POINT-SET DOMATIC NUMBERS OF GRAPHS

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Abstract. A subset D of the vertex set $V(G)$ of a graph G is called point-set dominating, if for each subset $S \subseteq V(G) - D$ there exists a vertex $v \in D$ such that the subgraph of G induced by $S \cup \{v\}$ is connected. The maximum number of classes of a partition of $V(G)$, all of whose classes are point-set dominating sets, is the point-set domatic number $d_p(G)$ of G. Its basic properties are studied in the paper.

Keywords: dominating set, point-set dominating set, point-set domatic number, bipartite graph

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The point-set domatic number of a graph is a variant of the domatic number $d(G)$ of a graph, which was introduced by E. J. Cockayne and S. T. Hedetniemi [1], and of the point-set domination number $\gamma_p(G)$, which was introduced by E. Sampathkumar and L. Pushpa Latha in [3] and [4]. We will describe its basic properties. All graphs considered are finite undirected graphs without loops and multiple edges.

A subset D of the vertex set $V(G)$ of a graph G is called dominating, if for each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. It is called point-set dominating (or shortly ps-dominating), if for each subset $S \subseteq V(G) - D$ there exists a vertex $v \in D$ such that the set $S \cup \{v\}$ induces a connected subgraph of G. A partition of $V(G)$ is called domatic (or point-set domatic), if all of its classes are dominating (or ps-dominating, respectively) sets in G. The maximum number of classes of a domatic (or point-set domatic) partition of $V(G)$ is called the domatic (or point-set domatic, respectively) number of G. The domatic number of G is denoted by $d(G)$, the point-set domatic number of G is denoted by $d_p(G)$. Instead of "point-set domatic" we will say shortly "ps-domatic".

For every graph G there exists at least one ps-domatic partition of $V(G)$, namely $\{V(G)\}$. Therefore $d_p(G)$ is well-defined for every graph G.

Evidently each ps-dominating set in G is a dominating set in G and thus we have a proposition.

Proposition 1. *For every graph G the inequality*

$$d_p(G) \leqslant d(G)$$

holds.

Each vertex of a complete graph K_n forms a one-element ps-dominating set and therefore the following proposition holds.

Proposition 2. *For every complete graph K_n its ps-domatic number satisfies*

$$d_p(K_n) = n.$$

A similar assertion holds for a complete bipartite graph $K_{m,n}$.

Proposition 3. *Let $K_{m,n}$ be a complete bipartite graph with $2 \leqslant m \leqslant n$. Then*

$$d_p(K_{m,n}) = m.$$

P r o o f. Let U, V be the bipartition classes of $K_{m,n}$. Let $u \in U$, $v \in V$ and consider the set $D = \{u, v\}$. Let $S \subseteq V(K_{m,n}) - D$. If $S \subseteq U$, then $S \cup \{v\}$ induces a subgraph which is a star and thus it is connected. If $S \subseteq V$, then so is $S \cup \{u\}$. Suppose that $S \cap U \neq \emptyset$, $S \cap V \neq \emptyset$. The set S itself induces a connected subgraph, namely a complete bipartite graph. The vertex u is adjacent to a vertex of $S \cap V$ and thus also $S \cup \{u\}$ induces a connected subgraph; the set $D = \{u, v\}$ is ps-dominating. If $U = \{u_1, \ldots, u_m\}$, $V = \{v_1, \ldots, v_n\}$, we take $D_i = \{u_i, v_i\}$ for $i = 1, \ldots, m-1$ and $D_m = \{u_m, v_m, \ldots, v_n\}$. Then $\{D_1, \ldots, D_m\}$ is a ps-domatic partition of $K_{m,n}$ and $d_p(K_{m,n}) \geqslant m$. On the other hand, $d_p(K_{m,n}) \leqslant d(K_{m,n}) = m$ and thus $d_p(K_{m,n}) = m$. □

Proposition 4. *Let n be an even integer, let G be obtained from the complete graph K_n by deleting edges of a linear factor. Then*

$$d_p(G) = n/2.$$

P r o o f. Evidently each pair of non-adjacent vertices in G is ps-dominating and there exists a partition of $V(G)$ into $n/2$ such sets. On the other hand, no one-vertex ps-dominating set exists. This implies the assertion. □

Now we will prove some theorems. By $d_G(x,y)$ we denote the distance between vertices x,y in a graph G. By $\operatorname{diam}(G)$ we denote the diameter of G.

