ON A HIGHER-ORDER HARDY INEQUALITY

David E. Edmunds, Sussex, Jiří Rákosník, Praha

(Received September 9, 1998)

Dedicated to Professor A. Kufner on the occasion of his 65th birthday

Abstract. The Hardy inequality $\int_\Omega |u(x)|^p d(x)^{-p}\,\mathrm{d}x \leqslant c\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x$ with $d(x) = \operatorname{dist}(x,\partial\Omega)$ holds for $u \in C_0^\infty(\Omega)$ if $\Omega \subset \mathbb{R}^n$ is an open set with a sufficiently smooth boundary and if $1 < p < \infty$. P. Hajłasz proved the pointwise counterpart to this inequality involving a maximal function of Hardy-Littlewood type on the right hand side and, as a consequence, obtained the integral Hardy inequality. We extend these results for gradients of higher order and also for $p = 1$.

Keywords: Hardy inequality, capacity, p-thick set, maximal function, Sobolev space

MSC 2000: 31C15, 46E35, 42B25

1. Introduction

Let Ω be a proper subdomain of $\mathbb{R}^n$ and let $d(x) = \operatorname{dist}(x,\partial\Omega)$, $x \in \Omega$, be the corresponding distance function.

It is well known that the Hardy inequality

$$\int_\Omega |u(x)|^p d(x)^{-p}\,\mathrm{d}x \leqslant c\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x, \tag{1.1}$$

holds for $u \in C_0^\infty(\Omega)$ if $1 < p < \infty$ and the boundary of Ω satisfies the Lipschitz condition or similar regularity conditions. For these results and further references we refer to [8], [10], [12].

The research was supported by the NATO Grant no. OUTR. CRG. 970071.

The second author was partly supported by the Grant no. 201/97/0744 of the Grant Agency of the Czech Republic.

Different authors introduced the notions of capacity and of thick sets in various ways (see, e.g. [1], [4]–[9], etc.) in order to find weaker sufficient conditions for inequalities of Hardy, Poincaré and other types. We shall concentrate mainly on [4] and [6].

Let K be a compact subset of Ω and let $1 \leqslant p < \infty$. The variational $(1,p)$-capacity $C_{1,p}(K,\Omega)$ of the condenser (K,Ω) is defined to be

$$C_{1,p}(K,\Omega) = \inf\left\{ \int_\Omega |\nabla u(x)|^p \,\mathrm{d}x \colon\ u \in C_0^\infty(\Omega), u(x) \geqslant 1 \text{ for } x \in K \right\}.$$

By $B(x,r)$ we denote the open ball in $\mathbb{R}^n$ of radius r, $0 < r < \infty$, centered at $x \in \mathbb{R}^n$.

Definition 1. A closed set $K \subset \mathbb{R}^n$ is *locally uniformly* $(1,p)$*-thick*, if there exist numbers $b > 0$ and r_0, $0 < r_0 \leqslant \infty$ such that

$$C_{1,p}\bigl(\overline{B}(x,r) \cap K, B(x,2r)\bigr) \geqslant b\, C_{1,p}\bigl(\overline{B}(x,r), B(x,2r)\bigr) \tag{1.2}$$

for all $x \in K$ and $0 < r < r_0$. If $r_0 = \infty$, then the set K is called *uniformly* $(1,p)$*-thick*.

Note that a scaling argument yields

$$C_{1,p}\bigl(\overline{B}(x,r), B(x,2r)\bigr) = c(n,p) r^{n-p}. \tag{1.3}$$

P. Hajłasz [4] used the Hardy-Littlewood maximal operator M and showed that for a domain Ω with a locally uniformly $(1,p)$-thick complement there exists $q \in (1,p)$ such that every function $u \in C_0^\infty(\Omega)$ satisfies the pointwise analogue of the Hardy inequality, which in a slightly simplified formulation reads

$$|u(x)| \leqslant c d(x) \bigl[M(|\nabla u|^q)(x)\bigr]^{1/q}.$$

As a corollary he obtained the integral Hardy inequality

$$\int_\Omega |u(x)|^p d(x)^{a-p} \,\mathrm{d}x \leqslant c \int_\Omega |\nabla u(x)|^p d(x)^a \,\mathrm{d}x,$$

for small positive numbers a. Similar results were obtained also by J. Kinnunen and O. Martio [6].

Our aim is to extend these results for derivatives of higher order.

