ON ONE-SIDED ESTIMATES FOR ROW-FINITE SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

Gerd Herzog, Karlsruhe

(Received July 11, 1997)

Abstract. We prove an existence and uniqueness theorem for row-finite initial value problems. The right-hand side of the differential equation is supposed to satisfy a one-sided matrix Lipschitz condition with a quasimonotone row-finite matrix which has an at most countable spectrum.

Keywords: Fréchet spaces, row-finite systems, one-sided estimates, row-finite matrices

MSC 2000: 34G20

1. Introduction

Let $\big(F, \|\cdot\|\big)$ be a real or complex Fréchet space where $\|\cdot\|\colon\ F \to \mathbb{R}^{\mathbb{N}}$ denotes a polynorm on F, i.e., $\big\{\|\cdot\|_j\colon\ j \in \mathbb{N}\big\}$ is a separating family of seminorms inducing the Fréchet space topology of F. Especially, for a sequence $E = \big((E_j, |\cdot|_j)\big)_{j=1}^{\infty}$ of Banach spaces we consider the Fréchet space $\big(F_E, \|\cdot\|\big)$ with $F_E = \prod\limits_{j=1}^{\infty} E_j$ and $\big\|(x_i)_{i=1}^{\infty}\big\|_j = \big|x_j\big|_j$, $j \in \mathbb{N}$. For a continuous function $f\colon\ [0,T] \times F \to F$ and $u_0 \in F$ we consider the initial value problem

$$u'(t) = f\big(t, u(t)\big), \quad u(0) = u_0. \tag{1}$$

Now let $A = (a_{ij})_{i,j\in\mathbb{N}}$ be a row-finite matrix, i.e., $a_{ij} \in \mathbb{C}$, i, $j \in \mathbb{N}$ and for every $i \in \mathbb{N}$ is $\sharp\{j \in \mathbb{N}\colon\ a_{ij} \neq 0\} < \infty$.

The row-finite matrices are exactly the continuous endomorphisms of the Fréchet space $\big(\mathbb{C}^{\mathbb{N}}, \|\cdot\|\big)$, $\|x\| = \big(|x_j|\big)_{j=1}^{\infty}$ and, according to a theorem of Ulm [20], the spectrum

$$\sigma(A) = \big\{\lambda \in \mathbb{C}\colon\ A - \lambda I \text{ is not invertible}\big\}$$

is either at most countable or has an at most countable complement $\mathbb{C} \setminus \sigma(A)$. For this and further properties of row-finite matrices we refer to [5], [6], [9], [10], [11], [12], [23].

In the sequel we will call a row-finite matrix $A = (a_{ij})_{i,j\in\mathbb{N}}$ *monotone* if $a_{ij} \in [0,\infty)$, i, $j \in \mathbb{N}$, and we will call A *quasimonotone* if $a_{ij} \in \mathbb{R}$, i, $j \in \mathbb{N}$ and $a_{ij} \geqslant 0$, $i \neq j$.

Now we define mappings $m_-\colon F \times F \to \mathbb{R}^{\mathbb{N}}$ and $m_+\colon F \times F \to \mathbb{R}^{\mathbb{N}}$ as

$$m_\pm[x,y] = \lim_{h\to 0\pm} \frac{\|x+hy\| - \|x\|}{h}.$$

The j-th coordinate of $m_\pm$ will be denoted by $m_{j\pm}$, $j \in \mathbb{N}$. The existence of these limits as well as the properties of the functions $m_\pm$ are consequences of the properties of convex functions on linear spaces (see [14], p. 36–46). Especially, for $x, y, z \in F$ we have that

$$\begin{aligned} \bigl|m_\pm[x,y]\bigr| &\leqslant \|y\|, \\ m_-[x,y] &\leqslant m_+[x,y], \\ m_\pm[x,y+z] &\leqslant m_\pm[x,y] + \|z\|. \end{aligned}$$

Here and in the sequel inequalities between elements of $\mathbb{R}^{\mathbb{N}}$ are intended componentwise. Now, if $I \subseteq \mathbb{R}$ is an interval and $u\colon I \to F$ is differentiable from the left-hand side or from the right-hand side at $t_0 \in I$, then $\|u\|\colon I \to \mathbb{R}^{\mathbb{N}}$ has the same property and

$$\begin{aligned} \|u\|'_-(t_0) &= m_-\bigl[u(t_0), u'_-(t_0)\bigr], \\ \|u\|'_+(t_0) &= m_+\bigl[u(t_0), u'_+(t_0)\bigr], \end{aligned}$$

respectively.

