# AVERAGES OF QUASI-CONTINUOUS FUNCTIONS

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*Abstract.* The goal of this paper is to characterize the family of averages of comparable (Darboux) quasi-continuous functions.

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## Preliminaries

The letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the real line, the set of rationals and the set of positive integers, respectively. The word *function* denotes a mapping from $\mathbb{R}$ into $\mathbb{R}$. We say that functions $\varphi$ and $\psi$ are *comparable* if either $\varphi < \psi$ on $\mathbb{R}$ or $\varphi > \psi$ on $\mathbb{R}$. For each $A \subset \mathbb{R}$ we use the symbols $\operatorname{cl} A$ and $\operatorname{bd} A$ to denote the closure and the boundary of $A$, respectively.

Let $f$ be a function. If $A \subset \mathbb{R}$ is nonvoid, then let $\omega(f, A)$ be the *oscillation of $f$ on $A$*, i.e., $\omega(f, A) = \sup\bigl\{|f(x) - f(t)|\colon\ x, t \in A\bigr\}$. For each $x \in \mathbb{R}$ let $\omega(f, x)$ be the *oscillation of $f$ at $x$*, i.e., $\omega(f, x) = \lim_{\delta \to 0^+} \omega\bigl(f, (x - \delta, x + \delta)\bigr)$. The symbol $\mathscr{C}_f$ denotes the set of points of continuity of $f$.

We say that a function $f$ is *quasi-continuous* in the sense of Kempisty [4] (*cliquish* [10]) at a point $x \in \mathbb{R}$ if for each $\varepsilon > 0$ and each open set $U \ni x$ there is a nonvoid open set $V \subset U$ such that $\omega\bigl(f, \{x\} \cup V\bigr) < \varepsilon$ ($\omega(f, V) < \varepsilon$ respectively). We say that $f$ is *quasi-continuous* (*cliquish*) if it is quasi-continuous (cliquish) at each point $x \in \mathbb{R}$. Cliquish functions are also known as *pointwise discontinuous*.

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Let $I$ be an interval and $f\colon I \to \mathbb{R}$. We say that $f$ is *Darboux* if it has the intermediate value property. We say that $f$ is *strong Świątkowski* [6] if whenever $a, b \in I$, $a < b$, and $y$ is a number between $f(a)$ and $f(b)$, there is an $x \in (a, b) \cap \mathscr{C}_f$ with $f(x) = y$. One can easily verify that strong Świątkowski functions are both Darboux and quasi-continuous, and that the converse is not true.

For brevity, if $f$ is a cliquish function and $x \in \mathbb{R}$, then we define

$$\underline{\mathrm{LIM}}(f, x) = \varliminf_{t \to x, t \in \mathscr{C}_f} f(t).$$

The symbols $\underline{\mathrm{LIM}}(f, x^-)$ and $\underline{\mathrm{LIM}}(f, x^+)$ are defined analogously.

## Introduction

In 1974 A. M. Bruckner, J. G. Ceder, and T. L. Pearson characterized the averages of comparable Darboux functions [1, Theorem 2]. In this paper we solve an analogous problem, namely we characterize the averages of comparable quasi-continuous functions.

A similar problem is to determine a necessary and sufficient condition that for a function $f$ there exists a quasi-continuous function $\psi$ such that $\psi > f$ on $\mathbb{R}$. (The answer to this question for Darboux functions can be easily obtained using the proof of [1, Theorem 2].) In both cases we ask whether there is a positive function $g$ such that both $f + g$ and $-f + g$ are quasi-continuous (the first problem) or such that $f + g$ is quasi-continuous (the second problem). This suggests a similar problem for larger classes of functions. Theorem 4.1 contains a solution of this problem for finite classes of cliquish functions. Recall that by [5, Example 2], we cannot in general allow infinite families in Theorem 4.1. Unlike [7, Theorem 4], we cannot conclude in condition (ii) of Theorem 4.1 that $g$ is a Baire one function; actually, we cannot even conclude that $g$ is Borel measurable (Corollary 4.5).

The Baire class one case makes no difficulty if we require only quasi-continuity of the sums, but it needs a separate argument if we require both the Darboux property and the quasi-continuity. Notice that by Proposition 4.3, the necessary and sufficient condition for Darboux quasi-continuous Baire one functions is essentially stronger.

