QUASILINEAR AND QUADRATIC SINGULARLY PERTURBED NEUMANN'S PROBLEM

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(Received June 6, 1997)

Abstract. The problem of existence and asymptotic behaviour of solutions of the quasilinear and quadratic singularly perturbed Neumann's problem as a small parameter at the highest derivative tends to zero is studied.

Keywords: singularly perturbed Neumann's problem

MSC 1991: 05C10, 05C75

1. Introduction

In the paper [2] the author established sufficient conditions for the existence and uniform convergence of the solutions of a semilinear singularly perturbed differential equation $\varepsilon y'' + ky = f(t, y)$ to a solution of the reduced problem $ku = f(t, u)$ as the small positive parameter ε tends to zero. The purpose of this paper is an extension of Theorem 1 of the above cited paper to more general cases. We will consider Neumann's problem

$$(\mathrm{NP}_0) \qquad \begin{aligned} \varepsilon y'' = F\left(t, y, y'\right), \quad a < t < b,\\ y'(a, \varepsilon) = 0, \quad y'(b, \varepsilon) = 0, \end{aligned}$$

where $F \in C^1\left([a, b] \times \mathbb{R}^2\right)$ and ε is a small positive parameter. The proofs of the theorems are based upon the method of lower and upper solutions.

As usual, we say that $\alpha \in C^2([a, b])$ is a lower solution for (NP_0) if $\alpha'(a, \varepsilon) \geqslant 0$, $\alpha'(b, \varepsilon) \leqslant 0$, and $\varepsilon\alpha''(t, \varepsilon) \geqslant F\left(t, \alpha(t, \varepsilon), \alpha'(t, \varepsilon)\right)$ for every $t \in [a, b]$. An upper solution $\beta \in C^2([a, b])$ satisfies $\beta'(a, \varepsilon) \leqslant 0$, $\beta'(b, \varepsilon) \geqslant 0$, and $\varepsilon\beta''(t, \varepsilon) \leqslant F\left(t, \beta(t, \varepsilon), \beta'(t, \varepsilon)\right)$ for every $t \in [a, b]$.

Definition 1. We say that a function F satisfies the Bernstein-Nagumo condition if for each $M > 0$ there exists a continuous function $h_M\colon [0,\infty) \to [a_M,\infty)$ with $a_M > 0$ and $\int_0^\infty \frac{s}{h_M(s)}\,\mathrm{d}s = \infty$ such that for all y, $|y| \leqslant M$, all $t \in [a,b]$ and all $z \in \mathbb{R}$ we have

$$|F(t,y,z)| \leqslant h_M(|z|).$$

Remark. As a remark we conclude that the functions of the form $F(t,y,y') = f(t,y)y' + g(t,y)$ and $F(t,y,y') = f(t,y)y'^2 + g(t,y)$ satisfy the Bernstein-Nagumo condition.

Lemma 1. *If α, β are lower and upper solutions for* (NP$_0$) *such that $\alpha(t,\varepsilon) \leqslant \beta(t,\varepsilon)$ on $[a,b]$ and F satisfies the Bernstein-Nagumo condition, then there exists a solution y of* (NP$_0$) *with $\alpha(t,\varepsilon) \leqslant y(t,\varepsilon) \leqslant \beta(t,\varepsilon)$, $a \leqslant t \leqslant b$.*

Notation. Let

$$D_\delta(u) = \bigl\{(t,y) \in \mathbb{R}^2 : a \leqslant t \leqslant b, |y - u(t)| < \delta\bigr\},$$
$$D_{\delta,a}(u) = \bigl\{(t,y) \in \mathbb{R}^2 : a \leqslant t \leqslant a+\delta, y \in \mathbb{R}\bigr\} \cap D_\delta(u),$$

and

$$D_{\delta,b}(u) = \bigl\{(t,y) \in \mathbb{R}^2 : b-\delta \leqslant t \leqslant b, y \in \mathbb{R}\bigr\} \cap D_\delta(u),$$

where $\delta \leqslant b - a$ is a positive constant and $u = u(t)$ is a solution of the reduced problem $F(t,u,u') = 0$ defined on $[a,b]$ such that $u \in C^2([a,b])$.

Let $h(t,y)$ denote $F(t,y,u'(t))$.

2. Quasilinear Neumann's problem

In this section we consider the quasilinear Neumann's problem

$$\text{(NP}_1\text{)}\qquad \begin{aligned} &\varepsilon y'' = f(t,y)y' + g(t,y), \quad a < t < b,\\ &y'(a,\varepsilon) = 0, \quad y'(b,\varepsilon) = 0, \end{aligned}$$

where $f, g \in C^1(D_\delta(u))$. Concerning the behaviour of solutions of (NP$_1$) for $\varepsilon \to 0^+$ we have the following result.