Now let $f\colon [0,T] \times F \to F$ be a continuous function. If there is a quasimonotone row-finite matrix L with $m_\pm[x-y, f(t,x)-f(t,y)] \leqslant L\|x-y\|$, $(t,x), (t,y) \in [0,T]\times F$ we will say that f is $L_\pm$-*dissipative*.

According to a theorem of Lemmert [12] we have

Theorem 1. *If $f\colon [0,T] \times F \to F$ is L_+-dissipative with $\sigma(L)$ at most countable, then problem* (1) *has at most one solution.*

R e m a r k. The same assertion holds if f is L_--dissipative.

According to [8] we have

Theorem 2. *If $f\colon [0,T] \times F \to F$ is L_--dissipative with $\sigma(L)$ at most countable and if there exists $j_0 \in \mathbb{N}$ such that*

$$\bigl\{f(t,x)\colon (t,x) \in [0,T] \times F, \|x\|_j \leqslant c_j, j = 1,\ldots,j_0\bigr\}$$

is bounded for every $(c_1,\ldots,c_{j_0})\in[0,\infty)^{\mathbb{N}}$, *then problem* (1) *is uniquely solvable on* $[0,T]$ *and the solution depends continuously on the initial value.*

Remarks:

1. Both theorems fail without the presupposition that $\sigma(L)$ is at most countable (for examples see [4], [8], [11]).
2. For the assertion in Theorem 2, in case that L is a diagonal matrix, and for differential equations on closed subsets of F, see [16], [17].
3. According to an example in [8], there is a Fréchet space $\bigl(F,\|\cdot\|\bigr)$ and a linear continuous operator $f\colon F\to F$ with $m_+[x,f(x)]=0$, $x\in F$, for which the problem $u'(t)=f\bigl(u(t)\bigr)$, $u(0)=u_0$ is locally unsolvable for some $u_0\in F$. Hence in general (and unlike the case in Banach spaces) $L_\pm$-dissipativity does not imply local existence of solutions for problem (1).

We now consider a Fréchet space $\bigl(F_E,\|\cdot\|\bigr)$ which is the direct product of Banach spaces $\bigl(E_j,|\cdot|_j\bigr)$, $j\in\mathbb{N}$. Let $P_m\colon F_E\to F_E$ denote the projection $P_m\bigl((x_j)_{j=1}^\infty\bigr)=(x_1,\ldots x_m,0,0,\ldots)$, $m\in\mathbb{N}$, and let $f=(f_j)_{j=1}^\infty\colon[0,T]\times F_E\to F_E$ be continuous. We call the function f, or the system of differential equations in problem (1), *row-finite* if there is a mapping $\varphi\colon\mathbb{N}\to\mathbb{N}$ with

$$f_j(t,x)=f_j\bigl(t,P_{\varphi(j)}(x)\bigr),\quad (t,x)\in[0,T]\times F_E,\quad j\in\mathbb{N}.$$

Row-finite systems of differential equations are an important type of differential equations in Fréchet spaces with various applications, for example, in polymerization chemistry, in semidiscretization of partial differential equations, in the study of delay-equations, and in modelling birth-death processes (see e.g. [2], [3], [5], [22] and the references given there).

In the sequel we will prove the following theorems:

Theorem 3. *If* $f\colon[0,T]\times F_E\to F_E$ *is* L_-*-dissipative, then* f *is row-finite.*

Theorem 4. *If* $f\colon[0,T]\times F_E\to F_E$ *is* L_-*-dissipative with* $\sigma(L)$ *at most countable, and if* $f([0,T]\times M)$ *is bounded for every bounded set* $M\subset F_E$*, then problem* (1) *is uniquely solvable on* $[0,T]$ *and the solution depends continuously on the initial value.*

Remarks:

1. Theorem 4, as well as Theorem 2, can be considered to be generalizations of Martin's theorem on ordinary differential equations in Banach spaces (see [14], p. 227–237). For a survey on existence and uniqueness results for Cauchy problems in locally convex spaces we refer to [1] and [13].

