MV-ALGEBRAS ARE CATEGORICALLY EQUIVALENT TO A CLASS OF $\mathcal{DR}l_{1(\mathrm{i})}$-SEMIGROUPS

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Abstract. In the paper it is proved that the category of MV-algebras is equivalent to the category of bounded *DRl*-semigroups satisfying the identity $1-(1-x)=x$. Consequently, by a result of D. Mundici, both categories are equivalent to the category of bounded commutative BCK-algebras.

Keywords: MV-algebra, *DRl*-semigroup, categorical equivalence, bounded BCK-algebra

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The notion of an MV-algebra was introduced by C. C. Chang in [1], [2] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic. D. Mundici in [9] proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras introduced by S. Tanaka in [12]. The notion of a dually residuated lattice ordered semigroup (*DRl*-semigroup) was introduced by K. L. N. Swamy in [11] as a common generalization of Brouwerian algebras and commutative lattice ordered groups (l-groups). Some connections between *DRl*-semigroups and MV-algebras were studied by the author in [10].

In this paper we will show that MV-algebras (and so also bounded commutative BCK-algebras) are categorically equivalent to some *DRl*-semigroups.

Let us recall the notions of an MV-algebra and a *DRl*-semigroup.

An *MV-algebra* is an algebra $A=(A,\oplus,\neg,0)$ of type $\langle 2,1,0\rangle$ satisfying the following identities. (See e. g. [3].)

(MV 1) $x\oplus(y\oplus z)=(x\oplus y)\oplus z$;

(MV 2) $x\oplus y=y\oplus x$;

(MV 3) $x\oplus 0=x$;

(MV 4) $\neg\neg x = x$;

(MV 5) $x \oplus \neg 0 = \neg 0$;

(MV 6) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$.

A *DRl-semigroup* is an algebra $A = (A, +, 0, \vee, \wedge, -)$ of type $\langle 2, 0, 2, 2, 2\rangle$ such that

(1) $(A, +, 0)$ is a commutative monoid;

(2) $(A, \vee, \wedge)$ is a lattice;

(3) $(A, +, \vee, \wedge)$ is a lattice ordered semigroup (l-semigroup), i. e. A satisfies the identities

$$x + (y \vee z) = (x + y) \vee (x + z),$$
$$x + (y \wedge z) = (x + y) \wedge (x + z).$$

(4) If $\leqslant$ denotes the order on A induced by the lattice $(A, \vee, \wedge)$ then for each $x, y \in A$, the element $x - y$ is the smallest $z \in A$ such that $y + z \geqslant x$.

(5) A satisfies the identity

$$\big((x - y) \vee 0\big) + y \leqslant x \vee y.$$

As is shown in [11], condition (4) is equivalent to the following system of identities:

$$(4') \qquad \begin{aligned} x + (y - x) &\geqslant y;\\ x - y &\leqslant (x \vee z) - y;\\ (x + y) - y &\leqslant x. \end{aligned}$$

Hence *DRl*-semigroups form a variety of type $\langle 2, 0, 2, 2, 2\rangle$.

N o t e. In Swamy's original definition of a *DRl*-semigroup, the identity $x - x \geqslant 0$ is also required. But by [6], Theorem 2, in any algebra satisfying (1)–(4) the identity $x - x = 0$ is always satisfied.

DRl-semigroups can be viewed as intervals of abelian l-groups. Indeed, let $G = (G, +, 0, -(\cdot), \vee, \wedge)$ be an abelian l-group and let $0 \leqslant u \in G$. For any x, $y \in [0, u] = \{x \in G;\ 0 \leqslant x \leqslant u\}$ set $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$. Put $\Gamma(G, u) = \big([0, u], \oplus, \neg, 0\big)$. Then $\Gamma(G, u)$ is an *MV*-algebra. The *MV*-algebras in the form $\Gamma(G, u)$ are sufficiently universal because by [7], if A is any *MV*-algebra then there exist an abelian l-group G and $0 \leqslant u \in G$ such that A is isomorphic to $\Gamma(G, u)$.