Theorem 1. *Consider the problem* (NP$_1$)*. Let there exist a solution $u \in C^2([a,b])$ of the reduced problem. Let δ, m be positive constants such that $\frac{\partial h(t,y)}{\partial y} \geqslant m$ for every $(t,y) \in D_\delta(u)$. Let $f(t,y) \leqslant 0$ and $f(t,y) \geqslant 0$ for every $(t,y) \in D_{\delta,a}(u)$*

and $(t, y) \in D_{\delta,b}(u)$, *respectively. Then there exists* ε_0 *such that for any* $\varepsilon \in (0, \varepsilon_0]$ *the problem* (NP_1) *has a solution satisfying the inequality*

$$|y(t,\varepsilon) - u(t)| \leqslant v_1(t,\varepsilon) + v_2(t,\varepsilon) + C\varepsilon$$

on $[a, b]$, *where*

$$v_1(t,\varepsilon) = |u'(a)| \frac{\exp\left[-\sqrt{\frac{m}{\varepsilon}}(b-t)\right] + \exp\left[-\sqrt{\frac{m}{\varepsilon}}(t-b)\right]}{\sqrt{\frac{m}{\varepsilon}}\left(\exp\left[\sqrt{\frac{m}{\varepsilon}}(b-a)\right] - \exp\left[-\sqrt{\frac{m}{\varepsilon}}(b-a)\right]\right)},$$

$$v_2(t,\varepsilon) = |u'(b)| \frac{\exp\left[-\sqrt{\frac{m}{\varepsilon}}(a-t)\right] + \exp\left[-\sqrt{\frac{m}{\varepsilon}}(t-a)\right]}{\sqrt{\frac{m}{\varepsilon}}\left(\exp\left[\sqrt{\frac{m}{\varepsilon}}(b-a)\right] - \exp\left[-\sqrt{\frac{m}{\varepsilon}}(b-a)\right]\right)}$$

and C *is a positive constant.*

P r o o f. We define the lower solutions by

$$\alpha(t,\varepsilon) = u(t) - v_1(t,\varepsilon) - v_2(t,\varepsilon) - \Gamma(\varepsilon)$$

and the upper solutions by

$$\beta(t,\varepsilon) = u(t) + v_1(t,\varepsilon) + v_2(t,\varepsilon) + \Gamma(\varepsilon).$$

Here $\Gamma(\varepsilon) = \frac{\varepsilon\gamma}{m}$, where γ is a constant which will be defined below. One can easily check that the functions α, β satisfy the boundary conditions required for the lower and upper solutions of (NP_1) and $\alpha \leqslant \beta$ on $[a, b]$. Now we show that $\varepsilon\alpha''(t,\varepsilon) \geqslant f(t, \alpha(t,\varepsilon))\alpha'(t,\varepsilon) + g(t, \alpha(t,\varepsilon))$ and $\varepsilon\beta''(t,\varepsilon) \leqslant f(t, \beta(t,\varepsilon))\beta'(t,\varepsilon) + g(t, \beta(t,\varepsilon))$ on $[a, b]$. By the Taylor theorem we obtain

$$\begin{aligned}
&\varepsilon\alpha'' - F(t,\alpha,\alpha') \\
&\qquad = \varepsilon\alpha'' - \big(F(t,\alpha,\alpha') - F(t,u,u')\big) \\
&\qquad = \varepsilon\alpha'' - \big[\big(F(t,\alpha,u') - F(t,u,u')\big) + \big(F(t,\alpha,\alpha') - F(t,\alpha,u')\big)\big] \\
&\qquad = \varepsilon\alpha'' - \Big[\frac{\partial h(t,\eta(t,\varepsilon))}{\partial y}(\alpha - u) + f(t,\alpha)(\alpha' - u')\Big] \\
&\qquad = \varepsilon u'' - \varepsilon v_1'' - \varepsilon v_2'' + \frac{\partial h(t,\eta)}{\partial y}(v_1 + v_2 + \Gamma) + f(t,\alpha)(v_1' + v_2') \\
&\qquad \geqslant \varepsilon u'' - \varepsilon v_1'' - \varepsilon v_2'' + m(v_1 + v_2 + \Gamma) + f(t,\alpha)(v_1' + v_2') \\
&\qquad = \varepsilon u'' + \varepsilon\gamma + f(t,\alpha)(v_1' + v_2') \\
&\qquad \geqslant -\varepsilon|u''| + \varepsilon\gamma + f(t,\alpha)(v_1' + v_2')
\end{aligned}$$

and

$$F(t,\beta,\beta') - \varepsilon\beta'' \geqslant -\varepsilon|u''| + \varepsilon\gamma + f(t,\beta)(v_1' + v_2'),$$

where $(t, \eta(t, \varepsilon))$ is a point between $(t, \alpha(t, \varepsilon))$ and $(t, u(t))$, $(t, \eta(t, \varepsilon)) \in D_\delta(u)$ for sufficiently small ε.