2. The existence part of Theorem 4 is relevant as well if F_E is a Montel space, i.e., if $\dim E_j < \infty$, $j \in \mathbb{N}$. According to an application of Tychonov's fixed point theorem, initial value problems with bounded right-hand sides are always solvable in Montel spaces (for $F_E = \mathbb{R}^{\mathbb{N}}$, see [19]), but if f is unbounded, problem (1) can be locally unsolvable. Consider, for example, in $\mathbb{R}^{\mathbb{N}}$ the initial value problem $u_j'(t) = u_j^2(t) + j$, $u_j(0) = 0$, $j \in \mathbb{N}$ (cf. [1], [4]).
3. If F_E is a Montel space then $f\big([0,T] \times M\big)$ is bounded for every bounded set $M \subset F_E$ since f is continuous.
4. The proof of Theorem 4 will show that the solution of problem (1) can be approximated by the solution of the truncated system (cf. [2], Chapter 7, Galerkin approximation).

In case F_E is a Montel space we will get the following assertion as an easy consequence of Theorem 4.

Theorem 5. *Let F_E be a Montel space. Let $g\colon [0,T] \times F_E \to F_E$ be L_--dissipative with $\sigma(L)$ at most countable, and countinuous. Further let $h\colon [0,T] \times F_E \to F_E$ be continuous with $h\big([0,T] \times F_E\big)$ bounded, and let $f := g + h$. Then problem* (1) *is solvable on* $[0,T]$.

R e m a r k . Theorem 5 also holds for general spaces F_E if $g\big([0,T]\times M\big)$ is bounded for every bounded set $M \subseteq F_E$ and $h\big([0,T]\times F_E\big)$ is assumed to be relatively compact. This can be considered a generalization of a theorem of Volkmann [21] on ordinary differential equations in Banach spaces (cf. also [18]). For the techniques to prove this assertion see [4], [8] and [21].

We will finally give some applications of our theorems to stability of solutions of $u'(t) = f\big(t, u(t)\big)$, $t \in [0,\infty)$, and to solvability of the problem $u'(t) = f\big(t, u(t)\big)$, $u(0) = u(T)$. Moreover, we will combine Theorem 5 with a uniqueness condition for certain L_--dissipative right-hand sides where L is allowed to have an uncountable spectrum.

2. Proof of Theorem 3

Since $L = (l_{ij})_{i,j\in\mathbb{N}}$ is row-finite, there exists a function $\varphi\colon \mathbb{N} \to \mathbb{N}$ with $l_{ij} = 0$, $j > \varphi(i)$, $i \in \mathbb{N}$, and $\varphi(i) \geqslant i$, $i \in \mathbb{N}$. We claim that $f_j(t,x) = f_j\big(t, P_{\varphi(j)}(x)\big)$, $(t,x) \in [0,T] \times F_E$, $j \in \mathbb{N}$. Assume that this is not true for some $j_0 \in \mathbb{N}$. Then there exists $(t_0, x_0) \in [0,T] \times F_E$ with $a := f_{j_0}(t_0, x_0) - f_{j_0}\big(t_0, P_{\varphi(j_0)}(x_0)\big) \neq 0$. Since f is continuous we can manage $|a|_{j_0} < 1$. For any $h < 0$ and $(t,x), (t,y) \in [0,T] \times F_E$ we have

$$\frac{1}{h}\Big(\big|x_{j_0}-y_{j_0}+h(f_{j_0}(t,x)-f_{j_0}(t,y))\big|_{j_0}-\big|x_{j_0}-y_{j_0}\big|_{j_0}\Big)$$
$$\leqslant m_{j_0-}[x-y,f(t,x)-f(t,y)]$$
$$\leqslant l_{j_0 1}|x_1-y_1|_1+\ldots+l_{j_0\varphi(j_0)}|x_{\varphi(j_0)}-y_{\varphi(j_0)}|_{\varphi(j_0)},$$

which implies for $x_{j_0}\neq y_{j_0}$ and $h=-\big|x_{j_0}-y_{j_0}\big|_{j_0}$ that

$$\begin{aligned}1-\left|\frac{x_{j_0}-y_{j_0}}{|x_{j_0}-y_{j_0}|_{j_0}}-\big(f_{j_0}(t,x)-f_{j_0}(t,y)\big)\right|_{j_0}&\\ \leqslant l_{j_0 1}\big|x_1-y_1\big|_1+\ldots+l_{j_0\varphi(j_0)}\big|x_{\varphi(j_0)}-y_{\varphi(j_0)}\big|_{\varphi(j_0)}.&\end{aligned}\tag{2}$$

