{B\cap[B_r(y)\setminus B_\varrho(y)]} q(x)|n^G(x) \cdot \operatorname{grad} h_y(x)| \, \mathrm{d}\mathcal{H}_{m-1}(x) > v_r^q(y) - \varepsilon k.$$

Next define

$$u(x) := \begin{cases} -q(x) \operatorname{sgn}[n^G(x) \cdot \operatorname{grad} h_y(x)] & \text{for } x \in B \cap [B_r(y) \setminus B_\varrho(y)], \\ 0 & \text{for the other } \ x \text{ in } B. \end{cases}$$

For σ-a.e. $z \in B_\varrho(y) \cap B$ we then have

$$\frac{1}{q(z)}(\mathcal{N}_\sigma - \tfrac{1}{2}I)'u(z) =$$
$$\frac{1}{q(z)} \int_{B\cap[B_r(y)\setminus B_\varrho(y)]} q(x) \operatorname{sgn}[n^G(x) \cdot \operatorname{grad} h_y(x)] \cdot [n^G(x) \cdot \ \operatorname{grad} h_z(x)] \, \mathrm{d}\mathcal{H}_{m-1}(x).$$

As z approaches y along the set

$$\{z \in B \cap [B_\varrho(y) \setminus H_0];\ q(z) < k\}$$

(which, in view of (52), intersects any ball $B_\tau(y)$ with $\tau \in (0, \varrho)$ in a set of positive σ-measure), the corresponding functions

$$x \mapsto n^G(x) \cdot \operatorname{grad} h_z(x)$$

converge (even uniformly w.r. to x) in $[B_r(y) \setminus B_\varrho(y)]$ to

$$x \mapsto n^G(x) \cdot \operatorname{grad} h_y(x),$$

whence

$$\begin{aligned}
&\int_{B\cap[B_r(y)\setminus B_\varrho(y)]} q(x) \operatorname{sgn}[n^G(x) \cdot \operatorname{grad} h_y(x)] \cdot [n^G(x) \cdot \operatorname{grad} h_z(x)] \, \mathrm{d}\mathcal{H}_{m-1}(x) \\
&\to \int_{B\cap[B_r(y)\setminus B_\varrho(y)]} q(x) |n^G(x) \cdot \operatorname{grad} h_y(x)| \, \mathrm{d}\mathcal{H}_{m-1}(x) > v_r^q(y) - \varepsilon k.
\end{aligned}$$

We see that the function

$$z \mapsto \frac{1}{q(z)} (\mathcal{N}_\sigma - \tfrac{1}{2} I)' u(z)$$

remains above the quantity $\frac{v_r^q(y)}{k} - \varepsilon$ on the set

$$\{z \in [B_\tau(y) \setminus H_0] \cap B;\ q(z) < k\}$$

of positive σ-measure for sufficiently small $\tau \in (0, \varrho)$. Consequently,

$$p'((\mathcal{N}_\sigma - \tfrac{1}{2} I)' u) \geqslant \frac{v_r^q(y)}{k} - \varepsilon.$$

Since (53) implies (51) we have

$$\begin{aligned}
p'((\mathcal{N}_\sigma - \tfrac{1}{2} I)' - T') &\geqslant p'((\mathcal{N}_\sigma - \tfrac{1}{2} I - T)' u) \geqslant p'((\mathcal{N}_\sigma - \tfrac{1}{2} I)' u) - p'(T' u) \\
&\geqslant \frac{v_r^q(y)}{k} - 2\varepsilon.
\end{aligned}$$

As k can be chosen arbitrarily close to $q(y)$ we obtain

$$p'((\mathcal{N}_\sigma - \tfrac{1}{2} I - T)') \geqslant \frac{v_r^q(y)}{q(y)} - 2\varepsilon$$

for $y \in B \setminus H_0$, i.e. for σ-a.e. $y \in B$. In view of (46), (48) we arrive at

$$p'(v_r^q) \leqslant p(\mathcal{N}_\sigma - \tfrac{1}{2} I - T) + 2\varepsilon \leqslant \omega_p(\mathcal{N}_\sigma - \tfrac{1}{2} I) + 3\varepsilon,$$

so that

$$\inf_{r>0} p'(v_r^q) \leqslant \omega_p(\mathcal{N}_\sigma - \tfrac{1}{2} I) + 3\varepsilon,$$

which yields (43) because $\varepsilon > 0$ was arbitrary. Combining this inequality with that obtained for $\alpha = \frac{1}{2}$ from Corollary 7 we arrive at (42). $\square$

R e m a r k. In [11], [1] examples have been constructed of simple sets $G \subset \mathbb{R}^3$ arising as unions of finitely many rectangular boxes such that for the operator $\mathcal{N}_\sigma$ corresponding to the surface measure $\sigma \equiv \mathcal{H}_2|_{\partial G}$ and the standard L^1-norm p_1 given by (2) the inequality $\omega_{p_1}(\mathcal{N}_\sigma - \alpha I) \geqslant |\alpha|$ holds for all $\alpha \in \mathbb{R}$ while for a suitable norm p given by (35) the estimate $\omega_p(\mathcal{N}_\sigma - \frac{1}{2}I) < \frac{1}{2}$ is true.

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