ON SPECIAL RIEMANNIAN 3-MANIFOLDS WITH DISTINCT CONSTANT RICCI EIGENVALUES

Oldřich Kowalski, Zdeněk Vlášek, Praha

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Abstract. The first author and F. Prüfer gave an explicit classification of all Riemannian 3-manifolds with distinct constant Ricci eigenvalues and satisfying additional geometrical conditions. The aim of the present paper is to get the same classification under weaker geometrical conditions.

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1. Introduction

The problem of how many Riemannian metrics exist on the open domains of $\mathbb{R}^3$ with prescribed constant Ricci eigenvalues $\varrho_1 = \varrho_2 \neq \varrho_3$ was completely solved in the series of papers [3], [2] and [7]. The main existence theorem says that the local isometry classes of these metrics are always parametrized by *two arbitrary functions of one variable*. Some nontrivial explicit examples are presented in [3], as well.

The case of distinct constant Ricci eigenvalues is more interesting. The problem of *how many* local isometry classes of solutions exist was definitely solved only recently in [8]. Here the local isometry classes are parametrized by *three arbitrary functions of two variables*. This improves essentially the earlier result by A. Spiro and F. Tricerri [9]. The first nontrivial examples have been presented by K. Yamato [11], and some others in [4]. Finally, in [5], nontrivial explicit examples have been constructed for *every choice* of the Ricci eigenvalues $\varrho_1 > \varrho_2 > \varrho_3$. (All examples in [11] are *complete* Riemannian manifolds but the range of the admissible triplets of

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Ricci eigenvalues is restricted by certain algebraic inequalities. Outside this range it seems that the corresponding metrics must be always incomplete.) In [6] an *explicit classification* was done under some additional geometric conditions, denoted as (G1), (G2) (see below). The aim of the present paper is to show that the *second condition is a consequence of the first.* This is a nontrivial fact which requires detailed analysis of the basic system of PDE for the problem.

Remark. A Riemannian manifold $(\mathscr{M}, g)$ is said to be *curvature homogeneous* if, for any pair of points p and q of $\mathscr{M}$, there is a linear isometry $F\colon T_p\mathscr{M} \to T_q\mathscr{M}$ between the corresponding tangent spaces such that $F^*R_q = R_p$ (where R denotes the curvature tensor of type $(0,4)$). I. M. Singer in 1960 (see [9]) asked the question whether there exist curvature homogeneous spaces which are not locally homogeneous. The first example was constructed by K. Sekigawa in 1973 (cf. [5], [6] and [1] for more details, further development and references). In dimension three, a Riemannian manifold is curvature homogeneous if and only if it has constant Ricci eigenvalues. The last fact remains the main motivation for our research, as well as the unsolved *conjecture of Gromov* (cf. Introduction in [10]).

2. The basic system of PDE for the problem

In this section we recall the basic preparatory results from [5] (omitting routine computational details) and we draw some simple consequences of them.

We assume here that $(\mathscr{M}, g)$ is a Riemannian 3-manifold of class C^∞ with distinct constant Ricci eigenvalues $\varrho_1, \varrho_2, \varrho_3$. Choose an open domain $\mathscr{U} \subset \mathscr{M}$ and a smooth orthonormal moving frame $\{E_1, E_2, E_3\}$ consisting of the corresponding Ricci eigenvectors at each point of $\mathscr{U}$. Denoting by R_{ijkl} and R_{ij} the corresponding components of the curvature tensor and the Ricci tensor respectively, we obtain

$$R_{ii} = \varrho_i \ (i = 1,2,3), \quad R_{ij} = 0 \ \text{ for } i \neq j, \tag{1}$$

$$\begin{aligned} &R_{1212} = \lambda_3,\ R_{1313} = \lambda_2,\ R_{2323} = \lambda_1, \ \text{ where } \lambda_i \text{ are constants,} \\ &R_{ijkl} = 0 \ \text{ if at least three indices are distinct.} \end{aligned} \tag{2}$$

Moreover, the numbers λ_i are connected with the numbers ϱ_i as follows:

$$\lambda_i - \lambda_j = -(\varrho_i - \varrho_j), \quad i,j = 1,2,3. \tag{3}$$

In a neighborhood $\mathscr{U}_p$ of any point $p \in \mathscr{U}$ one can construct a local coordinate system (w, x, y) such that

$$E_3 = \frac{\partial}{\partial y} \quad \text{on } \mathscr{U}_p. \tag{4}$$

Consider the orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ which is dual to $\{E_1, E_2, E_3\}$. Then the coordinate expression of the coframe $\{\omega^1, \omega^2, \omega^3\}$ in $\mathscr{U}_p$ must be of the form

$$\begin{aligned} \omega^1 &= A\,\mathrm{d}w + B\,\mathrm{d}x, \\ \omega^2 &= C\,\mathrm{d}w + D\,\mathrm{d}x, \\ \omega^3 &= \mathrm{d}y + G\,\mathrm{d}w + H\,\mathrm{d}x, \end{aligned} \tag{5}$$

where A, B, C, D, G, H are unknown functions to be determined.

Now, we shall compute the components ω^i_j of the connection form. These are determined by the standard formulas

$$\mathrm{d}\omega^i + \sum \omega^i_j \wedge \omega^j = 0, \quad \omega^i_j + \omega^j_i = 0, \quad i, j = 1, 2, 3. \tag{6}$$

We put

$$\omega^i_j = \sum_k a^i_{jk} \omega^k. \tag{7}$$

The components Ω^i_j of the curvature form are determined by the standard formula

$$\Omega^i_j = \mathrm{d}\omega^i_j + \sum \omega^i_k \wedge \omega^k_j. \tag{8}$$

From (2) we obtain at once

$$\begin{aligned} \mathrm{d}\omega^1_2 + \omega^1_3 \wedge \omega^3_2 &= \lambda_3 \omega^1 \wedge \omega^2, \\ \mathrm{d}\omega^1_3 + \omega^1_2 \wedge \omega^2_3 &= \lambda_2 \omega^1 \wedge \omega^3, \\ \mathrm{d}\omega^2_3 + \omega^2_1 \wedge \omega^1_3 &= \lambda_1 \omega^2 \wedge \omega^3. \end{aligned} \tag{9}$$

Differentiating (9) and substituting (9) and (6) in the new equations, we obtain

$$\begin{aligned} (\lambda_1 - \lambda_3)\omega^2 \wedge \omega^3 \wedge \omega^1_3 + (\lambda_3 - \lambda_2)\omega^1 \wedge \omega^3 \wedge \omega^2_3 = 0, \\ (\lambda_3 - \lambda_2)\omega^1 \wedge \omega^2 \wedge \omega^2_3 + (\lambda_2 - \lambda_1)\omega^2 \wedge \omega^3 \wedge \omega^1_2 = 0, \\ (\lambda_2 - \lambda_1)\omega^1 \wedge \omega^3 \wedge \omega^1_2 + (\lambda_1 - \lambda_3)\omega^1 \wedge \omega^2 \wedge \omega^1_3 = 0. \end{aligned} \tag{10}$$