## Auxiliary lemmas

**Lemma 3.1.** *If $f$ is a cliquish function, then the mapping $x \mapsto \underline{\mathrm{LIM}}(f, x)$ is lower semicontinuous, while the mapping $x \mapsto \underline{\mathrm{LIM}}(f, x^-)$ belongs to Baire class two.*

P r o o f. Let $y \in \mathbb{R}$. For every $x \in \mathbb{R}$, if $\underline{\mathrm{LIM}}(f, x) > y$, then there exist an open interval $I_x \ni x$ and a rational $q_x > y$ such that $f > q_x$ on $\mathscr{C}_f \cap I_x$, whence $\underline{\mathrm{LIM}}(f, t) \geqslant q_x > y$ for each $t \in I_x$. Thus the set $\{x \in \mathbb{R}\colon\ \underline{\mathrm{LIM}}(f, x) > y\}$ is open.

To prove the other assertion put $A_y = \{x \in \mathbb{R}\colon \underline{\mathrm{LIM}}(f, x^-) > y\}$ for each $y \in \mathbb{R}$. Let $y \in \mathbb{R}$. If $x \in A_y$, then proceeding as above we can find a closed interval $I_x \subset A_y$ with $x \in I_x$. So $A_y \cap \operatorname{bd} A_y$ is at most countable. Hence $A_y$ is an $F_\sigma$ set, while $\{x \in \mathbb{R}\colon \underline{\mathrm{LIM}}(f, x^-) < y\} = \bigcup\{\mathbb{R} \setminus A_q\colon\ q < y,\ q \in \mathbb{Q}\}$ is the difference of an $F_\sigma$ set and a countable one. $\square$

**Lemma 3.2.** *Let $I = [a, b]$ and $n \in \mathbb{N}$. Suppose that functions $f_1, \dots, f_k$ are cliquish and $\max\{\omega(f_1, I), \dots, \omega(f_k, I)\} < 1$. There is a positive Baire one function $g$ such that $g = 1$ on $\operatorname{bd} I$, $\mathscr{C}_g \supset \bigcap_{i=1}^{k} \mathscr{C}_{f_i}$, and for each $i$ the function $(f_i + g) \restriction I$ is strong Świątkowski and*

$$(f_i + g)\Big[I \cap \bigcap_{i=1}^{k} \mathscr{C}_{f_i}\Big] \supset \big[\inf f_i[I] + 1, \max\{\inf f_i[I] + 1, n\}\big].$$

P r o o f. Put $T = \max\{|n - \inf f_i[I]|\colon\ i \in \{1, \dots, k\}\} + 1$. Construct a nonnegative continuous function $\varphi$ such that $\varphi[I] = [0, T]$ and $\varphi = 0$ outside of $I$. For each $i$ define $\tilde{f}_i(x) = (f_i + \varphi)(x)$ if $x \in I$, and let $\tilde{f}_i$ be constant on $(-\infty, a]$ and $[b, \infty)$. By [7, Theorem 4], there is a Baire one function $\tilde{g}$ such that $\tilde{f}_i + \tilde{g}$ is strong Świątkowski for each $i$ (see condition (8) in the proof of [7, Theorem 4]), $\mathscr{C}_{\tilde{g}} \supset \bigcap_{i=1}^{k} \mathscr{C}_{f_i}$, and $|\tilde{g}| < 1$ on $\mathbb{R}$; by its proof, we can conclude that $\tilde{g} = 0$ on $\{a, b\}$. Put $g = \varphi + \tilde{g} + 1$. Then for each $i$, since $\tilde{f}_i + \tilde{g}$ is strong Świątkowski and $f_i + g = \tilde{f}_i + \tilde{g} + 1$ on $I$, we have

$$\begin{aligned}(f_i + g)\Big[I \cap \bigcap_{i=1}^{k} \mathscr{C}_{f_i}\Big] &\supset \big(\inf(f_i + g)[I], \sup(f_i + g)[I]\big)\\ &\supset \big(f_i(a), \inf f_i[I] + \sup g[I]\big)\\ &\supset \big[\inf f_i[I] + 1, \max\{\inf f_i[I] + 1, n\}\big].\end{aligned}$$

The other requirements are evident. $\square$

## Main results

**Theorem 4.1.** *Let $\mathscr{F}$ be one of the following classes of functions: all cliquish functions, Lebesgue measurable cliquish functions, cliquish functions in Baire class $\alpha$ ($\alpha \geqslant 1$), and suppose $f_1, \dots, f_k \in \mathscr{F}$. The following are equivalent:*

(i) *there is a positive function $g$ such that $f_i + g$ is quasi-continuous for each $i$;*

(ii) *there is a positive function $g \in \mathscr{F}$ such that $\mathscr{C}_g \supset \bigcap_{i=1}^{k} \mathscr{C}_{f_i}$ and $f_i + g$ is quasi-continuous for each $i$;*

(iii) *for each $x \in \mathbb{R}$ and each $i$ we have* $\underline{\mathrm{LIM}}(f_i, x) < \infty$.