Let $u'(a) \neq 0$, $u'(b) \neq 0$ (if $u'(a) = 0$ or $u'(b) = 0$, we proceed analogously). From the above assumptions we obtain that $f(t, \alpha)\,(v_1' + v_2') \geqslant 0$ and $f(t, \beta)\,(v_1' + v_2') \geqslant 0$ on $\left[a, a + \tilde{\delta}\right] \cup \left[b - \tilde{\delta}, b\right]$ for $\varepsilon \in (0, \varepsilon_1]$ where $\tilde{\delta} = \min\{\delta, \delta_1\}$, and δ_1, ε_1 are such that $v_1' + v_2' < 0$ ($v_1' + v_2' > 0$) on $[a, a + \delta_1]$ ($[b - \delta_1, b]$) and $(t, \alpha) \subset D_{\tilde{\delta}}(u)$, $(t, \beta) \subset D_{\tilde{\delta}}(u)$ for $\varepsilon \in (0, \varepsilon_1]$. On the interval $\left[a + \tilde{\delta}, b - \tilde{\delta}\right]$ we have $|f(t, \alpha)\,(v_1' + v_2')| \leqslant c_1\varepsilon$ and $|f(t, \beta)\,(v_1' + v_2')| \leqslant c_1\varepsilon$ for sufficiently small ε, for instance if $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 \leqslant \varepsilon_1$ and c_1 is a suitable positive constant (if $u'(a) = 0$ ($u'(b) = 0$) then $|f(t, \alpha)\,(v_1' + v_2')| \leqslant c_1\varepsilon$ and $|f(t, \beta)\,(v_1' + v_2')| \leqslant c_1\varepsilon$ on $\left[a, b - \tilde{\delta}\right]$ ($\left[a + \tilde{\delta}, b\right]$).

Thus if we choose a constant $\gamma \geqslant c_1 + \max\left\{|u''(t)|\,, t \in [a, b]\right\}$ then $\varepsilon\alpha''(t, \varepsilon) \geqslant f(t, \alpha(t, \varepsilon))\alpha'(t, \varepsilon) + g(t, \alpha(t, \varepsilon))$ and $\varepsilon\beta''(t, \varepsilon) \leqslant f(t, \beta(t, \varepsilon))\beta'(t, \varepsilon) + g(t, \beta(t, \varepsilon))$ on $[a, b]$. The existence of a solution of (NP_1) satisfying the above inequalities follows from Lemma. This completes the proof. $\square$

Example 1. As an illustrative example we consider the (NP_1) for the differential equation $\varepsilon y'' = yy' - \left(t - \frac{1}{2}\right)$ on $[0, 1]$. General solution of the reduced problem $uu' - \left(t - \frac{1}{2}\right) = 0$ is $u^2 = t^2 - t + k, k \in \mathbb{R}$; however, only $u(t) = t - \frac{1}{2}$ satisfies the assumptions asked on the solution of the reduced problem. On the basis of Theorem 1, there is ε_0 such that for every $\varepsilon \in (0, \varepsilon_0]$ the problem has a solution satisfying $\left|y(t, \varepsilon) - \left(t - \frac{1}{2}\right)\right| \leqslant v_1 + v_2 + c_1\varepsilon$ on $[0, 1]$.

3. Quadratic Neumann's problem

Now we will consider the quadratic Neumann's problem

$$(\mathrm{NP}_2) \qquad \begin{aligned} &\varepsilon y'' = f(t, y){y'}^2 + g(t, y), \quad a < t < b, \\ &y'(a, \varepsilon) = 0, \quad y'(b, \varepsilon) = 0, \end{aligned}$$

where $f, g \in C^1(D_\delta(u))$.