Using the notation (7) we obtain, more explicitly,

$$\begin{aligned} (\lambda_1 - \lambda_3)a^1_{31} + (\lambda_3 - \lambda_2)(-a^2_{32}) = 0, \\ (\lambda_3 - \lambda_2)a^2_{33} + (\lambda_2 - \lambda_1)a^1_{21} = 0, \\ (\lambda_2 - \lambda_1)(-a^1_{22}) + (\lambda_1 - \lambda_3)a^1_{33} = 0. \end{aligned} \tag{11}$$

Putting

$$\alpha = \frac{\lambda_1 - \lambda_3}{\lambda_3 - \lambda_2} = \frac{\varrho_1 - \varrho_3}{\varrho_3 - \varrho_2} \tag{12}$$

(where obviously $\alpha \neq 0, -1$), we get (11) in the unified form

$$a_{32}^2 = \alpha a_{31}^1,\ a_{33}^2 = (\alpha+1) a_{21}^1,\ a_{33}^1 = -\Big(\frac{\alpha+1}{\alpha}\Big) a_{22}^1. \tag{13}$$

Now, we shall calculate the coefficients a_{jk}^i using only (5) and (6). First we introduce new functions $\mathcal{D}, \mathcal{E}, \mathcal{F}$ (where $\mathcal{D} \neq 0$) by

$$\mathcal{D} = AD - BC,\ \mathcal{E} = AH - BG,\ \mathcal{F} = CH - DG. \tag{14}$$

We also define a bracket of two functions f, g by

$$[f, g] = f'_y g - f g'_y. \tag{15}$$

Then we obtain, by a routine calculation,

$$a_{21}^1 = \frac{1}{\mathcal{D}}(GB'_y - HA'_y + A'_x - B'_w),\ a_{31}^1 = \frac{1}{\mathcal{D}}(DA'_y - CB'_y), \tag{16}$$

$$a_{22}^1 = \frac{1}{\mathcal{D}}(GD'_y - HC'_y + C'_x - D'_w),\ a_{32}^2 = \frac{1}{\mathcal{D}}(AD'_y - BC'_y), \tag{17}$$

$$a_{33}^1 = \frac{1}{\mathcal{D}}(DG'_y - CH'_y),\ a_{33}^2 = \frac{1}{\mathcal{D}}(AH'_y - BG'_y), \tag{18}$$

$$\begin{aligned} a_{23}^1 &= \frac{1}{2\mathcal{D}}\{[C, D] + [A, B] - [G, H] + (G'_x - H'_w)\}, \\ a_{31}^2 &= \frac{1}{2\mathcal{D}}\{[C, D] - [A, B] + [G, H] - (G'_x - H'_w)\}, \\ a_{32}^1 &= \frac{1}{2\mathcal{D}}\{[C, D] - [A, B] - [G, H] + (G'_x - H'_w)\}. \end{aligned} \tag{19}$$

(In [5], there is a sign misprint in the last formula.)

Due to (13), we have only six basic coefficient functions, namely

$$a_{31}^1,\ a_{21}^1,\ a_{22}^1,\ a_{23}^1,\ a_{31}^2,\ a_{32}^1.$$

For the sake of brevity we put

$$p = a_{23}^1,\ q = a_{31}^2,\ r = a_{32}^1,\ s = a_{22}^1,\ t = a_{21}^1,\ u = a_{31}^1. \tag{20}$$

Now, taking into account the formulas (13), we can rewrite (16)–(19) as a system of partial differential equations

$$
\begin{aligned}
A'_y &= Au + C(r-p),\\
B'_y &= Bu + D(r-p),\\
C'_y &= A(p+q) + \alpha Cu,\\
D'_y &= B(p+q) + \alpha Du,\\
G'_y &= (\alpha+1)Ct - \frac{\alpha+1}{\alpha}As,\\
H'_y &= (\alpha+1)Dv - \frac{\alpha+1}{\alpha}Bs;
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
A'_x - B'_w &= \mathcal{D}t + \mathcal{E}u + \mathcal{F}(r-p),\\
C'_x - D'_w &= \mathcal{D}s + \mathcal{E}(p+q) + \alpha\mathcal{F}u,\\
G'_x - H'_w &= \mathcal{D}(r-q) - \frac{\alpha+1}{\alpha}\mathcal{E}s + (\alpha+1)\mathcal{F}t.
\end{aligned} \tag{22}
$$

Next, we express explicitly the conditions (9) for the curvature components. After lengthy but routine calculations we obtain the following system of partial differential equations (which is again re-arranged in two parts and in which the formulas (13) are used):

$$
\begin{aligned}
At'_y + Cs'_y + Gp'_y - p'_w - AS - CT = 0,\\
Bt'_y + Ds'_y + Hp'_y - p'_x - BS - CT = 0,\\
Au'_y + Cr'_y - \frac{\alpha+1}{\alpha}Gs'_y + \frac{\alpha+1}{\alpha}s'_w - A(N-m) - CP = 0,\\
Bu'_y + Dr'_y - \frac{\alpha+1}{\alpha}Hs'_y + \frac{\alpha+1}{\alpha}s'_x - B(N-m) - DP = 0,\\
Aq'_y + \alpha Cu'_y + (\alpha+1)Gt'_y - (\alpha+1)t'_w - AK - C(L-\ l) = 0,\\
Bq'_y + \alpha Du'_y + (\alpha+1)Ht'_y - (\alpha+1)t'_x - BK - D(L-\ l) = 0;
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
At'_x - Bt'_w + Cs'_x - Ds'_w + Gp'_x - Hp'_w - \mathcal{D}(R-n) - \mathcal{E}S - \mathcal{F}T = 0,\\
Au'_x - Bu'_w + Cr'_x - Dr'_w - \frac{\alpha+1}{\alpha}Gs'_x + \frac{\alpha+1}{\alpha}Hs'_w\\
-\mathcal{D}M - \mathcal{E}(N-m) - \mathcal{F}P = 0,\\
Aq'_x - Bq'_w + \alpha Cu'_x - \alpha Du'_w + (\alpha+1)Gt'_x - (\alpha+1)Ht'_w\\
-\mathcal{D}J - \mathcal{E}K - \mathcal{F}(L-l) = 0.
\end{aligned} \tag{24}
$$