The intervals of type $[0, u]$ of abelian l-groups can be also considered as (bounded) *DRl*-semigroups. Indeed, by [10], Theorem 1, if $G = (G, +, 0, -(\cdot), \vee, \wedge)$ is an abelian l-group, $0 \leqslant u \in G$, $B = [0, u]$, and if $x \oplus y = (x + y) \wedge u$ and $x \ominus y = (x - y) \vee 0$ for any

x, $y \in B$, then $(B, \oplus, 0, \vee, \wedge, \ominus)$ is a bounded *DRl*-semigroup in which, moreover, $u \ominus (u \ominus x) = x$ for each $x \in B$. So we have ([10], Corollary 2) that if $A = (A, \oplus, \neg, 0)$ is an *MV*-algebra and if we set $x \leqslant y \iff \neg(\neg x \oplus y) \oplus y = y$ for any x, $y \in A$, then $\leqslant$ is a lattice order on A (with the lattice operations $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$), for any r, $s \in A$ there exists a least element $r \ominus s$ with the property $s \oplus (r \ominus s) \geqslant r$, and $(A, \oplus, 0, \vee, \wedge, \ominus)$ is a bounded *DRl*-semigroup with the smallest element 0 and the greatest element $\neg 0$ in which $\neg 0 \ominus (\neg 0 \ominus x) = x$ for any $x \in A$. Further ([10], Theorem 3), if $(B, +, 0, \vee, \wedge, -)$ is a bounded *DRl*-semigroup with the greatest element 1 in which $1 - (1 - x) = x$ for any $x \in B$, and if we set $\neg x = 1 - x$ for any $x \in B$, then $(B, +, \neg, 0)$ is an *MV*-algebra.

Note. In [10], Theorem 3, the validity of the identity $x + (y - x) = y + (x - y)$ is also required. By [5], Theorem 1.2.3, if a *DRl*-semigroup A has the greatest element, then A is bounded also below and, moreover, 0 is the smallest element in A. And if this is the case then by [11], Lemma 2, $x + (y - x) = x \vee y$ for any x, $y \in A$, hence the identity $x + (y - x) = y + (x - y)$ is valid in A.

The following two propositions will make it possible to prove the main result of the paper. (The homomorphisms will be always meant with respect to the types and signatures mentioned.)

Proposition 1. *Let $A = (A, \oplus, \neg, 0)$ and $B = (B, \oplus, \neg, 0')$ be MV-algebras and $f\colon A \to B$ a homomorphism of MV-algebras. Then f is also a homomorphism of the induced DRl-semigroups $(A, \oplus, 0, \vee, \wedge, \ominus)$ and $(B, \oplus, 0', \vee, \wedge, \ominus)$.*

Proof. Let G and H be abelian l-groups with elements $0 \leqslant u \in G$ and $0 \leqslant v \in H$ such that A is isomorphic to the *MV*-algebra $\Gamma(G, u)$ and B is isomorphic to the *MV*-algebra $\Gamma(H, v)$. In [10], Proposition 11, it is proved that if $\bar{f}$ is a homomorphism of the abelian l-group G into an abelian l-group H then its restriction $f = \bar{f} \restriction \Gamma(G, u)$ is a homomorphism of the *MV*-algebra $\Gamma(G, u)$ into the *MV*-algebra $\Gamma\big(H, \bar{f}(u)\big)$. Further, by [8], Proposition 3.5, if G' and H' are abelian l-groups, $u' \in G'$ and $v' \in H'$ are strong order units in G' and H', respectively, and $f\colon \Gamma(G', u') \to \Gamma(H', v')$ is a homomorphism of *MV*-algebras such that $f(u') = v'$, then there exists a homomorphism $\bar{f}$ of the l-group G' into the l-group H' such that f is the restriction of $\bar{f}$ on $\Gamma(G', u')$. (Recall that an element u of an l-group G is called a *strong order unit* if $0 \leqslant u$ and for each $x \in G$ there exists $n \in \mathbb{N}$ such that $x \leqslant nu$.) If we consider in our case the convex l-subgroup of G generated by u and the convex l-subgroup of H generated by v instead of G and H, respectively, we get that f is a homomorphism of the *DRl*-semigroup $(A, \oplus, 0, \vee, \wedge, \ominus)$ into the *DRl*-semigroup $(B, \oplus, 0', \vee, \wedge, \ominus)$. $\square$