Now there is a sequence $\big(x_k\big)_{k=1}^{\infty}$ in F_E with $x_k\to x_0$ in F_E as $k\to\infty$ and with

$$\frac{x_{kj_0}-x_{0j_0}}{|x_{kj_0}-x_{0j_0}|_{j_0}}\to\frac{a}{|a|_{j_0}}$$

in E_{j_0} as $k\to\infty$. Since $\varphi(i)\geqslant i$, $i\in\mathbb{N}$, we get from (2) that

$$1-\left|\tfrac{a}{|a|_{j_0}}-a\right|_{j_0}=\lim_{k\to\infty}\left(1-\left|\tfrac{x_{kj_0}-x_{0j_0}}{|x_{kj_0}-x_{0j_0}|_{j_0}}-\big(f_{j_0}(t_0,x_k)-f_{j_0}(t_0,P_{\varphi(j_0)}(x_0))\big)\right|_{j_0}\right)\leqslant 0.$$

Since $|a|_{j_0}<1$, this yields that $|a|_{j_0}\leqslant 0$, which is a contradiction. □

R e m a r k. If in Theorem 3 the function f is assumed to be L_+-dissipative, then the proof is much easier.

3. Proof of Theorem 4

For the proof of Theorem 4 we need some preparations. For the first proposition, see the references in our introduction. The topological dual space of $\big(\mathbb{C}^{\mathbb{N}},\|\cdot\|\big)$ is the space $\mathbb{C}_{\mathbb{N}}$ of all finite complex sequences. We consider the duality $\langle x,y\rangle=\sum\limits_{j=1}^{\infty}x_jy_j$, $x\in\mathbb{C}^{\mathbb{N}}$, $y\in\mathbb{C}_{\mathbb{N}}$. The complex column-finite matrices are exactly the endomorphisms of $\mathbb{C}_{\mathbb{N}}$, and if A is a row-finite matrix the transposed matrix $A^{\top}$ is column-finite, and we have $\langle Ax,y\rangle=\langle x,A^{\top}y\rangle$, $x\in\mathbb{C}^{\mathbb{N}}$, $y\in\mathbb{C}_{\mathbb{N}}$. In the sequel, e_i always denotes the sequence $(\delta_{ij})_{j=1}^{\infty}$, $i\in\mathbb{N}$.

Proposition 6. *Let A be a row-finite matrix. The following assertions are equivalent.*

1. $\sigma(A)$ *is at most countable.*
2. $A^\top$ *is locally algebraic, i.e.,* $\{(A^\top)^k e_j\colon\ k \in \mathbb{N}_0\}$ *is linear dependent for every* $j \in \mathbb{N}$.
3. e^{tA} *exists,* $t \in \mathbb{R}$, *and it is a matrix function with continuous entries that has at most finitely many functions in every row which are not identically zero.*

Remark. If L is quasimonotone with $\sigma(L)$ at most countable, then e^{tL} is monotone, $t \in [0, \infty)$ (cf. [12]).

The next proposition shows that in the proof of Theorem 4 we can assume without loss of generality that L is monotone.

Proposition 7. *Let* $L = (l_{ij})_{i,j\in\mathbb{N}}$ *be a quasimonotone row-finite matrix with at most countable spectrum and let* $L_p = (\lambda_{ij})_{i,j\in\mathbb{N}}$ *denote the monotone matrix with* $\lambda_{ij} = l_{ij}$, $i \neq j$ *and* $\lambda_{ii} = \max\{l_{ii}, 0\}$. *Then* $\sigma(L_p)$ *is at most countable.*

Proof. Let $j \in \mathbb{N}$ be fixed and let U denote the linear hull of $\{(L^\top)^k e_j\colon\ k \in \mathbb{N}_0\}$ in $\mathbb{R}_\mathbb{N}$. By Proposition 1 we get $\dim U < \infty$. Now let $\gamma := \max\{|l_{ii}|\colon\ i \in \mathbb{N}\colon\ \exists y \in U\colon\ y_i \neq 0\}$. The linear hull of $\{(L^\top + \gamma I)^k e_j\colon\ k \in \mathbb{N}_0\}$ in $\mathbb{R}_\mathbb{N}$ equals U and therefore $0 \leqslant (L_p^\top)^k e_j \leqslant (L^\top + \gamma I)^k e_j$, $k \in \mathbb{N}_0$. Therefore $\{(L_p^\top)^k e_j\colon\ k \in \mathbb{N}_0\}$ is linear dependent (in $\mathbb{R}_\mathbb{N}$ as well as in $\mathbb{C}_\mathbb{N}$) for every $j \in \mathbb{N}$. Hence $\sigma(L_p)$ is at most countable, according to Proposition 1. $\square$

The next proposition is a consequence of a result due to Lemmert [12] on row-finite differential inequalities.