Here nine auxiliary functions are defined by

$$
\begin{aligned}
J &= \alpha tq - (\alpha-1)su - (\alpha+2)tr,\\
K &= \tfrac{(\alpha+1)(\alpha+2)}{\alpha} ts - (\alpha+1)uq - (\alpha-1)up,\\
L &= \tfrac{\alpha+1}{\alpha} s^2 - (\alpha+1)^2 t^2 - \alpha^2 u^2 + pq - rq + rp,\\
M &= \tfrac{1}{\alpha} sr + (\alpha-1)tu - \tfrac{2\alpha+1}{\alpha} sq,\\
N &= (\alpha+1)t^2 - u^2 - \tfrac{(\alpha+1)^2}{\alpha^2} s^2 - pq - pr - qr,\\
P &= (1-\alpha)pu - (\alpha+1)ru + \tfrac{(2\alpha+1)(\alpha+1)}{\alpha} ts,\\
R &= -t^2 - s^2 - \alpha u^2 + pq + qr - pr,\\
S &= \frac{1}{\alpha} sp - (\alpha+2)tu - \tfrac{2\alpha+1}{\alpha} sq,\\
T &= -\alpha tp - (\alpha+2)tr - (2\alpha+1)su
\end{aligned}
\tag{25}
$$

and three *constants* l, m, n are defined by

$$l = \lambda_1,\ m = \lambda_2,\ n = \lambda_3. \tag{26}$$

In the new notation, (12) takes on the form

$$\alpha = (l-n)/(n-m). \tag{27}$$

3. A special classification theorem

Let now E_γ $(\gamma = 1, 2, 3)$ be one of the vector fields consisting of unit Ricci eigenvectors and consider the following geometrical conditions:

(G1) The connection coefficients $a^i_{jk} = g(\nabla_{E_k} E_j, E_i)$ are constant along the trajectories of E_γ, i.e. $E_\gamma(a^i_{jk}) = 0$.

(G2) The local group of local diffeomorphisms defined by E_γ is volume-preserving, i.e. $\operatorname{div} E_\gamma = 0$.

Here we can assume, without loss of generality, that $E_\gamma = E_3$. This simplifies the calculations according to (4). From the condition (G1) we get at once

$$a^i_{jk} = a^i_{jk}(w, x) \quad (1 \leqslant i, j, k \leqslant 3), \tag{28}$$

i.e., all functions (20) depend only on w and x. Then the system (23) of PDE is reduced to

$$
\begin{aligned}
p'_w + AS + CT &= 0,\\
p'_x + BS + CT &= 0,\\
\tfrac{\alpha+1}{\alpha} s'_w - A(N-m) - CP &= 0,\\
\tfrac{\alpha+1}{\alpha} s'_x - B(N-m) - DP &= 0,\\
(\alpha+1)t'_w + AK + C(L-\ l) &= 0,\\
(\alpha+1)t'_x + BK + D(L-\ l) &= 0.
\end{aligned} \tag{29}
$$

Using (13) and the definition of the functions a^i_{jk} we get

$$
\operatorname{div} E_3 = \sum_j a^j_{3j} = (1+\alpha)a^1_{31}, \tag{30}
$$

and hence the condition (G2) means

$$
u = a^1_{31} = 0 \quad \text{on} \quad \mathscr{U}_p. \tag{31}
$$

The following theorem was been proved in [6] (here we keep the original notation for the curvatures λ_i):

Theorem 3.1. *Let $(\mathscr{M}, g)$ be a C^∞-Riemannian manifold of dimension three with distinct constant Ricci eigenavalues. If $(\mathscr{M}, g)$ satisfies the conditions* (G1) *and* (G2) *(with $\gamma = 3$) in a neighbourhood of each point $p \in \mathscr{M}$, then there is a dense open subset $\mathscr{U} \subset \mathscr{M}$ such that, for some neighbourhood $\mathscr{V}_q$ of any point $q \in \mathscr{U}$, one of the following three cases* **(i)–(iii)** *occurs:*

(i) *$(\mathscr{M}, g)$ restricted to $\mathscr{V}_q$ is locally isometric to a 3-dimensional Lie group with a left-invariant metric.*

(ii) *$(\mathscr{M}, g)$ restricted to $\mathscr{V}_q$ is locally isometric to a generalized Yamato space. This means that the adapted orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ is given with respect to an adapted system (w, x, y) of local coordinates by the formulas* (5) *and*

$$
\begin{aligned}
A &= C(\varphi_3 - \varphi_1)y + A_0,\\
B &= D(\varphi_3 - \varphi_1)y + B_0,\\
C &= \frac{(\varphi_1)'_w}{(\alpha\varphi_1 + (\alpha+2)\varphi_3)\varphi_2},\\
D &= \frac{(\varphi_1)'_x}{(\alpha\varphi_1 + (\alpha+2)\varphi_3)\varphi_2},\\
G &= (\alpha+1)C\varphi_2 y + G_0,\\
H &= (\alpha+1)D\varphi_2 y + H_0,
\end{aligned}
$$

where φ_1 is an arbitrary non-constant smooth function on $\mathbb{R}^2[w,x]$, $\varphi_1(w,x) \neq 0$, and the functions φ_2, φ_3 are defined by

$$(\alpha+1)(\varphi_2)^2 + (\varphi_1)^2 = \lambda_2,\ -\alpha\varphi_1\varphi_3 = (\alpha+1)\lambda_3 + \lambda_2,\ \varphi_2(w,x) > 0.$$

(Mind a misprint in [6] and [1].)

Further, A_0, B_0, G_0 and H_0 are any smooth functions of w and x satisfying the partial differential equations

$$\begin{aligned}(A_0)'_x - (B_0)'_w &= (DA_0 - CB_0)\varphi_2 + (DG_0 - CH_0)(\varphi_1 - \varphi_3),\\ (G_0)'_x - (H_0)'_w &= (DA_0 - CB_0)(\varphi_1 + \varphi_3) - (\alpha+1)(DG_0 - CH_0)\varphi_2.\end{aligned}$$

(iii) *$(\mathscr{M}, g)$ restricted to $\mathscr{V}_q$ has, in an adapted system of local coordinates, the following form: an orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ is again defined by (5) and*

$$\begin{aligned}&A = A(w,x),\ B = B(w,x),\ C = \varphi Ay + C_0,\ D = \varphi By + D_0,\\ &G = \tfrac{1}{2}(\alpha+1)\sqrt{-\lambda_3}\varphi Ay^2 + (\alpha+1)\sqrt{-\lambda_3}C_0 y + G_0,\\ &H = \tfrac{1}{2}(\alpha+1)\sqrt{-\lambda_3}\varphi By^2 + (\alpha+1)\sqrt{-\lambda_3}D_0 y + H_0,\end{aligned}$$

where A, B, C_0, D_0, G_0, H_0, φ are arbitrary smooth functions of w and x satisfying the system of quasilinear partial differential equations

$$\begin{aligned}A'_x - B'_w &= \sqrt{-\lambda_3}(AD_0 - BC_0),\\ (C_0)'_x - (D_0)'_w &= \varphi(AH_0 - BG_0),\\ (G_0)'_x - (H_0)'_w &= -\varphi(AD_0 - BC_0) + (\alpha+1)\sqrt{-\lambda_3}(C_0H_0 - D_0G_0),\\ A\varphi'_x - B\varphi'_w &= \alpha\sqrt{-\lambda_3}\varphi(AD_0 - BC_0).\end{aligned}$$

Here $\varphi = \varphi(w,x)$ is a non-constant function, and the equality $\lambda_1\lambda_3 = (\lambda_2)^2$ and the inequalities $\lambda_1 < 0$ and $\lambda_3 < 0$ must be satisfied.

