LOCAL SOLVABILITY AND REGULARITY RESULTS FOR A CLASS OF SEMILINEAR ELLIPTIC PROBLEMS IN NONSMOOTH DOMAINS

M. Bochniak, A.-M. Sändig, Stuttgart

(Received November 27, 1998)

Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. We consider a class of semilinear elliptic problems in two- and three-dimensional domains with conical points. We introduce Sobolev spaces with detached asymptotics generated by the asymptotical behaviour of solutions of corresponding linearized problems near conical boundary points. We show that the corresponding nonlinear operator acting between these spaces is Fréchet differentiable. Applying the local invertibility theorem we prove that the solution of the semilinear problem has the same asymptotic behaviour near the conical points as the solution of the linearized problem if the norms of the given right hand sides are small enough. Estimates for the difference between the solution of the semilinear and of the linearized problem are derived.

Keywords: semilinear elliptic problems, spaces with detached asymptotics, asymptotic behaviour near conical points

MSC 2000: 35A07, 35B65, 35C20, 35J60

1. Introduction

The theory of general linear elliptic problems in domains with a piecewise smooth boundary is well developed (see the monographs [10, 22] and the references therein). In papers by Kondrat'ev [11] and Maz'ya, Plamenevsky [17, 18] the Fredholm property of the corresponding linear operators in domains with conical points is investigated in several scales of functional spaces and the solutions u are decomposed into a linear combination of some singular functions S_i and a more regular remainder

$$u = \sum c_i S_i + u_{\text{reg}}. \tag{1}$$

Another simple approach based on the construction of a barrier function was used by Azzam [1] in order to prove regularity results for linear elliptic two-dimensional boundary value problems in domains with corners. This method leads to estimates of the solution u in the neighbourhood of the conical point P_i which have the form

$$|u(x) - u(P_i)| \leqslant c|x - P_i|^{\alpha}. \tag{2}$$

The theory of nonlinear elliptic problems in nonsmooth domains is much less developed. The barrier method was applied to several classes of semilinear and quasilinear problems [6, 25, 19, 3]. Asymptotic expansions near conical boundary points of the type (1) are known only for special nonlinear problems as semilinear perturbations of the biharmonic operator [2], degenerate p-harmonic problems [5], Navier Stokes equations [23, 20] and others [21]. More general results were obtained in [19] where the Dirichlet problem for quasilinear equations is investigated and asymptotic expansions near conical points are derived in domains with non-reentrant corner points under the assumption that a solution exists which is continuous up to the boundary in a neighbourhood of the corner point.

The solvability of nonlinear elliptic problems with small right hand sides can be investigated with the help of the implicit function theorem or the local invertibility theorem. Usually this approach demands an appropriate regularity of solutions to a (formally) linearized problem. This leads to difficulties if the domain is nonsmooth, if the coefficients have jumps or if mixed boundary conditions occur. These theorems were successfully applied to the Dirichlet and the Neumann problem in convex nonlinear elastic bodies [26, 4]. In [24] Recke used this approach to prove local $W^{1,p}(\Omega)$-solvability results with $p > 2$ for mixed boundary value problems for a class of two-dimensional quasilinear elliptic systems. In higher-dimensional cases he investigates the local $W^{1,2}(\Omega)$-solvability under special growth conditions for the coefficients.

In this paper we use the local invertibility theorem to investigate the local behaviour near conical boundary points for a class of semilinear elliptic systems. To this end we use an idea from [20] introducing Sobolev spaces with detached asymptotics which describe the asymptotic behaviour of solutions to the linearized problem. We prove that the operator of the semilinear problem acting between Sobolev spaces with detached asymptotics is Fréchet differentiable at $u = 0$ and that the Fréchet derivative coincides with the linearized operator. Thus the solution of the semilinear problem admits an asymptotic decomposition with singular terms of the same type as the solution of the linearized problem, provided that the norms of the right hand sides are small enough.

Let $\Omega \in \mathbb{R}^n$, $n = 2, 3$, be a bounded domain with a piecewise smooth boundary. We assume that there exists a finite set $P = \{P_1, \dots, P_q\} \subset \partial\Omega$ of boundary points such that $\partial\Omega \setminus P$ is smooth. Furthermore, let $\partial\Omega = \Gamma_1 \cup \Gamma_2$ be a given decomposition of the boundary with $\Gamma_2 \neq \emptyset$. We assume that in the two-dimensional case $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 \subset P$, whereas in the three-dimensional case $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. These assumptions guarantee that singularities of solutions occur only in the neighbourhood of isolated points of the boundary.