If $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n-tuple of non-negative integers, $|\alpha| = \sum_{i=1}^{n} \alpha_i$, $\alpha! = \alpha_1! \dots \alpha_n!$, and for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we set $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. The corresponding partial derivative operators will be denoted by

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

and the gradient of a real-valued function of order k, $k \in \mathbb{N}$, will be the vector $\nabla^k u = \{D^\alpha u\}_{|\alpha|=k}$. For $k = 1$, $\nabla^1 u = \nabla u$ is the usual gradient.

Given a measurable set $E \subset \mathbb{R}^n$, we denote its Lebesgue n-measure by $|E|$ and the characteristic function of E by χ_E. Constants c in estimates may vary during calculations but they always remain independent of all non-fixed entities.

2. The pointwise Hardy inequality

The fractional maximal function $M_{\gamma,R}u$, $0 \leqslant \gamma \leqslant n$, $0 < R \leqslant \infty$, is defined for every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$M_{\gamma,R}u(x) = \sup_{0<r<R} |B(x,r)|^{\gamma/n-1} \int_{B(x,r)} |u(y)| \,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

Note that $M_{0,\infty}u = Mu$ is the classical Hardy-Littlewood maximal function.

Theorem 1. *Let $1 \leqslant p < \infty$, let k be a positive integer and $0 \leqslant \gamma < k$. Let Ω be an open subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is locally uniformly $(1,p)$-thick and let b be the constant from Definition 1. Then there exists a constant $c = c(k,p,n,b) > 0$ such that every function $u \in C_0^\infty(\Omega)$ satisfies the inequality*

$$|u(x)| \leqslant c d(x)^{k-\gamma/p} \left[M_{\gamma,4d(x)}\big(|\nabla^k u|^p \chi_{B(\overline{x},2d(x))}\big)(x)\right]^{1/p}, \tag{2.1}$$

where $x \in \Omega$, $d(x) < r_0$, and $\overline{x} \in \partial\Omega$ is such that $|x - \overline{x}| = d(x)$.

This is the main result of this section which extends Theorem 2 of [4]. To prove it we shall need several auxiliary assertions. The first one is a generalization of [3, Lemma 7.16].

Lemma 1. *Let k be a natural number. There exists a constant $c = c(k,n) > 0$ such that for every ball $B \subset \mathbb{R}^n$ and for every function $u \in C^k(B)$ the inequality*

$$\left|u(x) - |B|^{-1} \int_B P(x,y) \,\mathrm{d}y\right| \leqslant c \int_B \frac{|\nabla^k u(y)|}{|x-y|^{n-k}} \,\mathrm{d}y, \qquad x \in B,$$

holds, where P is the polynomial of order $\leqslant k-1$ given by

$$P(x,y)=\sum_{|\alpha|\leqslant k-1}\frac{(-1)^{|\alpha|}}{\alpha!}D^{\alpha}u(y)(y-x)^{\alpha},\qquad x,y\in B. \tag{2.2}$$

Lemma 1 can be proved in a way similar to the proof of Lemma 7.16 in [3] using the Taylor expansion of the function $v(r)=u(x+r\theta)$, where $r=|x-y|$, $\theta=(y-x)/r$, $x,y\in\Omega$. Note that assertions of this type can be found for instance in [1, §8.1] and [8, §1.1.10].

The next assertion is a variation of a well-known result of L. I. Hedberg.

Lemma 2. *Let $0\leqslant\gamma<\kappa$ and let $B\subset\mathbb{R}^n$ be a ball of radius R. Then there exists a constant $c=c(n,\gamma,\kappa)>0$ such that every function $g\in L^1_{\mathrm{loc}}(B)$ satisfies the inequality*

$$\int_B\frac{|g(y)|\,\mathrm{d}y}{|x-y|^{n-\kappa}}\leqslant cR^{\kappa-\gamma}M_{\gamma,2R}(g)(x),\qquad x\in B.$$

Proof. Fix $x\in B$ and for $i\in\mathbb{N}$ set $A_i=\big(B(x,2^{1-i}R)\setminus B(x,2^{-i}R)\big)\cap B$. Then

$$\begin{aligned}\int_B\frac{|g(y)|}{|x-y|^{n-\kappa}}\,\mathrm{d}y&=\sum_{i=0}^{\infty}\int_{A_i}\frac{|g(y)|}{|x-y|^{n-\kappa}}\,\mathrm{d}y\\&\leqslant\max(1,2^{\kappa-n})\sum_{i=0}^{\infty}(2^{-i}R)^{\kappa-n}\int_{B(x,2^{1-i}R)}|g(y)|\,\mathrm{d}y\\&\leqslant|B(0,1)|^{-1}\max(1,2^{\kappa-n})2^{n-\gamma}R^{\kappa-\gamma}\sum_{i=0}^{\infty}2^{-i(\kappa-\gamma)}M_{\gamma,2R}(g)(x).\end{aligned}$$

□

We shall also need the following inequality of Poincaré type which follows from the considerations in [8, Sections 9.3 and 10.1.2].