The spaces in the items **(i)**–**(iii)** *are never locally isometric to each other; in particular, the cases* **(ii)** *and* **(iii)** *are never locally homogeneous.*

Remark 3.2. It follows from the considerations in [6] that the *converse* of Theorem 3.1 also holds, i.e., all spaces of types **(i)**, **(ii)**, **(iii)** satisfy both conditions (G1) and (G2) for some $\gamma \in \{1,2,3\}$. In particular, we have (see [6], Remark 1.3 and 1.2):

Proposition 3.3. *A locally homogeneous Riemannian 3-manifold with distinct Ricci eigenvalues satisfies the condition* (G2) *for some $\gamma \in \{1,2,3\}$. It belongs to type* **(i)** *of Theorem* 3.1.

Proposition 3.4. *A Riemannian 3-manifold $(\mathscr{M}, g)$ with distinct constant Ricci eigenvalues is locally homogeneous if and only if all connection coefficients a^i_{jk} are constant (i.e.,* (G1) *holds simultaneously for* $\gamma = 1, 2, 3$*).*

4. The main result

In the rest of this paper we will prove the following

Theorem 4.1. *Let $(\mathscr{M}, g)$ be a Riemannian 3-manifold with distinct constant Ricci eigenvalues satisfying the condition* (G1) *for* $\gamma = 3$. *Then $(\mathscr{M}, g)$ satisfies both conditions* (G1) *and* (G2) *for some* $\gamma \in \{1, 2, 3\}$.

Proof. The proof will be decomposed in a number of steps which are presented in the subsequent sections. We always suppose that $(\mathscr{M}, g)$ is a 3-manifold with distinct constant Ricci eigenvalues and satisfying (G1) for $\gamma = 3$. Our investigation will be always local and, therefore, we will be using all formulas and notation from the previous sections. As in [6], we will limit ourselves to a dense open subset $\mathscr{U}$ of $\mathscr{M}$ with the following property: for each "basic function" involved, the value of such function at $p \in \mathscr{U}$ is either nonzero or the function vanishes in a neighbourhood of p. Because the number of "basic functions" involved in the whole procedure is finite, we see that the set $\mathscr{U}$ is indeed open and dense. By the continuity, it suffices to prove the property (G2) on $\mathscr{U}$. A typical argument proceeds as follows: if (G2) is *not* satisfied for $\gamma = 3$, i.e., if $u \neq 0$ in some neighbourhood due to (31), then the space is locally homogeneous in this neighbourhood. Hence, according to Propositions 3.3 and 3.4, the conditions (G1) and (G2) are satisfied for $\gamma = 1$ or $\gamma = 2$. □

5. The classification of potential solutions

We start with

Proposition 5.1. *If* (G1) *holds on $(\mathscr{M}, g)$, then the functions A, B, C, D from* (5) *satisfy the same partial differential equation*

$$f''_{yy} - 2\omega f'_y + \mu f = 0 \tag{32}$$

where

$$\omega = \tfrac{1}{2}(1 + \alpha)u, \quad \mu = \alpha u^2 + (p - r)(p + q) \tag{33}$$

are coefficients not depending on y.

P r o o f. It follows by a routine calculation from the first four equations of (21). □

Proposition 5.2. *For the characteristic roots of the equation* (32), *we have the following cases:*

(i) *Elliptic case:* $\omega^2 - \mu < 0$. *Then each solution of* (32) *has the form*

$$f = \mathrm{e}^{\omega y}(f_1 \cos(\varphi y) + f_2 \sin(\varphi y)) \tag{34}$$

where f_1, f_2 *are functions of* w, x *only and*

$$\varphi = \sqrt{\mu - \omega^2}. \tag{35}$$

(ii) *General hyperbolic case:* $\omega^2 - \mu > 0,\ \mu \neq 0$. *Then each solution of* (32) *has the form*

$$f = f_1 \mathrm{e}^{\omega_1 y} + f_2 \mathrm{e}^{\omega_2 y} \tag{36}$$

where $\omega_1 \neq \omega_2$ *and* $\omega_1 \neq 0,\ \omega_2 \neq 0$ *depend only on* w *and* x.

(iii) *Special hyperbolic case:* $\omega \neq 0, \mu = 0$. *Then each solution of* (32) *has the form*

$$f = f_1 + f_2 \mathrm{e}^{2\omega y}. \tag{37}$$

(iv) *Parabolic case:* $\omega^2 - \mu = 0, \omega \neq 0$. *Then each solution of* (32) *has the form*

$$f = \mathrm{e}^{\omega y}(f_1 + f_2 y). \tag{38}$$

(v) *Planar case:* $\omega = \mu = 0$. *Then each solution of* (32) *has the form*

$$f = f_1 + f_2 y. \tag{39}$$

P r o o f. Obvious. □

Now we see from (33) that the planar case implies $u = 0$, i.e., $\operatorname{div} E_3 = 0$. This case is settled in Theorem 3.1. Similarly, we can exclude the equality $\omega = 0$ (i.e., $u = 0$) in the cases (i) and (ii). Now, we shall prove the following

Proposition 5.3. *In the cases* (i), (ii) *and* (iv), *if* $\omega \neq 0$ *and the corresponding metric* g *exists, it must be locally homogeneous.*

P r o o f. Substitute the corresponding expressions (34) or (36) or (38) for the functions A, B, C, D in the system of equations (29). An obvious argument using the inequality $\mathcal{D} = AD - BC \neq 0$ implies

$$p'_w = p'_x = s'_w = s'_x = t'_w = t'_x = 0, \tag{40}$$

$$K = L - l = N - m = P = S = T = 0. \tag{41}$$

From (40) it follows that p, s and t are constants. Then the first equation (24) gives, in addition,

$$R - n = 0. \tag{42}$$

Writing the algebraic equations (41), (42) explicitly according to the definition formulas (25), we obtain (substituting here for the moment $X := q$, $Y := r$ and $Z := u$)

$$\begin{aligned}
\tfrac{(\alpha+1)(\alpha+2)}{\alpha}ts - (\alpha-1)pZ - (\alpha+1)XZ &= 0,\\
-l + \tfrac{\alpha+1}{\alpha}s^2 - (\alpha+1)^2t^2 + p(X+Y) - XY - \alpha^2Z^2 &= 0,\\
-m - \tfrac{(\alpha+1)^2}{\alpha^2}s^2 + (\alpha+1)t^2 - p(X+Y) - XY - Z^2 &= 0,\\
\tfrac{(2\alpha+1)(\alpha+1)}{\alpha}ts + (1-\alpha)pZ - (\alpha+1)YZ &= 0,\\
-n - s^2 - t^2 + p(X-Y) + XY - \alpha Z^2 &= 0,\\
\tfrac{1}{\alpha}ps - \tfrac{(2\alpha+1)}{\alpha}sX - (\alpha+2)tZ &= 0,\\
-\alpha pt - (\alpha+2)tY - (2\alpha+1)sZ &= 0.
\end{aligned} \tag{43}$$

Our next goal is to get explicit expressions for X, Y, Z through the other quantities which are already constants. We will denote the corresponding equations in (43) by (E1)–(E7) whenever it will be convenient. Now, we proceed as follows: express Z^2 from the equation (E3)+(E5), express XY from the equation α(E3)$-$(E5) and substitute for Z^2 and XY into (E2). We obtain

$$2\alpha pX + 2\alpha^2pY + 2(\alpha^2+\alpha+1)(s^2-\alpha t^2) + \alpha^2(m+n) - \alpha(n+l) = 0. \tag{44}$$

Now, (E6), (E7) and (44) is a system of *linear* equations for X, Y and Z whose determinant is, up to a nonzero factor, equal to

$$p((\alpha+2)^2t^2 + (2\alpha+1)^2s^2). \tag{45}$$

If this determinant is nonzero, then X, Y, Z can be expressed explicitly as rational functions of p, s, t, m, l, n, α and hence they are constant. According to Proposition 3.4, the corresponding space (if it exists) is locally homogeneous in a neighbourhood and we are finished.