We consider a mixed boundary value problem for an elliptic system of k semilinear equations for the vector function $u = (u_1, \dots, u_k)$

$$
\begin{aligned}
-\partial_j[a_{ij\alpha\beta}(x)\partial_i u_\alpha] + b_{i\alpha\beta}(x,u)\partial_i u_\alpha + c_\beta(x,u) &= f_\beta \text{ in } \Omega,\ \beta = 1, \dots, k, \\
[a_{ij\alpha\beta}(x)\partial_i u_\alpha]\,\nu_j &= h_\beta \text{ on } \Gamma_1,\ \beta = 1, \dots, k, \\
u_\beta &= g_\beta \text{ on } \Gamma_2,\ \beta = 1, \dots, k.
\end{aligned} \tag{3}
$$

Here we denote by $\nu = (\nu_1, \dots, \nu_n)$ the unit outward normal on $\partial\Omega$. Moreover, we apply here and in the following Einstein's summation convention for the repeated indices $i, j = 1, \dots, n$ and $\alpha = 1, \dots, k$. We assume that $a_{ij\alpha\beta} \in C^1(\overline{\Omega}), b_{i\alpha\beta}, c_\beta \in C^2(\overline{\Omega} \times \mathbb{R}^n)$ and that $c_\beta(x, 0) = 0$. The last assumption guarantees that the problem (3) with homogeneous right hand sides has the trivial solution.

Semilinear systems of the form (3) appear in fluid mechanics (stationary Navier-Stokes equations, stationary convection-diffusion equations [7]) and in the theory of diffusion processes (reaction-diffusion systems [15, 8], advection equations in air quality modelling [9]). In practical applications the boundary of a domain is often piecewise smooth. Frequent examples are fluid or particle flow (transport) in an unbounded exterior domain around a rigid body with corner points and flow around nonsmooth obstacles at the bottom or at the walls of channels. In fluid mechanics one traditionally studies boundary conditions of the pure Dirichlet or Neumann type. The Dirichlet conditions express a non-slip behaviour of the fluid on the walls of the channel, whereas the Neumann conditions describe the flux or surface stresses on the surface of the obstacle. In order to describe the behaviour of more complicated models it is necessary to impose mixed boundary conditions.

In this paper we investigate the local solvability of (3) and the regularity of the solutions in the neighbourhood of conical points of the boundary. To this end we formulate (3) as an operator equation between appropriate spaces and apply the theorem on local invertibility of nonlinear operators.

Theorem 2.1. (Local Invertibility Theorem, see e.g. [4]) *Let $N\colon X \to Y$ be a continuous mapping between the Banach spaces X, Y and let $x_0 \in X$, $y_0 \in Y$ with $Nx_0 = y_0$. Suppose that the operator N is Fréchet differentiable at x_0 and that* *the Fréchet derivative $N'(x_0)\colon X \to Y$ is an isomorphism. Then there exist open neighbourhoods $U(x_0) \subset X$, $V(y_0) \subset Y$ such that the operator $N\colon U(x_0) \to V(y_0)$ is continuously invertible and its local inverse N^{-1} is Fréchet differentiable at y_0.*

3. Linear elliptic problems in spaces with detached asymptotics

We consider the formal linearization of the problem (3) at $u = 0$

$$
\begin{aligned}
-\partial_j[a_{ij\alpha\beta}(x)\partial_i v_\alpha] + b_{i\alpha\beta}(x,0)\partial_i v_\alpha + \langle \partial_u c_\beta(x,0), v\rangle &= f_\beta \text{ in } \Omega,\\
[a_{ij\alpha\beta}(x)\partial_i v_\alpha]\,\nu_j &= h_\beta \text{ on } \Gamma_1,\\
v_\beta &= g_\beta \text{ on } \Gamma_2.
\end{aligned}
\tag{4}
$$

Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^k$ and the derivatives are to be understood in the distributional sense.