Proposition 8. *Let* L *be a quasimonotone row-finite matrix with at most countable spectrum,* $g \in C([a,b], \mathbb{R}^\mathbb{N})$, *and* $u\colon\ [a,b] \to \mathbb{R}^\mathbb{N}$ *is continuous and differentiable from the left-hand side with* $u(a) \leqslant w \in \mathbb{R}^\mathbb{N}$. *Then*

$$u'_-(t) \leqslant Lu(t) + g(t), \quad t \in (a,b]$$

implies

$$u(t) \leqslant \mathrm{e}^{(t-a)L} w + \int_a^t \mathrm{e}^{(t-\tau)L} g(\tau)\,\mathrm{d}\tau, \quad t \in [a,b].$$

Proof of Theorem 4. According to Proposition 2 we assume without loss of generality that L is monotone. For $k \in \mathbb{N}$ let $f_k\colon\ [0,T] \times F_E \to F_E$ be defined as $f_{kj} = f_j$, $j = 1, \dots, k$, and $f_{kj} = 0$, $j > k$. Since L is monotone we have

$$m_-[x-y, f_k(t,x) - f_k(t,y)] \leqslant L\|x-y\|,$$

(t,x), $(t,y) \in [0,T] \times F_E$, $k \in \mathbb{N}$. Since $f([0,T] \times M)$ is bounded for every bounded set $M \subset F_E$, Theorem 2 can be applied. According to Theorem 3, we have that f is row-finite. Let $\varphi\colon \mathbb{N} \to \mathbb{N}$ be such that $f_j(t,x) = f_j\bigl(t, P_{\varphi(j)}(x)\bigr)$, $(t,x) \in [0,T] \times F_E$, $j \in \mathbb{N}$. For fixed $k \in \mathbb{N}$ we can choose $j_0 = \max\limits_{i=1,\dots,k} \varphi(i)$. Then by Theorem 2 we have a unique solution $u_k\colon [0,T] \to F_E$ of the initial value problem $u'(t) = f_k\bigl(t,u(t)\bigr)$, $u(0) = u_0$. For $k, l \in \mathbb{N}$ we have

$$\begin{aligned}
\bigl\|u_k - u_l\bigr\|'_-(t) &= m_-\bigl[u_k(t) - u_l(t), u'_k(t) - u'_l(t)\bigr] \\
&= m_-\bigl[u_k(t) - u_l(t), f_k\bigl(t,u_k(t)\bigr) - f_k\bigl(t,u_l(t)\bigr) + f_k\bigl(t,u_l(t)\bigr) - f_l\bigl(t,u_l(t)\bigr)\bigr] \\
&\leqslant L\bigl\|u_k(t) - u_l(t)\bigr\| + \bigl\|f_k\bigl(t,u_l(t)\bigr) - f_l\bigl(t,u_l(t)\bigr)\bigr\|, \quad t \in (0,T].
\end{aligned}$$

By Proposition 3 we get

$$\bigl\|u_k(t) - u_l(t)\bigr\| \leqslant \int_0^t \mathrm{e}^{(t-\tau)L}\bigl\|f_k\bigl(\tau,u_l(\tau)\bigr) - f_l\bigl(\tau,u_l(\tau)\bigr)\bigr\|\,\mathrm{d}\tau, \quad t \in [0,T].$$

Now consider the Fréchet space

$$\bigl(C([0,T],F_E), \|\!|\cdot|\!\|\bigr), \quad \|\!|u|\!\| = \Bigl(\max_{t\in[0,T]}\bigl\|u(t)\bigr\|_j\Bigr)_{j=1}^{\infty}.$$

Since L is monotone we have

$$\bigl\|\!\bigl|u_k - u_l\bigr|\!\bigr\| \leqslant \mathrm{e}^{TL}\bigl\|\!\bigl|f_k\bigl(\cdot,u_l(\cdot)\bigr) - f_l\bigl(\cdot,u_l(\cdot)\bigr)\bigr|\!\bigr\|.$$