Let now $p = 0$ in (45). Then the system of equations (43) can be simplified to the form

$$
(46)\qquad
\begin{aligned}
\tfrac{(\alpha+1)(\alpha+2)}{\alpha}ts - (\alpha+1)XZ &= 0,\\
-l + \tfrac{\alpha+1}{\alpha}s^2 - (\alpha+1)^2t^2 - XY - \alpha^2Z^2 &= 0,\\
-m - \tfrac{(\alpha+1)^2}{\alpha^2}s^2 + (\alpha+1)t^2 - XY - Z^2 &= 0,\\
\tfrac{(2\alpha+1)(\alpha+1)}{\alpha}ts - (\alpha+1)YZ &= 0,\\
-n - s^2 - t^2 + XY - \alpha Z^2 &= 0,\\
-\tfrac{(2\alpha+1)}{\alpha}sX - (\alpha+2)tZ &= 0,\\
-(\alpha+2)tY - (2\alpha+1)sZ &= 0.
\end{aligned}
$$

If $(\alpha+2)t \neq 0$, then the equation (E6) shows that $(2\alpha+1)s \neq 0$ (otherwise $Z = 0$, a contradiction). Then Z and Y can be expressed from (E6) and (E7) as nonzero constant multiples of X. Substituting in (E4) for YZ we see that X is a constant and hence Y and Z are also constants. If $(2\alpha+1)s \neq 0$, then the equation (E7) shows that $(\alpha+2)t \neq 0$ and we get the same conclusion.

Suppose now $(\alpha+2)t = (2\alpha+1)s = 0$ and p arbitrary. Then the equations (E1) and (E4) from (43) can be written in the form

$$
(47)\qquad
\begin{aligned}
Z((\alpha+1)X + (\alpha-1)p) &= 0,\\
Z((\alpha+1)Y + (\alpha-1)p) &= 0.
\end{aligned}
$$

Hence, because $Z \neq 0$, X and Y are constants. Then Z is also constant, as follows from (E3).

This obviously concludes the proof of Proposition 5.3. □

6. The special hyperbolic case

This is the only remaining (and most difficult) case. According to (37) we have

$$
(48)\qquad
\begin{aligned}
A &= A_1 + A_2\mathrm{e}^{2\omega y},\\
B &= B_1 + B_2\mathrm{e}^{2\omega y},\\
C &= C_1 + C_2\mathrm{e}^{2\omega y},\\
D &= D_1 + D_2\mathrm{e}^{2\omega y},
\end{aligned}
$$

where A_i, B_i, C_i, D_i are functions of w and x only. Substituting in (29), we obtain a system of 12 equations which are divided into two series:

$$\begin{aligned} A_2S + C_2T &= 0, \\ B_2S + D_2T &= 0, \\ A_2(N-m) + C_2P &= 0, \\ B_2(N-m) + D_2P &= 0, \\ A_2K + C_2(L-l) &= 0, \\ B_2K + D_2(L-l) &= 0, \end{aligned} \tag{49}$$

$$\begin{aligned} p'_w + A_1S + C_1T &= 0, \\ p'_x + B_1S + D_1T &= 0, \\ -\tfrac{\alpha+1}{\alpha}s'_w + A_1(N-m) + C_1P &= 0, \\ -\tfrac{\alpha+1}{\alpha}s'_x + B_1(N-m) + D_1P &= 0, \\ (\alpha+1)t'_w + A_1K + C_1(L-l) &= 0, \\ (\alpha+1)t'_x + B_1K + D_1(L-l) &= 0. \end{aligned} \tag{50}$$

Now, substituting from (48) in the first four differential equations (21), we obtain the following algebraic conditions for the coefficients A_i, B_i, C_i, D_i:

$$\begin{aligned} uA_1 + (r-p)C_1 &= 0, \\ uB_1 + (r-p)D_1 &= 0, \\ (p+q)A_1 + \alpha uC_1 &= 0, \\ (p+q)B_1 + \alpha uD_1 &= 0, \end{aligned} \tag{51}$$

$$\begin{aligned} \alpha uA_2 + (p-r)C_2 &= 0, \\ \alpha uB_2 + (p-r)D_2 &= 0, \\ (p+q)A_2 - uC_2 &= 0, \\ (p+q)B_2 - uD_2 &= 0. \end{aligned} \tag{52}$$

These conditions are not linearly independent because, in the special hyperbolic case,

$$\mu = \alpha u^2 + (p-r)(p+q) = 0. \tag{53}$$

Hence $\omega \neq 0$ implies

$$u \neq 0,\ p-r \neq 0,\ p+q \neq 0. \tag{54}$$

Then the conditions (51), (52) are equivalent to the formulas

$$C_1 = \frac{uA_1}{p-r},\ D_1 = \frac{uB_1}{p-r},\ C_2 = \frac{\alpha u A_2}{r-p},\ D_2 = \frac{\alpha u B_2}{r-p}. \tag{55}$$

Hence we see that one can never have $A_1 = B_1 = 0$, or $A_2 = B_2 = 0$. Indeed, using (48) we obtain in each case $\mathcal{D} = AD - BC = 0$, which is a contradiction.

In particular, substituting from (55) into (49) we get

$$\begin{aligned} (r-p)S + \alpha u T &= 0, \\ (r-p)(N-m) + \alpha u P &= 0, \\ (r-p)K + \alpha u(L-l) &= 0. \end{aligned} \tag{56}$$

Using (56) and (55), we can rewrite (50) in the form

$$\begin{aligned} \alpha p'_w + (\alpha+1)SA_1 &= 0, \\ \alpha p'_x + (\alpha+1)SB_1 &= 0, \\ s'_w - (N-m)A_1 &= 0, \\ s'_x - (N-m)B_1 &= 0, \\ \alpha t'_w + KA_1 &= 0, \\ \alpha t'_x + KB_1 &= 0. \end{aligned} \tag{57}$$

Next, we integrate the last two equations of (21) using the expressions (48). We get, using also (55),

$$G = G_1 + \frac{\alpha+1}{\alpha} A_1 \left(\frac{\alpha u t}{p-r} - s \right) y - \frac{A_2}{\alpha u} \left(\frac{\alpha^2 u t}{p-r} + s \right) \mathrm{e}^{2\omega y}, \tag{58}$$

$$H = H_1 + \frac{\alpha+1}{\alpha} B_1 \left(\frac{\alpha u t}{p-r} - s \right) y - \frac{B_2}{\alpha u} \left(\frac{\alpha^2 u t}{p-r} + s \right) \mathrm{e}^{2\omega y}. \tag{59}$$

Here G_1 and H_1 are new functions of the variables w, x.