The regularity of solutions of linear elliptic problems of the form (4) in domains with conical points was thoroughly investigated in a series of papers by Kondrat'ev [11], Maz'ya and Plamenevsky [17, 18]. From their results it follows that the behaviour of the solutions of (4) in the neighbourhood of a singular point $P_i \in P$ can be described with the help of the singular functions

$$
w_j = r^{\lambda_j} \sum_{s=0}^{R(\lambda_j)-1} \frac{1}{s!}(\log r)^s \varphi_{j,s}(\omega),
\tag{5}
$$

$$
v_j = r^j \sum_{s=0}^{R(j)} \frac{1}{s!}(\log r)^s \psi_{j,s}(\omega).
\tag{6}
$$

Here $r = |x - P_i|$ is the distance from the singular point P_i and ω are $n-1$ coordinates on the unit sphere S^{n-1}. The functions w_j satisfy the homogeneous linear problem

$$
a_{ij\alpha\beta}(0)\partial_{ij} v_\alpha = 0, \quad \beta = 1, \dots, k,
\tag{7}
$$

and corresponding homogeneous boundary conditions in the infinite wedge $W(P_i)$ with origin on P_i which coincides with Ω in some neighbourhood of the corner point P_i. The exponents λ_j and the functions $\varphi_{j,s}$ can be interpreted as eigenvalues and (generalized) eigenfunctions of a certain operator pencil $\mathcal{A}(P_i)$ (see [11, 17] for details). $R(\lambda_j)$ denotes the Riesz-index (the maximal length of the Jordan chains to λ_j). Moreover, $R(j)$ vanishes if j is not an eigenvalue of the operator pencil $\mathcal{A}(P_i)$. Otherwise it coincides with the Riesz index to j. The functions v_j are special solutions of the same problem with non-vanishing (polynomial) right hand sides [18].

The regularity of weak solutions $u \in W_2^1(\Omega)$ of (4) depends on the distribution of eigenvalues of $\mathcal{A}(P_i)$ for every corner point $P_i \in P$. Let $a_0 = \min\{\operatorname{Re}\alpha_j\}$, where the minimum is taken over all eigenvalues α_j of $\mathcal{A}(P_i)$ with $1 - \frac{n}{2} \leqslant \operatorname{Re}\alpha_j$ for every singular point $P_i \in P$. The weak solution u belongs to $W_p^1(\Omega)$ if $p < n/(1 - a_0)$. It means, we can guarantee that $u \in W_p^1(\Omega)$ if the strip $1 - \frac{n}{2} < \operatorname{Re}\lambda < 1 - \frac{n}{p}$ is free of eigenvalues of the operator pencil $\mathcal{A}(P_i)$ for all corner points $P_i \in P$. In the following we assume:

(E) *The problem* (4) *is elliptic and has a unique weak solution* $u \in W_2^1(\Omega)$.

(R) *The weak solution u belongs to $W_p^1(\Omega)$, where $p = n + \varepsilon$ with a small real $\varepsilon > 0$.*

The regularity condition (R) is always satisfied if $n = 2$ [11]. In the three-dimensional case this can be often guaranteed if $(Av)_\beta := \partial_j\,[a_{ij\alpha\beta}(0)\partial_i v_\alpha]$ is strongly elliptic. This follows from $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$ and the following Theorem:

Theorem 3.1. [12, 13, 14] *Let $n = 3$, $P_i \in P$ and suppose that pure Dirichlet or pure Neumann conditions are given in a neighbourhood of P_i. If the operator $(Av)_\beta = \partial_j\,[a_{ij\alpha\beta}(0)\partial_i v_\alpha]$ is strongly elliptic, then the strip $|\operatorname{Re}\lambda + \frac{1}{2}| < \frac{1}{2}$ is free of eigenvalues of the operator pencil $\mathcal{A}(P_i)$.*

In fact, the value $a_0 = 0$ is an eigenvalue for the pure Neumann problem in a neighbourhood of a conical point P_i, but the corresponding asymptotic term w_j does not appear in the asymptotics due to the condition (E) and only logarithmic terms v_0 remain in this case. For right hand sides which vanish in the neighborhood of P_i the term v_0 vanishes, too. Therefore the condition (R) is satisfied for pure Dirichlet conditions and even reasonable for Neumann conditions under assumptions for the right hand sides.

Now we can formulate the regularity results for the linear system (4).