P r o o f. The implication (ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (iii). Let $x \in \mathbb{R}$ and $i \in \{1, \ldots, k\}$. Since $f_i + g$ is quasi-continuous, so by [2] (see also [3, Lemma 2]) we obtain

$$\underline{\mathrm{LIM}}(f_i, x) \leqslant \underline{\mathrm{LIM}}(f_i + g, x) \leqslant (f_i + g)(x) < \infty.$$

(iii) $\Rightarrow$ (ii). Put $A = \bigcup_{i=1}^{k} \{x \in \mathbb{R}\colon\ \omega(f_i, x) \geqslant 1\}$. Then $A$ is closed and nowhere dense. Find a family $\{I_n\colon\ n \in \mathbb{N}\}$ consisting of nonoverlapping compact intervals, such that $\bigcup_{n \in \mathbb{N}} I_n = \mathbb{R} \setminus A$ and each $x \notin A$ is an interior point of $I_n \cup I_m$ for some $n, m \in \mathbb{N}$. Since each $I_n$ is compact and $\omega(f_i, x) < 1$ for each $x \in I_n$ and $i \in \{1, \ldots, k\}$, so we may assume that $\omega(f_i, I_n) < 1$ for each $i$ and $n$. For each $n \in \mathbb{N}$ use Lemma 3.2 to construct a positive Baire one function $g_n$ such that $g_n = 1$ on $\operatorname{bd} I_n$, $\mathscr{C}_{g_n} \supset \bigcap_{i=1}^{k} \mathscr{C}_{f_i}$, and for each $i$ the function $(f_i + g_n) \restriction I_n$ is strong Świątkowski and

$$(\star) \qquad (f_i + g_n)\left[I_n \cap \bigcap_{i=1}^{k} \mathscr{C}_{f_i}\right] \supset \bigl[\inf f_i[I_n] + 1, \max\{\inf f_i[I_n] + 1, n\}\bigr].$$

Define $g(x) = g_n(x)$ if $x \in I_n$ for some $n \in \mathbb{N}$, and

$$g(x) = \max\bigl\{\max\{\, \underline{\mathrm{LIM}}(f_i, x) - f_i(x)\colon\ i \in \{1, \ldots, k\}\}, 0\bigr\} + 1$$

if $x \in A$. By Lemma 3.1, each mapping $x \mapsto \underline{\mathrm{LIM}}(f_i, x)$ is Baire one, so $g \in \mathscr{F}$.

Fix an $i \in \{1, \ldots, k\}$. Clearly $f_i + g$ is quasi-continuous outside of $A$. On the other hand, if $x \in A$, then by $(\star)$, for each $\delta > 0$ we have

$$(f_i + g)\bigl[(x - \delta, x + \delta) \cap \mathscr{C}_{f_i + g}\bigr] \supset \bigl(\underline{\mathrm{LIM}}(f_i, x) + 1, \infty\bigr).$$

Hence $f_i + g$ is quasi-continuous. $\square$

**Theorem 4.2.** *Let $\mathscr{F}$ be one of the following classes of functions: all cliquish functions, Lebesgue measurable cliquish functions, cliquish functions in Baire class $\alpha$ ($\alpha \geqslant 2$), and suppose $f_1, \ldots, f_k \in \mathscr{F}$. The following are equivalent:*

(i) *there is a positive function $g$ such that $f_i + g$ is both Darboux and quasi-continuous for each $i$;*

(ii) *there is a positive function $g \in \mathscr{F}$ such that $\mathscr{C}_g \supset \bigcap_{i=1}^{k} \mathscr{C}_{f_i}$ and $f_i + g$ is strong Świątkowski for each $i$;*

(iii) *for each $x \in \mathbb{R}$ and each $i$ we have* $\max\bigl\{\underline{\mathrm{LIM}}(f_i, x^-), \underline{\mathrm{LIM}}(f_i, x^+)\bigr\} < \infty$.