Moreover, $\bigl\|\!\bigl|f_k\bigl(\cdot,u_l(\cdot)\bigr) - f_l\bigl(\cdot,u_l(\cdot)\bigr)\bigr|\!\bigr\|_j = 0$, $j = 1,\dots,\min\{k,l\}$. Therefore $(u_k)_{k=1}^{\infty}$ is a Cauchy sequence in $C([0,T],F_E)$, and $u = \lim\limits_{k\to\infty} u_k$ is a solution of problem (1). Now let $u\colon [0,T] \to F_E$ and $v\colon [0,T] \to F_E$ fulfil $u'(t) = f(t,u(t))$, $u(0) = u_0$ and $v'(t) = f(t,v(t))$, $v(0) = v_0$, respectively. Then $\|u - v\|'_-(t) \leqslant L\|u(t) - v(t)\|$, $t \in (0,T]$, which implies $\|\!|u - v|\!\| \leqslant \mathrm{e}^{TL}\|u_0 - v_0\|$ by Proposition 3. Therefore the solution of problem (1) is unique and depends continuously on the initial value. $\square$

4. Proof of Theorem 5

According to Proposition 2, we assume without loss of generality that L is monotone. Since h is bounded there exists $b \in [0,\infty)^{\mathbb{N}}$ with $\bigl\|h(t,x)\bigr\| \leqslant b$, $(t,x) \in [0,T] \times F_E$. We consider the operator $Q\colon C\bigl([0,T],F_E\bigr) \to C\bigl([0,T],F_E\bigr)$, where $Q(v)$ is the solution of $u'(t) = g\bigl(t,u(t)\bigr) + h\bigl(t,v(t)\bigr)$, $u(0) = u_0$, which exists and is unique, according to Theorem 4. Here again $C\bigl([0,T],F_E\bigr)$ is endowed with the polynorm $\|\!|u|\!\| = \Bigl(\max\limits_{t\in[0,T]}\bigl\|u(t)\bigr\|_j\Bigr)_{j=1}^{\infty}$. Since L is monotone, according to Proposition 3, for v, $w \in C\bigl([0,T],F_E\bigr)$

$$\bigl\|\!\bigl|Q(v) - Q(w)\bigr|\!\bigr\| \leqslant \mathrm{e}^{TL}\bigl\|\!\bigl|h\bigl(\cdot,v(\cdot)\bigr) - h\bigl(\cdot,w(\cdot)\bigr)\bigr|\!\bigr\|,$$

which implies that Q is continuous and

$$|||Q(v)||| \leqslant |||Q(0)||| + 2\mathrm{e}^{TL}b.$$

Hence $Q\big(C([0,T],F_E)\big)$ is bounded. Since $g\big([0,T]\times M\big)$ is bounded for every bounded set $M\subset F_E$, we have that $\big\{\big(Q(v)\big)'\colon\ v\in C([0,T],F_E)\big\}$ is bounded in $C([0,T],F_E)$. Therefore $Q\big(C([0,T],F_E)\big)$ is equicontinuous. Altogether $Q(C([0,T],F_E))$ is relatively compact, since F_E is a Montel space, and, according to Tychonov's fixed point theorem, there is a solution of problem (1). □

5. Applications

Let L be a quasimonotone row-finite matrix with $\sigma(L)\subset\big\{z\in\mathbb{C}\colon\ \mathrm{Re}(z)<0\big\}$, which implies that $\sigma(L)$ is at most countable. Then $\sigma(\mathrm{e}^{tL})\subset\big\{z\in\mathbb{C}\colon\ |z|<1\big\}$, $t\in(0,\infty)$, and Theorem 4 together with Proposition 3 leads to the following assertions:

1. If $f\colon\ [0,\infty)\times F_E\to F_E$ is L_--dissipative, then $u'(t)=f\big(t,u(t)\big)$, $u(0)=u_0$ is solvable on $[0,\infty)$ for every $u_0\in F_E$, and $\big\|v(t)-w(t)\big\|\leqslant\mathrm{e}^{tL}\big\|v(0)-w(0)\big\|\to 0$ as $t\to\infty$ for any two solutions v, w of $u'(t)=f\big(t,u(t)\big)$.
2. If $f\colon\ [0,T]\times F_E\to F_E$ is L_--dissipative, then $\big\|v(T)-w(T)\big\|\leqslant\mathrm{e}^{TL}\big\|v(0)-w(0)\big\|$ for any two solutions v,w of $u'(t)=f\big(t,u(t)\big)$. Hence, according to Lemmert's fixed point theorem (see [12]), the problem $u'(t)=f\big(t,u(t)\big)$, $u(0)=u(T)$ has a unique solution.
3. As a consequence of 1. and 2. we get: If $f\colon\ [0,\infty)\times F_E\to F_E$ is L_--dissipative and if $f(t,x)=f(t+T,x)$, $(t,x)\in[0,\infty)\times F_E$, then $u'(t)=f\big(t,u(t)\big)$ has a unique T-periodic solution which is asymptotically stable.