Lemma 3. *Let $1\leqslant p<\infty$. Let $B=B(x,R)$ be a ball in $\mathbb{R}^n$ and let K be a closed subset of $\overline{B}$. Then every function $u\in C^{\infty}(\overline{B})$ such that $\operatorname{dist}(\operatorname{supp}u,K)>0$ satisfies the inequality*

$$\int_{\overline{B}}|u(x)|^p\,\mathrm{d}x\leqslant c\,\frac{R^n}{C_{1,p}(K,B(x,2R))}\int_{\overline{B}}|\nabla u(x)|^p\,\mathrm{d}x,$$

where c is a positive constant independent of B, K and u.

Proof of Theorem 1. Let $x \in \Omega$ be such that $d(x) < r_0$, where r_0 is the number from Definition 1. Let $\overline{x} \in \partial\Omega$ satisfy $|x-\bar{x}| = d(x) = R$ and let $u \in C_0^\infty(\Omega)$. Set $B = B(\overline{x}, 2R)$. Then $x \in B$ and

$$|u(x)| \leqslant |u(x) - P_B(x)| + |P_B(x)|, \tag{2.3}$$

where $P_B(x) = |B|^{-1} \int_B P(x,y)\,\mathrm{d}y$ and P is the polynomial from Lemma 1. Using Lemma 1, Lemma 2 and the Hölder inequality we obtain

$$\begin{aligned} |u(x) - P_B(x)| &\leqslant c \int_B \frac{|\nabla^k u(y)|}{|x-y|^{n-k}}\,\mathrm{d}y \leqslant cR^{k-\gamma} M_{\gamma,4R}(|\nabla^k u|\chi_B)(x) \\ &\leqslant cR^{k-\gamma/p}\big[M_{\gamma,4R}(|\nabla^k u|^p\chi_B)(x)\big]^{1/p}. \end{aligned} \tag{2.4}$$

From (2.2) we have

$$\begin{aligned} |P_B(x)| &\leqslant |B|^{-1} \int_B |P(x,y)|\,\mathrm{d}y \leqslant c \sum_{i=0}^{k-1} R^i |B|^{-1} \int_B |\nabla^i u(y)|\,\mathrm{d}y \\ &\leqslant c \sum_{i=0}^{k-1} R^i \left(|B|^{-1} \int_B |\nabla^i u(y)|^p\,\mathrm{d}y\right)^{1/p}. \end{aligned}$$

Repeated application of Lemma 3 and of (1.2) and (1.3) yields

$$\begin{aligned} \int_B |\nabla^i u(x)|^p\,\mathrm{d}x &\leqslant c\,\frac{R^n}{C_{1,p}\big((\mathbb{R}^n\setminus\Omega)\cap\overline{B}, B(\overline{x},4R)\big)} \int_B |\nabla^{i+1} u(x)|^p\,\mathrm{d}x \\ &\leqslant c\,R^p \int_B |\nabla^{i+1} u(x)|^p\,\mathrm{d}x \\ &\leqslant c\,R^{(k-i)p} \int_B |\nabla^k u(x)|^p\,\mathrm{d}x, \qquad i = 0,\dots,k-1. \end{aligned}$$

Hence,

$$\begin{aligned} |P_B(x)| &\leqslant cR^k \left(|B|^{-1} \int_B |\nabla^k u(x)|^p\,\mathrm{d}x\right)^{1/p} \\ &\leqslant cR^{k-\gamma/p}\big[M_{\gamma,4R}\big(|\nabla^k u|^p\chi_B\big)(x)\big]^{1/p}. \end{aligned} \tag{2.5}$$

The inequality (2.1) follows from (2.3)–(2.5). $\square$

3. Integral inequalities

In this section we shall use Theorem 1 to obtain higher-order analogues of the classical Hardy inequality. As in [4] and [6], in further considerations we shall essentially use the openness of the $(1,p)$-thickness with respect to p. This deep property was originally proved by J. L. Lewis [7, Theorem 1] and later on in another way by P. Mikkonen [9, Theorem 8.2]. The following lemma can be obtained as a particular case of Lewis' and Mikkonen's results. It is not important for our purpose that Lewis dealt with another type of capacity.