The equations (21) are thus completely solved by the formulas (48), (58), (59) together with (55). The functions A_1, A_2, B_1, B_2, G_1, H_1 remain arbitrary functions of two variables w, x with the only inequality $A_1B_2 - A_2B_1 \neq 0$ (see formula (61) below).

Introduce new determinant functions

$$U = A_1B_2 - A_2B_1,\ V = A_1H_1 - B_1G_1,\ W = A_2H_1 - B_2G_1. \tag{60}$$

Then we can calculate the determinant functions from (14) as follows:

$$\mathcal{D} = \frac{(\alpha+1)Uue^{2\omega y}}{r-p}, \tag{61}$$

$$\mathcal{E} = V + \left[W - \frac{(\alpha^2 ut + (p-r)s)U}{\alpha u(p-r)}\right]e^{2\omega y} + \frac{(\alpha+1)U}{\alpha(p-r)}\big((p-r)s - \alpha ut\big)ye^{2\omega y}, \tag{62}$$

$$\mathcal{F} = \frac{uV}{p-r} - \left[\frac{\alpha uW}{p-r} + \frac{\big(\alpha^2 ut + (p-r)s\big)U}{\alpha(p-r)^2}\right]e^{2\omega y} + \frac{(\alpha+1)u\big(\alpha ut - (p-r)s\big)U}{(p-r)^2}ye^{2\omega y}. \tag{63}$$

Now, let us substitute in the first equation (22) from (48), (58), (59), (60)–(63). We hence obtain three equations which are *independent of* y, namely

$$(A_1)'_x - (B_1)'_w = 0, \tag{64}$$

$$(A_2)'_x - (B_2)'_w + \frac{(\alpha+1)utU}{p-r} - (\alpha+1)uW = 0, \tag{65}$$

$$\alpha\big(A_2u'_x - B_2u'_w\big)(p-r) + (\alpha+1)u\big(\alpha tu - (p-r)s\big)U = 0. \tag{66}$$

Similarly, substitute in the second equation (22). We obtain only two additional equations, namely

$$(C_1)'_x - (D_1)'_w = 0, \tag{67}$$

$$(C_2)'_x - (D_2)'_w = (\alpha+1)(\alpha u^2 W + usU)/(r-p). \tag{68}$$

Now, (67), (55) and (64) imply

$$A_1\left(\frac{u}{p-r}\right)'_x - B_1\left(\frac{u}{p-r}\right)'_w = 0. \tag{69}$$

Further, (68), (55) and (65) imply

$$A_2\left(\frac{u}{p-r}\right)'_x - B_2\left(\frac{u}{p-r}\right)'_w = \frac{(\alpha+1)u\big(\alpha ut + (p-r)s\big)U}{\alpha(p-r)^2}. \tag{70}$$

Next, we shall need

Proposition 6.1. *For each function* $f \in \{p, q, r, s, t, u\}$ *we have*

$$A_1 f'_x - B_1 f'_w = 0. \tag{71}$$

Hence the functions p, q, r, s, t, u *are functionally dependent.*

P r o o f. From (57) we obtain

$$A_1 p'_x - B_1 p'_w = 0,\ A_1 s'_x - B_1 s'_w = 0,\ A_1 t'_x - B_1 t'_w = 0. \tag{72}$$

Moreover, the integrability condition for the last two pairs of equations (57) can be written, using also (64), in the form

$$A_1 N'_x - B_1 N'_w = 0,\ A_1 K'_x - B_1 K'_w = 0. \tag{73}$$

Now, let us introduce new functions

$$Y = p + q,\ Z = p - r,\ F = u/Z. \tag{74}$$

According to (54), all three functions are nonzero. From (53) we get $\alpha u^2 + YZ = 0$ and hence we can express

$$u = FZ,\ Y = -\alpha F^2 Z. \tag{75}$$

The six unknown functions p, q, r, s, t, u are now reduced to five unknown functions F, Z, p, s, t on the account of the identity (53).

The equation (69) now reads

$$A_1 F'_x - B_1 F'_w = 0. \tag{76}$$

If we write down the equations (73) explicitly (using the definition formulas (25)), we can still simplify them by the identities (72). Next, using the substitutions (74), (75) and the identity (76), we are left with the following two equations:

$$\begin{aligned} \big(\alpha p - (\alpha+1)Z\big)\big(A_1 Z'_x - B_1 Z'_w\big) &= 0,\\ \big(p + \alpha(\alpha+1)ZF^2\big)\big(A_1 Z'_x - B_1 Z'_w\big) &= 0. \end{aligned} \tag{77}$$

An obvious linear combination gives

$$(\alpha+1)Z(1+\alpha^2F^2)\big(A_1 Z'_x - B_1 Z'_w\big) = 0 \tag{78}$$

and, because the coefficient is nonzero, we obtain

$$A_1 Z'_x - B_1 Z'_w = 0. \tag{79}$$

Proposition 6.1 now follows from (72), (76), (79) and the transformation formulas (74), (75). □

Our next goal is to express explicitly the derivatives of the basic functions p, q, r, s, t, u. We will again take advantage of the transformation formulas (74), (75). First, using (70), (76) and Cramer's rule, we obtain

$$\begin{aligned} F'_w &= -\tfrac{\alpha+1}{\alpha}(\alpha Ft+s)FA_1, \\ F'_x &= -\tfrac{\alpha+1}{\alpha}(\alpha Ft+s)FB_1. \end{aligned} \tag{80}$$

Next, the equation (66) can be written in the form

$$\alpha Z(A_2u'_x - B_2u'_w) + (\alpha+1)EUu = 0, \tag{81}$$

where

$$E = \alpha tu - Zs. \tag{82}$$

Substituting $u = FZ$ into (81) and then using (80), we get easily

$$A_2Z'_x - B_2Z'_w = -2(\alpha+1)FZtU. \tag{83}$$

Using (79),(83) and Cramer's rule, we get

$$\begin{aligned} Z'_w &= 2(\alpha+1)FZtA_1, \\ Z'_x &= 2(\alpha+1)FZtB_1. \end{aligned} \tag{84}$$