Theorem 3.2. [11, 18] *Assume that the conditions* (E) *and* (R) *are satisfied. Let $f_\beta \in L_p(\Omega)$, $h_\beta \in W_p^{1-1/p}(\Gamma_1)$, $g_\beta \in W_p^{2-1/p}(\Gamma_2)$, $\beta = 1, \dots, k$, $p = n + \varepsilon$. Furthermore let $w_1, \dots, w_{l_1}$ be the sequence of all singular functions* (5) *to eigenvalues λ of $\mathcal{A}(P_i)$ with $1 - \frac{n}{p} < \lambda < 2 - \frac{n}{p}$ for every $P_i \in P$. Let us denote by $v_1, \dots, v_{l_2}$ the sequence of all singular functions* (6) *corresponding to the eigenvalue* 1 *of $\mathcal{A}(P_i)$ for every $P_i \in P$. Then the unique weak solution $u \in W_2^1(\Omega)$ of* (4) *allows the decomposition*

$$u = \sum_{j=1}^{l_1} c_j w_j + \sum_{j=1}^{l_2} d_j v_j + \tilde{u} \tag{8}$$

with $\tilde{u} \in W_p^2(\Omega)$.

This theorem motivates the introduction of Sobolev spaces with detached asymptotics. Let $\{z_1,\dots,z_m\}$ be a basis of the finite dimensional linear space generated by the functions $\{w_1,\dots,w_{l_1},v_1,\dots,v_{l_2}\}$. We define

$$D_p^2(\Omega) := \operatorname{span}\langle z_1,\dots,z_m\rangle \otimes W_p^2(\Omega). \tag{9}$$

The norm of a function $u\in D_p^2(\Omega)$ with the decomposition $u=\sum\limits_{j=1}^{m} c_j z_j+\tilde u$ is defined by

$$\|u\|_{D_p^2(\Omega)} = \sum_{j=1}^{m}|c_j| + \|\tilde u\|_{W_p^2(\Omega)}. \tag{10}$$

Lemma 3.3. *The imbedding $D_p^2(\Omega)\to W_p^1(\Omega)$ is continuous.*

P r o o f. Since $z_j\in W_p^1(\Omega)$ and since $W_p^2(\Omega)$ is continously imbedded in $W_p^1(\Omega)$ we obtain for $u=\sum c_j z_j+\tilde u$

$$\begin{aligned}\|u\|_{W_p^1(\Omega)} &\leqslant \sum_{j=1}^{m}|c_j|\|z_j\|_{W_p^1(\Omega)} + \|\tilde u\|_{W_p^1(\Omega)}\\ &\leqslant \max\{1,\|z_1\|_{W_p^1(\Omega)},\dots,\|z_m\|_{W_p^1(\Omega)}\}\Big(\sum_{j=1}^{m}|c_j| + \|\tilde u\|_{W_p^1(\Omega)}\Big)\\ &\leqslant d\|u\|_{D_p^2(\Omega)}.\end{aligned}$$

$\square$

Let the operator L be defined by

$$\begin{aligned}(Lv)_\beta = \big(&-\partial_j\left[a_{ij\alpha\beta}(x)\partial_i v_\alpha\right] + b_{i\alpha\beta}(x,0)\partial_i v_\alpha + \langle\partial_u c_\beta(x,0),v\rangle,\\ &\left[a_{ij\alpha\beta}(x)\partial_i v_\alpha\right]\nu_j|_{\Gamma_1}, v_\beta|_{\Gamma_2}\big).\end{aligned} \tag{11}$$

The Theorem (3.2) can be formulated as

Theorem 3.4. *Let the assumptions of Theorem* 3.2 *be satisfied. Then the operator*

$$L\colon\ D_p^2(\Omega)\to L_p(\Omega)\times W_p^{1-1/p}(\Gamma_1)\times W_p^{2-1/p}(\Gamma_2) \tag{12}$$

defined by (11) *is an isomorphism.*

4. Continuity and Fréchet-differentiability of the semilinear operators

We write the problem (3) in the form of an operator equation

$$Nu = (f, h, g) \tag{13}$$

with $f = (f_1, \dots, f_k), h = (h_1, \dots, h_k), g = (g_1, \dots, g_k)$ and the operator N defined by

$$\begin{aligned}(Nu)_\beta = \big(- \partial_j \left[a_{ij\alpha\beta}(x)\partial_i u_\alpha\right] + b_{i\alpha\beta}(x,u)\partial_i u_\alpha + c_\beta(x,u), \\ \left[a_{ij\alpha\beta}(x)\partial_i u_\alpha\right] \nu_j|_{\Gamma_1}, u_\beta|_{\Gamma_2}\big).\end{aligned} \tag{14}$$

The continuity and the Fréchet differentiability of the operator N follows from the corresponding properties of the composition operator.