Next, let F_E be a Montel space, let g,h satisfy the conditions in Theorem 5, and let $f:=g+h$ be S_- dissipative, where S is a quasimonotone row-finite matrix with arbitrary spectrum. Remember that g is L_--dissipative with $\sigma(L)$ at most countable. According to Theorem 5, problem (1) is solvable on $[0,T]$. From [7], Theorem 2, we get uniqueness conditions for the solution of problem (1) in this case (for other types of matrix Lipschitz conditions we refer to [15]). These uniqueness conditions in [7] are proved for L_+-dissipativity, but they hold for L_--dissipativity with analogous proofs. To this end let S_p denote the positive part of S as in Proposition 2 and let $D:=S_p-S$. Moreover, let $b\in[0,\infty)^{\mathbb{N}}$ such that $\big\|h(t,x)-h(t,y)\big\|\leqslant b$, (t,x), $(t,y)\in[0,T]\times F_E$. Then the solution of problem (1) is unique if one of the following conditions is satisfied:

1. There exist $\alpha\in[0,\infty)^{\mathbb{N}}$ and $\beta\geqslant 0$ with

$$(S_p)^n\int_0^T \mathrm{e}^{\tau L}b\,\mathrm{d}\tau\leqslant\beta^n n^n\alpha,\quad n\in\mathbb{N}.$$

2. The diagonal matrix D is invertible, and there exist $\alpha \in [0,\infty)^{\mathbb{N}}$ and $\beta \geqslant 0$ with

$$(D^{-1}S_p^2)^n \int_0^T \mathrm{e}^{\tau L} b \,\mathrm{d}\tau \leqslant \beta^n n^n \alpha, \quad n \in \mathbb{N}.$$

Example. We consider the Fréchet space $(\mathbb{R}^{\mathbb{N}}, \|\cdot\|)$ with $\|x\| = \big(|x_j|\big)_{j=1}^{\infty}$, and the following row-finite initial value problem:

$$u_j'(t) = -\gamma_j\gamma_{j+1}\sinh\big(u_j(t)\big) - u_j^{1/(2j-1)}(t) + \gamma_j \arctan\big(u_{j+1}(t)\big) + g_j(t),$$

$u_j(0) = u_{0j}$, $j \in \mathbb{N}$, with $\gamma_j \geqslant 1$, $g_j \in C\big([0,T],\mathbb{R}\big)$, $j \in \mathbb{N}$, and $u_0 \in \mathbb{R}^{\mathbb{N}}$. Setting $g(t,x) = -\gamma_j\gamma_{j+1}\sinh(x_j) - x_j^{1/(2j-1)}$, $h(t,x) = \gamma_j \arctan(x_{j+1}) + g_j(t)$, and $f := g + h$, we have that g is L_--dissipative and that f is S_--dissipative, with $L = \operatorname{diag}(-\gamma_j\gamma_{j+1})$ and

$$S = \begin{pmatrix} -\gamma_1\gamma_2 & \gamma_1 & 0 & 0 & 0 & \dots \\ 0 & -\gamma_2\gamma_3 & \gamma_2 & 0 & 0 & \dots \\ 0 & 0 & -\gamma_3\gamma_4 & \gamma_3 & 0 & \dots \\ 0 & 0 & 0 & -\gamma_4\gamma_5 & \gamma_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

We have that $\sigma(L) = \big\{-\gamma_j\,\gamma_{j+1} \colon j \in \mathbb{N}\big\}$, and $\sigma(S) = \mathbb{C}$ (cf. [10]). Moreover, $\big\|h(t,x) - h(t,y)\big\| \leqslant \pi(\gamma_j)_{j=1}^{\infty} =: b$. We have

$$\big(D^{-1}S_p^2\big)^n \int_0^T \mathrm{e}^{\tau L} b \,\mathrm{d}\tau \leqslant \big(\pi\big)_{j=1}^{\infty}, \quad n \in \mathbb{N},$$

and therefore our problem is uniquely solvable on $[0,T]$. Note that a fast to $-\infty$ tending diagonal of S is helpful for uniqueness. The initial value problem $u_j'(t) = \gamma_j \arctan\big(u_{j+1}(t)\big)$, $u_j(0) = u_{0j}$, $j \in \mathbb{N}$, can have an infinite number of solutions, if $(\gamma_j)_{j=1}^{\infty}$ is growing fast (see [7]).

