A REMARK ON GRAPH OPERATORS

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Abstract. A theorem is proved which implies affirmative answers to the problems of E. Prisner. One problem is whether there are cycles of the line graph operator L with period other than 1, the other whether there are cycles of the 4-edge graph operator ∇_4 with period greater than 2. Then a similar theorem follows.

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In [1], page 71, E. Prisner suggests the problem whether there are L-cycles with period greater than 1. In the same book on page 131 the problem whether ∇_4-periods greater than 2 are possible is given. We shall prove a general theorem which implies affirmative answers to both these questions.

Let Γ be a class of graphs. A graph operator on Γ is a mapping Φ which assigns to every graph $G \in \Gamma$ a graph $\Phi(G) \in \Gamma$.

We consider the class Γ of all undirected graphs (finite and infinite) without loops and multiple edges. We denote by K_0 the empty graph, i.e. the graph in which both the vertex set and the edge set are empty.

The operator L is the line graph operator which to every graph G from Γ assigns its line graph $L(G)$, i.e. the graph whose vertex set is the edge set of G and in which two vertices are adjacent if and only if there exists a vertex in G incident to both of them (as edges).

The operator ∇_k is the k-edge graph operator. For an integer $k \geqslant 2$, a k-edge of a graph G is either a clique (i.e. a maximal complete subgraph) in G with at most k vertices, or a complete subgraph of G with k vertices. The k-edge graph $\nabla_k(G)$ of a graph G is the graph whose vertex set is the set of all k-edges of G and in which two vertices are adjacent if and only if they have at least one common vertex (as subgraphs).

Note that if a graph G has no triangles and no isolated vertices, then $\nabla_k(G) = L(G)$ for any k. A different situation occurs for graphs with isolated vertices; namely an isolated vertex is not an edge, but it is a clique. We have $\nabla_k(K_1) = K_1$ while $L(K_1) = K_0$.

A graph operator Φ on Γ will be called additive, if $\Phi(K_0) = K_0$ and for every graph $G \in \Gamma$ the image $\Phi(G)$ is the disjoint union of graphs $\Phi(C_i)$, where C_i for i from some index set I are connected components of G. (Most of commonly used graph operators have this property.)

If Φ is a graph operator on Γ, then we define Φ^0 to be the identical mapping on Γ and Φ^n for a positive integer n to be the operator such that $\Phi^n(G) = \Phi(\Phi^{n-1}(G))$ for every graph $G \in \Gamma$.

By P_n we denote the path of length n, i.e. with $n+1$ vertices. In particular, $P_0 = K_1$.

Theorem 1. *Let Φ be an additive graph operator on Γ and let r be a positive integer. If there is an infinite sequence $(H_n)_{n=0}^{\infty}$ of pairwise non-isomorphic graphs such that $\Phi(H_0) = H_0$ and $\Phi(H_n) = H_{n-1}$ for any $n \geqslant 1$, then there are r pairwise non-isomorphic graphs G_i, $0 \leqslant i \leqslant r-1$, such that the sequence $(\Phi^n(G_i))_{n=0}^{\infty}$ is periodic with period r.*

P r o o f. The graph G_i for $0 \leqslant i \leqslant r-1$ will be defined as the disjoint union of all graphs H_j such that $j \equiv i \pmod r$ and of infinitely many disjoint copies of H_0. Evidently the graphs $G_0, \ldots, G_{r-1}$ are pairwise non-isomorphic. If i, p are positive integers and $p \leqslant i$, then $\Phi^p(H_i) = H_{i-p}$; if $p > i$, then $\Phi^p(H_i) = H_0$. This implies that for $0 \leqslant i \leqslant r-1$ we have $\Phi^p(G_i) = G_q$, where $0 \leqslant q \leqslant r-1$ and $q \equiv i - p \pmod r$. This implies the assertion. □

Corollary 1. *Let L be the line graph operator and let r be a positive integer. Then there exist at least r graphs G_i, $0 \leqslant i \leqslant r-1$, such that the sequence $(L^n(G_i))_{n=0}^{\infty}$ is periodic with period r.*

P r o o f. The assertion follows from Theorem 1 if we put $H_0 = K_0$ and $H_i = P_{i-1}$ for every positive integer i. □

Corollary 2. *Let ∇_k be the k-edge graph operator for an integer $k \geqslant 2$, let r be a positive integer. Then there exist at least r graphs G_i, $0 \leqslant i \leqslant r-1$, such that the sequence $(\nabla_k^n(G_i))_{n=0}^{\infty}$ is periodic with period r.*

P r o o f. This again follows from Theorem 1 if we put $H_i = P_i$ for every non-negative integer i. □

We will prove another theorem similar to the preceding one.

Theorem 2. *Let Φ be an additive operator on Γ, let H be a graph such that $\Phi^n(H)$ is a proper subgraph of $\Phi^{n+1}(H)$ for each non-negative integer n. Then there exists a graph G such that $\Phi^{n+1}(G)$ is a proper subgraph of $\Phi^n(G)$ for each non-negative integer n.*

Proof. The graph G is the disjoint union of all graphs $\Phi^n(H)$ for non-negative integers n. The graph $\Phi^{n+1}(G)$ is obtained from $\Phi^n(G)$ be deleting the subgraph $\Phi^n(H)$ for any n. $\square$

At the end we remark that in the proof of Corollary 2 the paths need not necessarily occur.

We may define graphs P_n^k analogous to the paths P_n. Let $k \geqslant 2$. We have $P_0^k = K_1$ and $P_1^k = K_k$. The graph P_2^k has k blocks which are complete graphs with k vertices each and a unique articulation common to all of them. If the graph P_{n-2}^k is constructed for an integer $n \geqslant 3$, then to each vertex v of P_{n-2}^k which belongs to only one block we assign $k-1$ new copies of K_k, choose one vertex in each of them and identify it with v. (In the case $k = 2$ we have $P_n^k = P_n$ for any n.) We have $\nabla_k(P_0^k) = P_0^k$, $\nabla_k(P_n^k) = P_{n-1}^k$ for any positive integer n.

References

[1] *Prisner E.*: Graph Dynamics. Longman House Ltd., Burnt Mill, Harlow, Essex 1995.

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