For a *DRl*-semigroup with the greatest element 1 we can consider the identity

$$1 - (1 - x) = x\,. \tag{i}$$

Proposition 2. ([10], Proposition 12) *Let $A = (A, +, 0, \vee, \wedge, -)$ and $B = (B, +, 0', \vee, \wedge, -)$ be DRl-semigroups with the greatest elements 1 and $1'$, respectively, satisfying identity* (i) *and let $g\colon A \to B$ be a homomorphism of DRl-semigroups such that $g(1) = 1'$. Then g is a homomorphism of the induced MV-algebras.*

Consequently, in what follows, for the class of bounded *DRl*-semigroups, we will consider the greatest element 1 as a nullary operation and so we will extend the signature of such *DRl*-semigroups to $\langle +, 0, \vee, \wedge, -, 1\rangle$ of type $\langle 2, 0, 2, 2, 2, 0\rangle$. Further, the morphisms of the categories of algebras considered will be always all homomorphisms of the corresponding signatures. Then we get the following theorem.

Theorem 3. *MV-algebras are categorically equivalent to bounded DRl-semigroups satisfying identity* (i).

Proof. If $A = (A, \oplus, \neg, 0)$ is an *MV*-algebra, set $\mathcal{F}(A) = (A, \oplus, 0, \vee, \wedge, \ominus, \neg 0)$. For any *MV*-algebras A and B and any *MV*-homomorphism $f\colon A \to B$ set $\mathcal{F}(f) = f$. If we denote by $\mathcal{MV}$ the category of all *MV*-algebras and by $\mathcal{DR}l_{1(\mathrm{i})}$ the category of all bounded *DRl*-semigroups satisfying (i) then Propositions 1 and 2 imply that $\mathcal{F}\colon \mathcal{MV} \to \mathcal{DR}l_{1(\mathrm{i})}$ is a functor which is an equivalence. $\square$

Now, let us recall the notion of a bounded commutative *BCK*-algebra.

A *bounded commutative BCK-algebra* is an algebra $A = (A, *, 0, 1)$ of type $\langle 2, 0, 0\rangle$ satisfying the following identitites:

(1) $(x * y) * z = (x * z) * y$;

(2) $x * (x * y) = y * (y * x)$;

(3) $x * x = 0$;

(4) $x * 0 = x$;

(5) $x * 1 = 0$.

Bounded commutative *BCK*-algebras were introduced in [12] and, as varieties, in [14]. In [4] it was proved that such a *BCK*-algebra forms a lattice with respect to the order relation $x \leqslant y \Longleftrightarrow x * y = 0$ and in [13] it was proved that this lattice is distributive. Mundici in [9] showed that MV-algebras and bounded commutative BCK-algebras are categorically equivalent. If we denote by $\mathcal{BCK}_{01}$ the category of bounded commutative BCK-algebras, the following theorem is an immediate consequence of [9] and our Theorem 3.

Theorem 4. *The following three categories are equivalent:*

a) The category $\mathcal{MV}$ of MV-algebras.

b) The category $\mathcal{DR}l_{1(\mathrm{i})}$ of bounded DRl-semigroups satisfying condition (i).

c) The category $\mathcal{BCK}_{01}$ of bounded commutative BCK-algebras.

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