Acknowledgement. The author wishes to express his sincere gratitude to Prof. Roland Lemmert for many helpful remarks to improve this paper.

References

[1] *Bogachev, V. I.*: Deterministic and stochastic differential equations in infinite dimensional spaces. Acta Appl. Math. *40* (1995), 25–93.

[2] *Deimling, K.*: Ordinary Differential Equations in Banach Spaces. Lecture Notes in Mathematics 596, Springer, 1977.

[3] *Feldstein, A., Iserles, A., Levin, D.*: Embedding of delay equations into an infinite-dimensional ODE system. J. Differential Equations *117* (1995), 127–150.

[4] *Herzog, G.*: Über gewöhnliche Differentialgleichungen in Fréchetträumen. Dissertation, Univ. Karlsruhe, 1992.

[5] *Herzog, G.*: On ordinary linear differential equations in $\mathbb{C}^J$. Demonstratio Math. *28* (1995), 383–398.

[6] *Herzog, G.*: On linear time-dependent row-finite systems of differential equations. Travaux Math. Sem. Math. Luxemb. *8* (1996), 167–176.

[7] *Herzog, G.*: On existence and uniqueness conditions for ordinary differential equations in Fréchet spaces. Studia Sci. Math. Hungar. *23* (1996), 367–375.

[8] *Herzog, G.*: One-sided estimates for ordinary differential equations in Fréchet spaces. Ricerche Mat. *46* (1997), 479–487.

[9] *Kaplansky, I.*: Infinite Abelian Groups. The University of Michigan Press, 4^{th} printing, 1962.

[10] *Körber, K. H.*: Das Spektrum zeilenfiniter Matrizen. Math. Ann. *181* (1969), 8–34.

[11] *Lemmert, R., Weckbach, Ä.*: Charakterisierungen zeilenendlicher Matrizen mit abzählbarem Spektrum. Math. Z. *188* (1984), 119–124.

[12] *Lemmert, R.*: On ordinary differential equations in locally convex spaces. Nonlinear Analysis *10* (1986), 1385–1390.

[13] *Lobanov, S. G., Smolyanov, O. G.*: Ordinary differential equations in locally convex spaces. Russ. Math. Surv. *49* (1994), 97–175.

[14] *Martin, R. H.*: Nonlinear Operators and Differential Equations in Banach Spaces. Robert E. Krieger Publishing Company, Malabar, Florida, 1987.

[15] *Moszyński, K., Pokrzywa, A.*: Sur les systèmes infinis d'équations différentielles ordinaires dans certains espaces de Fréchet. Dissertationes Math. *115* (1974), 1–37.

[16] *Polewczak, J.*: Ordinary differential equations on closed subsets of Fréchet spaces with applications to fixed point theorems. Proc. Amer. Math. Soc. *107* (1989), 1005–1012.

[17] *Polewczak, J.*: Ordinary differential equations on closed subsets of locally convex spaces with applications to fixed point theorems. J. Math. Anal. Appl. *151* (1990), 208–225.

[18] *Schmidt, S.*: Existenzsätze für gewöhnliche Differentialgleichungen in Banachräumen. Funkcial. Ekvac. *35* (1992), 199–222.

[19] *Tychonov, A.N.*: Ein Fixpunktsatz. Math. Ann. *111* (1935), 767–776.

[20] *Ulm, H.*: Elementarteilertheorie unendlicher Matrizen. Math. Ann. *114* (1937), 493–505.

[21] *Volkmann, P.*: Ein Existenzsatz für gewöhnliche Differentialgleichungen in Banachräumen. Proc. Amer. Math. Soc. *80* (1980), 297–300.

[22] *Wagner, E.*: Über ein abzählbares System gewöhnlicher linearer Differentialgleichungen von Bandstruktur. Demonstratio Math. *23* (1990), 753–773.

[23] *Williamson, J. H.*: Spectral representation of linear transformations in ω. Proc. Cambridge Philos. Soc. *47* (1951), 461–472.

Author's address: *Gerd Herzog*, Mathematisches Institut I, Universität Karlsruhe, D-76128 Karlsruhe, Germany, e-mail: `Gerd.Herzog@math.uni-karlsruhe.de`.