Now, we summarize the formulas (57), (80), (84) and come back to our original functions by the transformation formulas (74), (75). After a routine calculation we finally get

$$\begin{aligned} &p'_w = -\tfrac{(\alpha+1)}{\alpha}SA_1,\ p'_x = -\tfrac{(\alpha+1)}{\alpha}SB_1, \\ &s'_w = (N-m)A_1,\ s'_x = (N-m)B_1, \\ &t'_w = -\tfrac{KA_1}{\alpha},\ t'_x = -\tfrac{KB_1}{\alpha}, \\ &q'_w = \tfrac{\alpha+1}{\alpha}\big(S-2w(p+q)\big)A_1,\ q'_x = \tfrac{\alpha+1}{\alpha}\big(S-2w(p+q)\big)B_1, \\ &r'_w = -\tfrac{\alpha+1}{\alpha}\big(S+2\alpha ut\big)A_1,\ r'_x = -\tfrac{\alpha+1}{\alpha}\big(S+2\alpha ut\big)B_1, \\ &u'_w = \tfrac{\alpha+1}{\alpha(p-r)}uEA_1,\ u'_x = \tfrac{\alpha+1}{\alpha(p-r)}uEB_1. \end{aligned} \tag{85}$$

Up to now, we have not investigated the differential equations (24) and the last equation (22). Substituting in the last equation (22) from (58), (59), (61)–(63), (64), (65) and (85), we obtain after a lengthy but routine computation the only new condition

$$\big(G_1\big)'_x - \big(H_1\big)'_w = -\frac{(\alpha+1)EV}{\alpha(p-r)}. \tag{86}$$

Substituting in the first equation of (24), we obtain a new algebraic equation

$$(\alpha+1)(R-n)+\alpha(N-m)+(L-l)=0. \tag{87}$$

A careful check shows that there are no new consequences of the system (24). We can summarize:

Proposition 6.2. *In the special hyperbolic case the basic system of partial differential equations* (21)–(24) *is equivalent to the formulas* (48), (58), (59), *the system of five algebraic equations* (53), (56), (87) *and the system of partial differential equations* (64), (65), (85) *and* (86) *for the functions of two variables.*

We shall express the algebraic equations (56) and (87) in the new variables F, Z, p, s, t (eliminating hence the equation (53)). We obtain four equations

$$\alpha\big((2\alpha+1)Fs-(\alpha+2)t\big)FZ+2p(\alpha^2tF+s)=0, \tag{88-1}$$

$$\begin{aligned}&\alpha^2(\alpha+1)^2F^2Z^2-2\alpha^3(\alpha+1)pF^2Z+\alpha^2(2\alpha+1)(\alpha+1)tsF\\&\qquad-\alpha^2p^2+(\alpha+1)^2s^2-\alpha^2(\alpha+1)t^2+\alpha^2m=0,\end{aligned} \tag{88-2}$$

$$\begin{aligned}&\alpha^2(\alpha+1)^2F^3Z^2+2\alpha(\alpha+1)pFZ\\&\qquad+\alpha\big(\alpha(\alpha+1)^2t^2-\alpha p^2-(\alpha+1)s^2-\alpha^2m+\alpha(\alpha+1)n\big)F\\&\qquad+(\alpha+1)(\alpha+2)ts=0,\end{aligned} \tag{88-3}$$

$$\alpha(\alpha+1)F^2Z^2+\alpha p(F^2-1)Z+(\alpha+1)(p^2+s^2+t^2+n)=0. \tag{88-4}$$

In the last two equations we have expressed the parameter l from the formula (27) through α, m and n.

Now, we continue with

Proposition 6.3. *If $p=$ constant on an open domain $\mathscr{V}\subset\mathscr{M}$ then the functions q, r, s, t, u are also constant in $\mathscr{V}$ and the corresponding metric g on $\mathscr{V}$ (if it exists) is locally homogeneous.*

Proof. From (56) and (57) we see that $S=T=0$ in $\mathscr{V}$. In the auxiliary variables F, Z, p, s, t it means that

$$\begin{aligned}&\big((2\alpha+1)Fs-(\alpha+2)t\big)FZ+\tfrac{2(\alpha+1)}{\alpha}sp=0,\\&\big((2\alpha+1)Fs-(\alpha+2)t\big)Z+2(\alpha+1)tp=0.\end{aligned} \tag{89}$$

This automatically satisfies the equation (88-1). Taking suitable linear combinations of the equations (89) we obtain

$$p(s-\alpha Ft)=0,\quad (s-\alpha Ft)\big((2\alpha+1)Fs-(\alpha+2)t\big)=0. \tag{90}$$

At each point $x \in \mathscr{V}$ we obtain either

(i) $s - \alpha F t = 0$,

or

(ii) $p = 0$ and $(2\alpha + 1)Fs - (\alpha + 2)t = 0$.

In the first case (i) we make the substitution $s = \alpha F t$ in the second equation of (89). We get

$$t\big[\big((2\alpha+1)\alpha F^2 - (\alpha+2)\big)Z + 2(\alpha+1)p\big] = 0. \tag{91}$$

Let first $t = 0$ and hence $s = 0$. We substitute $t = s = 0$ in the equations (88-2) and (88-3). The new equations are equivalent to

$$\begin{gathered}(\alpha+1)^2F^2Z^2 - 2\alpha(\alpha+1)pF^2Z - p^2 + m = 0,\\ \alpha(\alpha+1)^2F^2Z^2 + 2(\alpha+1)pZ + \alpha^2(n-m) - \alpha p^2 + \alpha n = 0.\end{gathered} \tag{92}$$

The resultant equation of (92) with respect to the variable F^2 gives

$$2(\alpha+1)^2pZ^2 + \big(\alpha(\alpha+1)^2(n-m) - 4\alpha(\alpha+1)p^2\big)Z + 2\alpha^2\big(p^2 + \alpha m - (\alpha+1)n\big)p = 0. \tag{93}$$

Because $n - m \neq 0$, either the coefficient of Z^2 or the coefficient of Z is nonzero and Z must be a constant. From the second equation of (92) we see that also F is a constant.

Let now $t \neq 0$, then (91) implies

$$(2\alpha+1)\alpha F^2 Z - (\alpha+2)Z + 2(\alpha+1)p = 0. \tag{94}$$

Making the substitution $s = \alpha F t$ into (88-2) and (88-3) we obtain

$$\begin{aligned}&(\alpha+1)^2F^2Z^2 - 2\alpha(\alpha+1)pF^2Z + (2\alpha^2+2\alpha+1)(\alpha+1)t^2F^2\\ &\qquad - (\alpha+1)t^2 - p^2 + m = 0,\end{aligned} \tag{95}$$

$$\begin{aligned}&\alpha(\alpha+1)^2F^2Z^2 + 2(\alpha+1)pZ - \alpha^2(\alpha+1)t^2F^2\\ &\qquad + (\alpha+1)(\alpha^2+2\alpha+2)t^2 + \alpha^2(n-m) + \alpha(n-p^2) = 0.\end{aligned} \tag{96}$$

Next we calculate the resultant R_1 of (94) and (95) with respect to the variable F^2 and the resultant R_2 of (94) and (96) with respect to the same variable. Finally, we calculate the resultant equation of the equations $R_1 = 0$ and $R_2 = 0$ with respect to the variable t^2. We obtain

$$\begin{aligned}&(\alpha-1)^2pZ^2 + \big[(2\alpha+1)(\alpha^2+\alpha+1)(n-m) + 2(\alpha-1)p^2\big]Z\\ &\qquad + (2\alpha+1)\big[2\alpha^2m - (2\alpha^2+2\alpha+1)n\big]p + p^3 = 0.\end{aligned} \tag{97}$$

We see again that either the coefficient of Z^2 or the coefficient of Z is nonzero unless $2\alpha + 1 = 0, p = 0$. In the last case the equations (89) show that $t = 0$, which is a contradiction. We see again that $Z = \text{const}$ and $F = \text{const}$ in the corresponding domain.