Theorem 4.1. (Marcus/Mizel [16], Valent [26]) *Let $p > n$ and $c \in C^2(\overline{\Omega}, \mathbb{R}^k)$. Then the composition operator $C\colon W_p^1(\Omega) \to W_p^1(\Omega)$ defined by $(Cu)(x) = c(x, u(x))$ is continuous and Fréchet differentiable. The Fréchet derivative of C is given by*

$$C'(u)v = \big\langle \partial_u c(x,u), v \big\rangle. \tag{15}$$

Lemma 4.2. *Let $p > n$ and $b \in C^2(\overline{\Omega}, \mathbb{R}^k)$. Then the operators $B_i\colon W_p^1(\Omega) \to L_p(\Omega)$ defined by $(B_i u)(x) = b(x, u(x))\partial_i u$, $i = 1, \dots, n$, are continuous and Fréchet differentiable. The Fréchet derivative of B_i is given by*

$$B_i'(u)v = \big\langle \partial_u b(x,u), v \big\rangle \partial_i u + b(x,u)\partial_i v. \tag{16}$$

P r o o f. If $u \in W_p^1(\Omega)$ then $b(x,u) \in W_p^1(\Omega)$ and $\partial_i u \in L_p(\Omega)$. Therefore we have $b(x, u(x))\partial_i u \in L_p(\Omega)$ and B_i is continuous due to multiplicative properties of Sobolev spaces ([26, Corollary II.2.3]). The relation (16) follows directly from Theorem 4.1 applied to $u \mapsto b(x,u)$ and the Leibniz rule for the Fréchet derivatives. □

Theorem 4.3. *Let $p > n + \varepsilon$. Then the operator $N\colon D_p^2(\Omega) \to L_p(\Omega) \times W_p^{1-1/p}(\Gamma_1) \times W_p^{2-1/p}(\Gamma_2)$ is Fréchet differentiable and its Fréchet derivative at $u = 0$ coincides with the operator L defined by* (11).

P r o o f. We decompose the operator N into the sum $N = N_1 + N_2 + N_3$, where

$$\begin{aligned}(N_1 u)_\beta &= \big(- \partial_j \left[a_{ij\alpha\beta}(x)\partial_i u_\alpha\right], \left[a_{ij\alpha\beta}(x)\partial_i u_\alpha\right] \nu_j|_{\Gamma_1}, u_\beta|_{\Gamma_2}\big),\\ (N_2 u)_\beta &= (b_{i\alpha\beta}(x,u)\partial_i u_\alpha, 0, 0),\\ (N_3 u)_\beta &= (c_\beta(x,u), 0, 0).\end{aligned}$$

The operator N_1 is linear and continuous and therefore $N_1'(0)v = N_1 v$.

Note, if an operator $N\colon X \to Y$ is continuous and Fréchet differentiable at $x_0 \in \widetilde{X}$, where $\widetilde{X}$ is continuously imbedded in X then $N\colon \widetilde{X} \to Y$ is also continuous and Fréchet differentiable at x_0. In our case the space $D^2_p(\Omega)$ can be continuously imbedded into $W^1_p(\Omega)$. Using this fact we conclude from Lemma 4.2 that N_2 is Fréchet differentiable and

$$(N_2'(0)v)_\beta = (b_{i\alpha\beta}(x,0)\partial_i v_\alpha, 0, 0). \tag{17}$$

From Theorem 4.1 it follows that

$$(N_3'(0)v)_\beta = \big(\langle \partial_u c_\beta(x,0), v\rangle, 0, 0\big). \tag{18}$$

Summing up the Fréchet derivatives of N_1, N_2 and N_3 we obtain the assertion. □

5. Asymptotic behaviour of the solution of the semilinear problem

Let us summarize the assumptions which we need in order to apply the Theorem 2.1 to the nonlinear problem (3):

A1. $a_{ij\alpha\beta} \in C^1(\overline{\Omega}), b_{i\alpha\beta}, c_\beta \in C^2(\overline{\Omega} \times \mathbb{R}^n)$.

A2. $c_\beta(x,0) = 0$.