Theorem 1. *Let G be a graph. If $d_p(G) \geqslant 3$, then $\operatorname{diam}(G) \leqslant 2$.*

Proof. Let $d_p(G) = k \geqslant 3$. Then there exists a ps-domatic partition $\{D_1,\dots,D_k\}$ of G. Let x,y be two vertices of G. As $k \geqslant 3$, at least one of the sets $D_1,\dots,D_k$ contains neither x nor y. Without loss of generality let it be D_1. We have $\{x,y\} \subseteq V(G) - D_1$ and therefore there exists a vertex $v \in D_1$ such that $\{v,x,y\}$ induces a connected subgraph of G. If x,y are adjacent, then $d_G(x,y) = 1$. If x,y are not adjacent, then v must be adjacent to both x and y and $d_G(x,y) = 2$. As x,y were chosen arbitrarily, we have $\operatorname{diam}(G) \leqslant 2$. □

Theorem 2. *Let G be a graph. If $d_p(G) = 2$, then $\operatorname{diam}(G) \leqslant 3$.*

Proof. Let $d_p(G) = 2$. There exists a ps-domatic partition $\{D_1,D_2\}$ of $V(G)$. Let x,y be two vertices of G. If both x,y are in D_1, then $\{x,y\} \subseteq V(G) - D_2$ and $d_G(x,y) \leqslant 2$ analogously as in the proof of Theorem 1. Similarly in the case when both x,y are in D_2. Now let $x \in D_1$, $y \in D_2$. As $\{y\} \subseteq V(G) - D_1$, there exists $v \in D_1$ adjacent to y. As both x, v are in D_1, we have $d_G(x,v) \leqslant 2$, $d_G(v,y) = 1$ and thus $d_G(x,y) \leqslant 3$. As x,y were chosen arbitrarily, we have $\operatorname{diam}(G) \leqslant 3$. □

Now we shall consider bipartite graphs.

Corollary. *Let G be a bipartite graph. If $d_p(G) \geqslant 3$, then G is a complete bipartite graph.*

This follows from the fact that every non-complete bipartite graph has the diameter at least 3.

Theorem 3. *Let G be a non-complete bipartite graph. Then $d_p(G) = 2$ if and only if G has a spanning tree T with $\operatorname{diam}(T) \leqslant 3$.*

Proof. Let T be a tree with $\operatorname{diam}(T) \leqslant 3$. If D_1, D_2 are the bipartition classes of T, then $\{D_1,D_2\}$ is a ps-domatic partition of T and $d_p(T) \leqslant 2$ and thus $d_p(T) = 2$. If G is a graph such that T is its spanning tree and G is a non-complete bipartite graph, then obviously also $d_p(G) = 2$.

Now suppose that $d_p(G) = 2$ and let $\{D_1,D_2\}$ be a ps-domatic partition. Let V_1, V_2 be the bipartition classes of G. First suppose that D_1 is a proper subset of V_1. Then $V_1 - D_1 \subseteq V(G) - D_1$ and for each $v \in D_1$ the set $(V_1 - D_1) \cup \{v\}$ is independent, i.e. it does not induce a connected subgraph of G. Hence this case is impossible and moreover D_1 cannot be a proper subset of V_2 and D_2 cannot be a proper subset of V_1

or of V_2. Now consider the case $D_1 = V_1$. Then $D_2 = V_2$. We have $V_2 \subseteq V(G) - D_1$ and there exists a vertex $v_1 \in V_1$ adjacent to all vertices of V_2. Analogously, there exists a vertex $v_2 \in V_2$ adjacent to all vertices of V_1. All edges joining v_1 with vertices of V_2 and all edges joining v_2 with vertices of V_1 form the spanning tree T; its central edge is v_1v_2 and its diameter is 3. The case $D_1 = V_2$, $D_2 = V_1$ is analogous. Now the case remains when $D_1 \cap V_1 \neq \emptyset$, $D_1 \cap V_2 \neq \emptyset$, $D_2 \cap V_1 \neq \emptyset$, $D_2 \cap V_2 \neq \emptyset$. Let $V_1 \in D_1 \cap V_1$, $x_2 \in D_1 \cap V_2$. We have $\{x_1, x_2\} \subseteq V(G) - D_2$ and there exists a vertex $v \in D_2$ such that $\{v, x_1, x_2\}$ induces a connected subgraph of G. As x_1, x_2 belong to distinct bipartition classes of G, the vertex v cannot be adjacent to both of them and thus x_1, x_2 are adjacent. Therefore D_2 induces a complete bipartite subgraph on the sets $D_2 \cap V_1$, $D_2 \cap V_2$ and analogously, D_1 induces a complete bipartite subgraph on the sets $D_1 \cap V_1$, $D_1 \cap V_2$. We have $D_1 \cap V_1 \subseteq V(G) - D_2$ and therefore there exists a vertex $w_2 \in D_2$ adjacent to all vertices of $D_2 \cap V_1$; evidently $w_2 \in D_2 \cap V_2$. Analogously, there exists a vertex $w_1 \in D_1 \cap V_1$ adjacent to all vertices of $D_1 \cap V_2$. The vertex w_1 is adjacent to all vertices of V_2 and the vertex w_2 is adjacent to all vertices of V_1. Obviously w_1, w_2 are adjacent. There exists a spanning tree T with the central edge w_1w_2 which has the diameter 3. $\square$

Now we turn to circuits. By C_n we denote the circuit of the length n.