P r o o f. The proof of the implication (iii) $\Rightarrow$ (ii) is a repetition of the argument used in Theorem 4.1, and the implication (ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (iii). Let $x \in \mathbb{R}$ and $i \in \{1, \ldots, k\}$. Since $f_i + g$ is both Darboux and quasi-continuous, so by [9, Lemma 2] we obtain

$$\underline{\mathrm{LIM}}(f_i, x^-) \leqslant \underline{\mathrm{LIM}}(f_i + g, x^-) \leqslant (f_i + g)(x) < \infty.$$

Similarly $\underline{\mathrm{LIM}}(f_i, x^+) < \infty$. □

**Proposition 4.3.** *There is a Baire one function $f$ such that $f+g$ is strong Świątkowski for some positive function $g$ in Baire class two, but $f+g$ is Darboux for no positive Baire one function $g$.*

P r o o f. Let $F$ be the Cantor ternary set and let $\mathscr{I} = \{(a_n, b_n) \colon n \in \mathbb{N}\}$ and $\mathscr{J}$ be disjoint families of components of $\mathbb{R} \setminus F$ such that $F = \left(\mathrm{cl} \bigcup \mathscr{I}\right) \cap \left(\mathrm{cl} \bigcup \mathscr{J}\right)$. Define $f(x) = n$ if $x \in (a_n, b_n)$ for some $n \in \mathbb{N}$ and $f(x) = 0$ otherwise. Clearly $f$ belongs to Baire class one.

Let $x \in \mathbb{R}$. If $x \in (a_n, b_n]$ for some $n \in \mathbb{N}$, then $\underline{\mathrm{LIM}}(f, x^-) = n$, otherwise $\underline{\mathrm{LIM}}(f, x^-) = 0$. Similarly $\underline{\mathrm{LIM}}(f, x^+) < \infty$. By Theorem 4.2 there is a positive Baire two function $g$ such that $f + g$ is strong Świątkowski.

On the other hand, by [8, Proposition 6.10], $f+g$ is Darboux for no positive Baire one function $g$. □

In Proposition 4.4 the symbol $\mathfrak{c}$ denotes the first ordinal equipollent with $\mathbb{R}$.

**Proposition 4.4.** *Given a family of positive functions, $\{g_\xi \colon \xi < \mathfrak{c}\}$, we can find a cliquish function $f$ which fulfils condition* (iii) *of Theorem* 4.2 *and such that $f + g_\xi$ is not quasi-continuous for each $\xi < \mathfrak{c}$.*

P r o o f. Let $F$ be the Cantor ternary set and let $\{x_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration of $F$. Define $f(x) = -g_\xi(x) - 1$ if $x = x_\xi$ for some $\xi < \mathfrak{c}$, and $f(x) = 0$ otherwise. Clearly $f$ is cliquish, and for each $x \in \mathbb{R}$ we have $\underline{\mathrm{LIM}}(f, x^-) = \underline{\mathrm{LIM}}(f, x^+) = 0$.

Let $\xi < \mathfrak{c}$. Then $(f + g_\xi)(x_\xi) = -1$ and $f + g_\xi$ is positive on a dense open set. Thus $f + g_\xi$ is not quasi-continuous at $x_\xi$. □

**Corollary 4.5.** *There is a cliquish function $f$ which fulfils condition* (iii) *of Theorem* 4.2 *and such that $f + g$ is not quasi-continuous for each positive Borel measurable function $g$.*

**Theorem 4.6.** *Let $f_1, \ldots, f_k$ be Baire one functions. The following are equivalent:*

(i) *there is a positive Baire one function $g$ such that $f_i + g$ is both Darboux and quasi-continuous for each $i$;*

(ii) *there is a positive Baire one function $g$ such that* $\mathscr{C}_g \supset \bigcap_{i=1}^{k} \mathscr{C}_{f_i}$ *and* $f_i + g$ *is strong Świątkowski for each* $i$*;*

(iii) *there is a Baire one function $h$ such that for each $x \in \mathbb{R}$ and each $i$ we have* $\max\bigl\{\underline{\mathrm{LIM}}(f_i, x^-), \underline{\mathrm{LIM}}(f_i, x^+)\bigr\} \leqslant h(x)$.

Proof. The implication (i) ⇒ (iii) can be proved similarly as in Theorem 4.2 (we let $h = \max\{f_1, \dots, f_k\} + g$), and the implication (ii) ⇒ (i) is obvious.

(iii) ⇒ (ii). The proof of this implication is a repetition of the argument used in Theorem 4.1. The only difference is in the definition of the function $g$ on the set $A$. More precisely, we put

$$g(x) = \max\bigl\{\max\{h(x) - f_i(x)\colon\ i \in \{1, \dots, k\}\}, 0\bigr\} + 1$$

if $x \in A$. Then clearly $g$ is a Baire one function. □

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