Lemma 4. *Let $1 < p < \infty$ and let $K \subset \mathbb{R}^n$ be a closed locally uniformly (k,p)-thick set. Then there exists q, $1 < q < p$, depending only on n, k, p and b, such that K is locally uniformly (k,q)-thick with the same value of r_0 as for p.*

For $r > 0$ we set

$$\Omega_r = \{x \in \Omega\colon\ d(x) < r\}.$$

Theorem 2. *Let $1 < p < \infty$ and let k be a positive integer. Let Ω be an open subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is locally uniformly $(1,p)$-thick. Then there exists a positive constant $c = c(k,p,n,b)$ such that the inequality*

$$\int_{\Omega_r} \left(\frac{|u(x)|}{d(x)^k}\right)^p \,\mathrm{d}x \leqslant c \int_{\Omega_r} |\nabla^k u(x)|^p \,\mathrm{d}x \tag{2.6}$$

holds for every function $u \in C_0^\infty(\Omega)$ and for every $r \in (0, r_0)$, where r_0 is the parameter given in Definition 1.

P r o o f. Let $p > 1$ and let $q \in (1,p)$ be from Lemma 4, and suppose that $r \in (0, r_0)$. It follows from (2.1) that for all $u \in C_0^\infty(\Omega)$,

$$|u(x)| d(x)^{-k} \leqslant c \left[M\big(|\nabla^k u|^q \chi_{\Omega_r}\big)(x)\right]^{1/q}, \quad x \in \Omega_r. \tag{2.7}$$

We use the boundedness of $M\colon\ L^{p/q} \to L^{p/q}$ and the Hölder inequality to obtain

$$\int_{\Omega_r} \left(\frac{|u(x)|}{d(x)^k}\right)^p \,\mathrm{d}x \leqslant c \int_{\Omega_r} \left[M\big(|\nabla^k u|^q \chi_{\Omega_r}\big)(x)\right]^{p/q} \,\mathrm{d}x \leqslant c \int_{\Omega_r} |\nabla^k u(x)|^p \,\mathrm{d}x. \tag{2.8}$$

Note that the norm of the maximal operator M and, consequently, also the constant c depend on the value of p/q. $\square$

If $p = 1$, we cannot use Lemma 4. Instead we use the fact that for Ω with $|\Omega| < \infty$ the maximal operator M is a bounded mapping of $L\log L(\Omega)$ in $L^1(\Omega)$ (see [2], p. 74). Recall that $L\log L(\Omega)$ is the Zygmund space which consists of all measurable functions u with $\int_\Omega |u(x)|\log_+ |u(x)|\,\mathrm{d}x < \infty$, endowed with the norm

$$\|u\|_{L\log L(\Omega)} = \int_0^{|\Omega|} u^*(t)\log\frac{|\Omega|}{t}\,\mathrm{d}t,$$

where u^* is the non-increasing rearrangement of u.

Theorem 3. *Let $p = 1$ and let k be a positive integer. Let Ω be a bounded open subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is locally uniformly $(1,1)$-thick. Then there exists a positive constant $c = c(k,n,b)$ such that the inequality*

$$\int_{\Omega_r} \frac{|u(x)|}{d(x)^k}\,\mathrm{d}x \leqslant c\,\|\nabla^k u\|_{L\log L(\Omega_r)} \tag{2.9}$$

holds for every function $u \in C_0^\infty(\Omega)$ and for every $r \in (0, r_0)$, where r_0 is the parameter given in Definition 1.

P r o o f. From the estimate (2.1) we have

$$|u(x)|d(x)^{-k} \leqslant cM\bigl(|\nabla^k u|\chi_{\Omega_r}\bigr)(x), \quad x \in \Omega_r.$$

Integrating both sides of the inequality over Ω_r and using the boundedness of $M\colon L\log L(\Omega) \to L^1(\Omega)$ we arrive at the inequality (2.9). $\square$

Corollary 1. *Let $1 < p < \infty$ and let k be a positive integer. Let Ω be an open subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is locally uniformly $(1,p)$-thick. Then there exists a number $\varepsilon_0 > 0$ such that the inequality*

$$\int_{\Omega_r} \left(\frac{|u(x)|}{d(x)^k}\right)^p d(x)^{\varepsilon p}\,\mathrm{d}x \leqslant c\int_{\Omega_r} |\nabla^k u(x)|^p d(x)^{\varepsilon p}\,\mathrm{d}x \tag{2.10}$$

holds for all $u \in C_0^\infty(\Omega)$, $r \in (0, r_0)$ and $0 \leqslant \varepsilon < \varepsilon_0$. The constant $c > 0$ depends on n, p, k, b and on the number q from Lemma 4.