Now, let us discuss the second case (ii). First, suppose that $\alpha + 2 = 0$, which implies $s = 0$. Substituting $\alpha = -2$, $s = p = 0$ in (88-2) and (88-3), we obtain at once a contradiction with the inequality $n - m \neq 0$.

Let now $\alpha + 2 \neq 0$ and let us make the substitutions $p = 0$, $t = (2\alpha + 1)Fs/(\alpha + 2)$ into (88-2) and (88-3). A simple elimination shows again a contradiction with the inequality $n - m \neq 0$. We see that the functions F, Z, p, s, t are constants in $\mathscr{V}$ and hence, passing over to the original variables, and using Proposition 3.4, we conclude the proof of Proposition 6.3. $\square$

As a consequence we obtain

Proposition 6.4. *Let $(\mathscr{M}, g)$ be not locally homogeneous on any open subset. Then F, Z, s and t are functions of p on a dense open subset $\mathscr{U} \subset \mathscr{M}$.*

Proof. Due to Proposition 6.3, p is not constant on any open subset of $\mathscr{M}$. Then on a dense open subset $\mathscr{U}$ of $\mathscr{M}$ the derivatives p'_w and p'_x are not simultaneously zero. According to Proposition 6.1 and its proof, the functions F, Z, s and t must be on $\mathscr{U}$ smooth functions of p (see a classical theorem from analysis). $\square$

From (85) we now obtain the following formulas:

$$\begin{aligned} t'(p) &= \frac{K}{(\alpha+1)S}, \quad & s'(p) &= \frac{-\alpha(N-m)}{(\alpha+1)S}, \\ Z'(p) &= -2\frac{\alpha F Z t}{S}, \quad & F'(p) &= \frac{(\alpha t F + s)F}{S}. \end{aligned} \tag{98}$$

Here we know that $S \neq 0$ on $\mathscr{U}$ because otherwise (57) implies $p'_w = p'_x = 0$, a contradiction. $\square$

Now, we shall consider the equations (88-1)–(88-4) as algebraic equations in which p is an independent variable and F, Z, s, t are functions of p. After differentiation with respect to p we substitute for the derivatives from the formulas (98).

We obtain the following equations:

$$
\begin{aligned}
&\alpha^3(\alpha^2-1)F^4Z^3 \quad -2\alpha^2(\alpha^3F^2+1)pF^2Z^2\\
&+\big(\big[-\alpha^3(\alpha+1)(\alpha-1)t^2-\alpha^3(2\alpha+1)p^2\\
&\qquad+\alpha(2\alpha+1)\big(3(\alpha+1)^2s^2+\alpha^2m\big)\big]F^2\\
&\qquad-2\alpha s(\alpha+1)(\alpha+2)^2tF\big)Z\\
&-2\alpha^4(\alpha+1)pt^2F^2 \quad +4\alpha^2(\alpha+1)(2\alpha+1)pstF\\
&-2\big[\alpha^2p^2+\alpha^2(\alpha+1)t^2-3(\alpha+1)^2s^2-\alpha^2m\big]p=0,
\end{aligned}
\tag{99-1}
$$

$$
\begin{aligned}
&\alpha^2(\alpha+1)\big[\alpha^2(2\alpha+1)sF^2+\alpha^2tF-2(\alpha+1)^2s\big]F^2Z^2\\
&-2\alpha^3\big[\alpha^2tF^2-4(\alpha+1)^2sF+\alpha t\big]pFZ\\
&+\alpha^4(\alpha+1)(2\alpha+1)st^2F^2\\
&-\alpha^2\big[-\alpha^2(\alpha+1)(2\alpha-1)t^2-\alpha^2(2\alpha-1)p^2\\
&\qquad+(\alpha+1)^2(6\alpha+1)s^2+\alpha^2(2\alpha-1)m\big]tF\\
&-2(\alpha+1)\big[(\alpha+1)^2s^2+\alpha^2m-\alpha^2(2\alpha+3)t^2-3\alpha^2p^2\big]s=0,
\end{aligned}
\tag{99-2}
$$

$$
\begin{aligned}
&\alpha^2(\alpha+1)\big[\alpha^2(\alpha+1)tF^2-(3\alpha^2+6\alpha+4)sF+(\alpha+2)t\big]F^2Z^2\\
&+2\alpha\big[\alpha^2sF^2+\alpha^2(\alpha+1)tF-(3\alpha^2+6\alpha+4)s\big]pFZ\\
&-\alpha^3\big[-\alpha(\alpha+1)^2t^2+\alpha p^2+(\alpha+1)s^2+\alpha^2m-\alpha(\alpha+1)n\big]tF^2\\
&+\alpha\big[(\alpha+1)(3\alpha+2)s^2-\alpha^2(\alpha+1)(3\alpha+7)t^2+\alpha(4+3\alpha)p^2\\
&\qquad+\alpha^2(\alpha+2)m-\alpha^2(\alpha+1)n\big]sF\\
&+(\alpha+2)\big[\alpha^2(\alpha+1)t^2-(\alpha+1)(2\alpha+3)s^2+\alpha^2p^2-\alpha^2m\big]t=0,
\end{aligned}
\tag{99-3}
$$

$$
\begin{aligned}
&\alpha^2\big[(2\alpha+1)sF^3-\alpha(2\alpha+1)tF^2+(2\alpha+3)sF+(\alpha+2)t\big]FZ^2\\
&+2\alpha\big[(5\alpha+2)sF^2-3\alpha tF-(\alpha+1)s\big]pZ\\
&+2\big[(\alpha+1)^2s^2+(\alpha^2+4\alpha+2)p^2-(\alpha+1)^2(\alpha-2)t^2+\alpha^2m\big]s=0.
\end{aligned}
\tag{99-4}
$$

Now, repeating the resultant operation more times one can see that the formulas (99-1), (99-2) and (99-3) are *algebraic consequences* of (88-1), (88-2) and (88-3). On the other hand, the five equations (88-1)–(88-4) and (99-4) form an *algebraically independent system* (for any admissible choice of the parameters α, m and n). In other words, the functions F, Z, p, s, t are constant in the neighbourhood of any generic point of $(\mathscr{M}, g)$. According to Proposition 3.4, such a Riemannian manifold, if it exists, must be locally homogeneous. This concludes the proof of Theorem 4.1.

Let us mention that the final procedure yields rather long formulas (for which a computer assistance and the Maple software was used). We do not reproduce these formulas in full to keep this article in reasonable limits.

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Authors' addresses: *O. Kowalski, Z. Vlášek,* Faculty of Mathematics and Physics Charles University, Sokolovská 83 186 00 Praha, Czech Republic, e-mail: `kowalski@karlin.mff.cuni.cz`.