Theorem 5. *For the circuits we have*

$$\begin{aligned} d_p(C_3) &= 3, \\ d_p(C_4) &= 2, \\ d_p(C_5) &= 2, \\ d_p(C_n) &= 1 \qquad \text{for} \quad n \geqslant 6. \end{aligned}$$

P r o o f. The circuit C_3 is the complete graph K_3 and thus $d_p(C_3) = 3$. The circuit C_4 contains a spanning tree which is a path P_3 of length 3 and therefore $d_p(C_4) = 2$; note that C_4 is a bipartite graph. Consider C_5 and let its vertices be $u_1, \ldots, u_5$ and edges u_iu_{i+1} for $i = 1, \ldots, 4$ and u_5u_1. There exists a *ps*-domatic partition $\{D_1, D_2\}$, where $D_1 = \{u_1, u_2, u_4\}$, $D_2 = \{u_3, u_5\}$; thus $d_p(C_5) \geqslant 2$. As the domatic number $d(C_5) = 2$, we have $d_p(C_5) = 2$ as well. The circuit C_6 is a bipartite graph and does not contain any spanning tree of diameter 3, therefore $d_p(C_6) = 1$. Now consider C_7. Suppose that in C_7 there exists a *ps*-domatic partition $\{D_1, D_2\}$ and denote its vertices by $u_1, \ldots, u_7$ in the usual way. Any two vertices with the distance 3 are in distinct classes of $\{D_1, D_2\}$; this follows from the proofs of Theorem 1 and of Theorem 2. If $u_1 \in D_1$ (without loss of generality), then $u_4 \in D_2$, $u_7 \in D_1$, $u_3 \in D_2$, $u_6 \in D_1$, $u_2 \in D_2$, $u_5 \in D_1$, $u_1 \in D_2$, which is a contradiction and thus $d_p(C_7) = 1$. For $n \geqslant 8$ we have $\operatorname{diam}(C_n) \geqslant 4$ and thus $d_p(C_n) = 1$. $\square$

Theorem 6. *For the complement $\overline{C}_n$ of a circuit C_n we have*

$$
\begin{aligned}
&d_p(\overline{C}_3) = 1,\\
&d_p(\overline{C}_4) = 1,\\
&d_p(\overline{C}_n) = \lfloor n/2 \rfloor \quad \textit{for } n \geqslant 5.
\end{aligned}
$$

Proof. The graphs $\overline{C}_3$ and $\overline{C}_4$ are disconnected and therefore they have the *ps*-domatic number 1. If $n \geqslant 5$, then any pair of non-adjacent vertices in $\overline{C}_n$ is a *ps*-dominating set, which can be easily verified by the reader. There exists a partition of $V(\overline{C}_n)$ into $\lfloor n/2 \rfloor$ sets, each of which is a pair of non-adjacent vertices, except at most one which has three vertices from which only two are adjacent. There exists no one-element *ps*-dominating set, therefore $d_p(\overline{C}_n) = \lfloor n/2 \rfloor$. □

In the end we will prove an existence theorem.

Theorem 7. *Let V be a finite set, let k be an integer, $1 \leqslant k \leqslant |V|$, let $\{D_1, \ldots, D_k\}$ be a partition of V. Then there exists a graph G such that $V(G) = V$, $d_p(G) = k$ and $\{D_1, \ldots, D_k\}$ is a ps-domatic partition of G.*

Proof. For $i = 1, \ldots, k$ choose a vertex $v_i \in D_i$ and join it by edges with all vertices not belonging to D_i. The resulting graph is the graph G. For each subset $S \subseteq V(G) - D_i$ there exists a vertex of D_i which is adjacent to all vertices of S, namely v_i. Therefore $\{D_1, \ldots, D_k\}$ is a *ps*-domatic partition of G and $d_p(G) \geqslant k$. If $|D_i| = 1$ for all i, then G is K_k and $d_p(G) = k$. If $|D_i| \geqslant 2$ for some i, then a vertex $u \in D_i - \{v_i\}$ has the degree $k-1$ and thus the domatic number satisfies $d(G) \leqslant k$ (by [1]) and $d_p(G) \leqslant d(G) \leqslant k$. This implies $d_p(G) = k$. □

In the end we will give a motivation for introducing the point-set domination. The concept of a dominating set is usually motivated by the displacement of certain service stations (medical, police, fire-brigade) which have to provide service for certain places (vertices of a graph). In the case of the point-set dominating set we want that for any chosen region (set of vertices) there might exist a station providing services for the whole region. Note that the point-set domination number is also a variant of the set domination number introduced in [5] and mentioned in [2].

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