P r o o f. Fix $\varepsilon > 0$ and let $u \in C_0^\infty(\Omega)$ be such that the integral on the right hand side of (2.10) is finite.

If $k = 1$, we set $v(x) = |u(x)|d(x)^\varepsilon$. Then

$$|\nabla v(x)| \leqslant |\nabla u(x)|d(x)^\varepsilon + \varepsilon|u(x)|d(x)^{\varepsilon-1} \quad \text{for a.e. } x \in \Omega, \tag{2.11}$$

and (2.10) implies that v belongs to the Sobolev space $W_0^{1,p}(\Omega)$. Applying Theorem 2 to functions from $C_0^\infty(\Omega)$ which approximate v in $W_0^{1,p}(\Omega)$ and passing to the limit we obtain

$$\int_{\Omega_r} \left(\frac{|u(x)|}{d(x)}\right)^p d(x)^{\varepsilon p}\,\mathrm{d}x = \int_{\Omega_r} \left(\frac{|v(x)|}{d(x)}\right)^p \mathrm{d}x \leqslant c \int_{\Omega_r} |\nabla v(x)|^p\,\mathrm{d}x$$

for $0 \leqslant \varepsilon < \varepsilon_0$. By (2.11), we have

$$\int_{\Omega_r} \left(\frac{|u(x)|}{d(x)}\right)^p d(x)^{\varepsilon p}\,\mathrm{d}x \\ \leqslant c\left(\int_{\Omega_r} |\nabla u(x)|^p d(x)^{\varepsilon p}\,\mathrm{d}x + \varepsilon^p \int_{\Omega_r} \left(\frac{|u(x)|}{d(x)}\right)^p d(x)^{\varepsilon p}\,\mathrm{d}x\right).$$

Thus, the inequality (2.10) holds for $0 \leqslant \varepsilon < \varepsilon_0 = c^{-1/p}$.

Let $k > 1$ and suppose that the inequality (2.10) holds for $j = 1, 2, \ldots, k-1$ and $0 \leqslant \varepsilon < \varepsilon_0$. Let ϱ be the regularized distance function equivalent to d and satisfying the estimate

$$|\nabla^j \varrho(x)| \leqslant c_j d(x)^{1-j}, \quad x \in \Omega, \quad j = 1, 2, \ldots,$$

(see, e.g., [11, p. 171]). Set $v(x) = |u(x)|\varrho(x)^\varepsilon$. Then

$$|\nabla^k v(x)| \leqslant |\nabla^k u(x)|\varrho(x)^\varepsilon + \varepsilon \sum_{j=1}^k Q_j(\varepsilon)|\nabla^{k-j} u(x)|\varrho(x)^{\varepsilon-j},$$

where Q_j are polynomials of degree j. Thus, we have

$$\begin{aligned}\int_{\Omega_r} \left(\frac{|u(x)|}{d(x)^k}\right)^p d(x)^{\varepsilon p}\,\mathrm{d}x &\leqslant c \int_{\Omega_r} \left(\frac{|v(x)|}{\varrho(x)^k}\right)^p \mathrm{d}x \\ &\leqslant c \int_{\Omega_r} |\nabla^k u(x)|^p \varrho(x)^{\varepsilon p}\,\mathrm{d}x + c\varepsilon^p \sum_{j=1}^k |Q_j(\varepsilon)|^p \int_{\Omega_r} \left(\frac{|u(x)|}{\varrho(x)^{k-j}}\right)^p \varrho(x)^{\varepsilon p}\,\mathrm{d}x \\ &\leqslant c \int_{\Omega_r} |\nabla^k u(x)|^p \varrho(x)^{\varepsilon p}\,\mathrm{d}x + c\varepsilon^p \int_{\Omega_r} \left(\frac{|u(x)|}{\varrho(x)^k}\right)^p \varrho(x)^{\varepsilon - p}\,\mathrm{d}x \\ &\leqslant c \int_{\Omega_r} |\nabla^k u(x)|^p d(x)^{\varepsilon p}\,\mathrm{d}x + c\varepsilon^p \int_{\Omega_r} \left(\frac{|u(x)|}{d(x)^k}\right)^p d(x)^{\varepsilon - p}\,\mathrm{d}x,\end{aligned}$$