A3. $f_\beta \in L_p(\Omega)$, $h_\beta \in W_p^{1-1/p}(\Gamma_1)$, $g_\beta \in W_p^{2-1/p}(\Gamma_2)$, $\beta = 1,\dots,k$, $p = n + \varepsilon$ with a small positive ε, and the norms of f_β, h_β, g_β are small enough.

A4. The linearized problem (4) is elliptic and has a unique weak solution in $W^1_p(\Omega)$.

Now we can formulate the main result of this paper:

Theorem 5.1. *Let the assumptions* A1–A4 *be satisfied. Then the semilinear problem* (3) *has a unique solution $u \in D^2_p(\Omega)$.*

P r o o f. According to Theorem 4.3 the operator N of the boundary value problem (3) is Fréchet differentiable at $u = 0$ and its Fréchet derivative coincides with the linear operator L defined by (11). On the other hand we know from Theorem 3.4 that the operator L is an isomorphism. Thus the operator N is locally invertible in the neighbourhood of $u = 0 \in D^2_p(\Omega)$. □

Furthermore, using an idea from [4, Theorem 6.8.1] we can estimate the difference between the solution of the semilinear and of the linearized problem.

Theorem 5.2. *Let $u = \sum\limits_{j=1}^{m} c_j z_j + \tilde{u}$ be the solution of the semilinear problem* (3) *and let $u^{\rm lin} = \sum\limits_{j=1}^{m} c_j^{\rm lin} z_j + \tilde{u}^{\rm lin}$ be the solution of the linearized problem* (4). *Then* *the following estimates are valid*

$$\text{(19)}\quad \|u-u^{\mathrm{lin}}\|_{D_p^2(\Omega)} \leqslant d_1\|(f,h,g)\|_{L_p(\Omega)\times W_p^{1-1/p}(\Gamma_1)\times W_p^{2-1/p}(\Gamma_2)},$$

$$\text{(20)}\quad |c_j-c_j^{\mathrm{lin}}| \leqslant d_2\|(f,h,g)\|_{L_p(\Omega)\times W_p^{1-1/p}(\Gamma_1)\times W_p^{2-1/p}(\Gamma_2)},\ j=1,\ldots,m,$$

$$\text{(21)}\quad \|u\|_{D_p^2(\Omega)} \leqslant d_3\|(f,h,g)\|_{L_p(\Omega)\times W_p^{1-1/p}(\Gamma_1)\times W_p^{2-1/p}(\Gamma_2)}$$

with positive real constants d_1, d_2, d_3.

P r o o f. From the local invertibility theorem follows that

$$N(N^{-1}(f,h,g))=(f,h,g) \text{ and } N'(N^{-1}(f,h,g))(N^{-1})'(f,h,g)=I$$

for $(f,h,g)\in L_p(\Omega)\times W_p^{1-1/p}(\Gamma_1)\times W_p^{2-1/p}(\Gamma_2)$ whose norm are small enough. Thus

$$(N^{-1})'(0)=(N')(0)^{-1}=L^{-1}.$$

From the Fréchet differentiability of N^{-1} at 0 we obtain then

$$\lim_{\|(f,h,g)\|\to 0}\frac{\|u-u^{\mathrm{lin}}\|_{D_p^2(\Omega)}}{\|(f,h,g)\|}=\lim_{\|(f,h,g)\|\to 0}\frac{\|N^{-1}(f,h,g)-(N^{-1})'(0)(f,h,g)\|_{D_p^2(\Omega)}}{\|(f,h,g)\|}=0.$$

Thus (19) is proved. The estimate (20) is a direct consequence of (19) and the definition of the norm in $D_p^2(\Omega)$. Furthermore we have

$$\|u\|_{D_p^2(\Omega)} \leqslant \|u^{\mathrm{lin}}\|_{D_p^2(\Omega)}+\|u-u^{\mathrm{lin}}\|_{D_p^2(\Omega)}.$$

Therefore the estimate (21) follows from (19) and the continuity of the inverse operator L^{-1}. $\square$

R e m a r k. The estimates (19) and (20) can be further improved if we can guarantee that the nonlinear mapping N is twice Fréchet differentiable at 0 (see [4, Theorem 6.8.1]).

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Authors' addresses: *M. Bochniak, A.-M. Sändig*, Mathematisches Institut A/6, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany, e-mails: `bochniak@mathematik.uni-stuttgart.de`, `saendig@mathematik.uni-stuttgart.de`.