Theorem 2. *Consider the problem* (NP_2). *Let there exist a solution $u \in C^2([a, b])$ of the reduced problem. Let δ, m be positive constants such that $\frac{\partial h(t,y)}{\partial y} \geqslant m$ for every $(t, y) \in D_\delta(u)$. Let $f(t, y) \leqslant 0$ ($f(t, y) \geqslant 0$) for $(t, y) \in D_{\delta,a}(u)$ when $u'(a) > 0$ ($u'(a) < 0$) and $f(t, y) \leqslant 0$ ($f(t, y) \geqslant 0$) for $(t, y) \in D_{\delta,b}(u)$ when $u'(b) < 0$ ($u'(b) > 0$). Then there exists ε_0 such that for any $\varepsilon \in (0, \varepsilon_0]$ the problem* (NP_2) *has a solution satisfying the inequality*

$$|y(t, \varepsilon) - u(t)| \leqslant v_1(t, \varepsilon) + v_2(t, \varepsilon) + C\varepsilon$$

on $[a,b]$ where v_1, v_2 are the functions from Theorem 1 *and C is a positive constant.*

P r o o f. The idea of the proof is essentially the same as in the proof of Theorem 1. Let us define the lower solutions by

$$\alpha(t,\varepsilon) = u(t) - v_1(t,\varepsilon) - v_2(t,\varepsilon) - \Gamma(\varepsilon)$$

and the upper solutions by

$$\beta(t,\varepsilon) = u(t) + v_1(t,\varepsilon) + v_2(t,\varepsilon) + \Gamma(\varepsilon).$$

Analogously as in Theorem 1 we obtain

$$\begin{aligned}\varepsilon\alpha'' - F\left(t,\alpha,\alpha'\right) &\geqslant -\varepsilon\left|u''\right| + \gamma\varepsilon - f(t,\alpha)\left({\alpha'}^2 - {u'}^2\right)\\ &= -\varepsilon\left|u''\right| + \gamma\varepsilon + f(t,\alpha)\left(v_1' + v_2'\right)\left(2u' - v_1' - v_2'\right)\end{aligned}$$

and

$$\begin{aligned}F\left(t,\beta,\beta'\right) - \varepsilon\beta'' &\geqslant -\varepsilon\left|u''\right| + \gamma\varepsilon + f(t,\beta)\left({\beta'}^2 - {u'}^2\right)\\ &= -\varepsilon\left|u''\right| + \gamma\varepsilon + f(t,\beta)\left(v_1' + v_2'\right)\left(2u' + v_1' + v_2'\right).\end{aligned}$$

Similarly as in the previous theorem we conclude (for $u'(a) \neq 0$, $u'(b) \neq 0$) that

$$f(t,\alpha)\left(v_1' + v_2'\right)\left(2u' - v_1' - v_2'\right) \geqslant 0,\ \ f(t,\beta)\left(v_1' + v_2'\right)\left(2u' + v_1' + v_2'\right) \geqslant 0$$

on $\left[a, a+\hat{\delta}\right] \cup \left[b-\hat{\delta}, b\right]$ and

$$\begin{aligned}&\left|f(t,\alpha)\left(v_1' + v_2'\right)\left(2u' - v_1' - v_2'\right)\right| \leqslant c_2\varepsilon,\\ &\left|f(t,\beta)\left(v_1' + v_2'\right)\left(2u' + v_1' + v_2'\right)\right| \leqslant c_2\varepsilon\end{aligned}$$

on $\left[a+\hat{\delta}, b-\hat{\delta}\right]$ for $\varepsilon \in (0,\varepsilon_0]$, sufficiently small $\hat{\delta} > 0$ and a suitable positive constant c_2. Therefore, for $\gamma \geqslant c_2 + \max\left\{\left|u''(t)\right|, t \in [a,b]\right\}$ we have

$$\varepsilon\alpha''(t,\varepsilon) \geqslant f(t,\alpha(t,\varepsilon)){\alpha'}^2(t,\varepsilon) + g(t,\alpha(t,\varepsilon))$$

and

$$\varepsilon\beta''(t,\varepsilon) \leqslant f(t,\beta(t,\varepsilon)){\beta'}^2(t,\varepsilon) + g(t,\beta(t,\varepsilon))$$

on $[a,b]$. Hence Theorem 2 is proved. $\square$

E x a m p l e 2. Consider problem (NP_2) for differential equation $\varepsilon y'' = yy'^2 - (t+1)$ on $[-2,1]$. Obviously $u(t) = t+1$ is the only solution of the reduced problem $uu'^2 - (t+1) = 0$ satisfying the assumptions of Theorem 2. Hence, there is ε_0 such that for every $\varepsilon \in (0,\varepsilon_0]$ the problem has a solution satisfying

$$|y(t,\varepsilon) - (t+1)| \leqslant v_1 + v_2 + c_2\varepsilon$$

on $[-2,1]$.

References

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