and the inequality (2.10) holds for $0 \leqslant \varepsilon < c^{-1/p}$. □

Corollary 2. *Let Ω be such that $\mathbb{R}^n \setminus \Omega$ is locally uniformly $(1,p)$-thick with $r_0 > \frac{1}{2}\operatorname{diam}(\Omega)$. Then the inequality (2.1) holds for every $x \in \Omega$ and the assertions of Theorem 2, Theorem 3 and Corollary 1 hold with Ω in place of Ω_r and for all functions u from the corresponding Sobolev spaces $W_0^{k,p}$ on Ω.*

Proof. It suffices to observe that $\Omega_r = \Omega$ for $r > \frac{1}{2}\operatorname{diam}(\Omega)$ and that the constant c does not depend on the parameter r_0. □

Note that the assumption of Corollary 2 holds, in particular, if $\mathbb{R}^n \setminus \Omega$ is uniformly $(1,p)$-thick (i.e., $r_0 = \infty$).

An open problem. Additional weights could be introduced into the inequality (2.6) by applying a weighted inequality for the maximal function. Following the proof of Theorem 2 we can multiply both sides of inequality (2.7) (or, more precisely, of inequality (2.1)) by $d(x)^\varepsilon$ and integrate over Ω_r. However, to make the final step in (2.8) we have to know that the maximal function satisfies the weighted inequality

$$\int_{\Omega_r} \left[M\left(|\nabla^k u|^q \chi_{\Omega_r}\right)(x)\right]^{p/q} d(x)^{\varepsilon p}\,\mathrm{d}x \leqslant c \int_{\Omega_r} |\nabla^k u(x)|^p d(x)^{\varepsilon p}\,\mathrm{d}x.$$

Note that we are dealing with the global maximal function (the balls in the construction of $M_{\gamma,4d(x)}$ from inequality (2.1) cross the complement of Ω) and so to use the known weighted inequalities for M we would have to consider $d(x)$ extended properly outside Ω. The question is, if the sufficient conditions for such weighted estimate would not override the condition of $(1,p)$-thickness of $\mathbb{R}^n \setminus \Omega$.

References

[1] *D. R. Adams, L. I. Hedberg*: Function spaces and potential theory. Springer, Berlin, 1996.

[2] *D. E. Edmunds, H. Triebel*: Function spaces, entropy numbers and differential operators. Cambridge Tracts in Mathematics, vol. 120, Cambridge University Press, Cambridge, 1996.

[3] *D. Gilbarg, N. S. Trudinger*: Elliptic partial differential equations of second order. (2nd ed.), Springer, Berlin, 1983.

[4] *P. Hajłasz*: Pointwise Hardy inequalities. Proc. Amer. Math. Soc. *127* (1999), 417–423.

[5] *J. Heinonen, T. Kilpeläinen, O. Martio*: Nonlinear potential theory of degenerate elliptic equations. Oxford Science Publications, Clarendon Press, Oxford, 1993.

[6] *J. Kinnunen, O. Martio*: Hardy's inequalities for Sobolev functions. Math. Res. Lett. *4* (1997), no. 4, 489–500.

[7] *J. L. Lewis*: Uniformly fat sets. Trans. Amer. Math. Soc. *308* (1988), no. 1, 177–196.

[8] *V. G. Maz'ya*: Sobolev spaces. Springer, Berlin, 1985.

[9] *P. Mikkonen*: On the Wolff potential and quasilinear elliptic equations involving measures. Ann. Acad. Sci. Fenn. Ser. A. I. Math. Dissertationes *104* (1996), 1–71.

[10] *B. Opic, A. Kufner*: Hardy-type inequalities. Pitman Research Notes in Math. Series 219, Longman Sci.&Tech., Harlow, 1990.

[11] *E. M. Stein*: Singular integrals and differentiability properties of functions. Princeton University Press, Princeton, N.J., 1970.

[12] *A. Wannebo*: Hardy inequalities. Proc. Amer. Math. Soc. *109* (1990), no. 1, 85–95.

Authors' addresses: *D. E. Edmunds*, Centre for Mathematical Analysis and Its Applications, School of Mathematical and Physical Sciences, University of Sussex, Falmer, Brighton, BN1 9QH, United Kingdom, e-mail: `D.E.Edmunds@sussex.ac.uk`; *J. Rákosník*, Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: `rakosnik@math.cas.cz`.