ON MAXIMAL OVERDETERMINED HARDY'S INEQUALITY OF SECOND ORDER ON A FINITE INTERVAL

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. A characterization of the weighted Hardy inequality

$$\|Fu\|_2 \leqslant C \left\|F''v\right\|_2, \ F(0) = F'(0) = F(1) = F'(1) = 0$$

is given.

Keywords: weighted Hardy's inequality

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Introduction

Let $I = [0,1]$, $1 < p$, $q < \infty$, let $k \geqslant 1$ be an integer and let AC_p^k denote the space of all functions on I with absolutely continuous $(k-1)$-th derivative $F^{(k-1)}(x)$ and such that

$$\|F\|_{AC_p^k} := \|F^{(k)}v\|_p < \infty,$$
$$F(0) = F'(0) = \ldots = F^{(k-1)}(0) = F(1) = \ldots = F^{(k-1)}(1) = 0,$$

where $v(x)$ is a locally integrable weight function and $\left\|g\right\|_p := \left(\int_0^1 |g(x)|^p \,\mathrm{d}x\right)^{1/p}$.

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We consider the characterization problem for the inequality

$$\|Fu\|_q \leqslant C\|F^{(k)}v\|_p,\ F \in AC_p^k. \tag{1}$$

The case $k = 1$ has been solved by P. Gurka [2] (see also [13]) and many works have been performed in this area by A. Kufner [6] and by A. Kufner with co-authors [1], [5], [7–10]. In particular, following Kufner's terminology we call the inequality (1) "maximal overdetermined Hardy's inequality", that is when a function F and its derivatives vanish at both ends of the interval up to $(k-1)$-th order. A part of analysis related to the weighted Hardy inequality for functions vanishing at both ends of an interval was also given by G. Sinnamon [15] and the authors [11], [12]. In particular, the maximal inequality (1) on semiaxis was characterized in [11], [12].

The aim of the present paper is twofold. At first we prove an alternative version of (1) (see Theorem 1) and it allows, using the results of [4], to characterize the inequality (1), when $p = q = 2$, $k = 2$ (Theorem 3).

Without loss of generality we assume throughout the paper that the undeterminates of the form $0 \cdot \infty, 0/0, \infty/\infty$ are equal to zero.

An alternate version

Denote $I_k f(x)$ and $J_k f(x)$ the Riemann-Liouville operators of the form

$$I_k f(x) = \frac{1}{\Gamma(k)} \int_0^x (x-y)^{k-1} f(y)\,\mathrm{d}y,\ x \in I,$$
$$J_k f(x) = \frac{1}{\Gamma(k)} \int_x^1 (y-x)^{k-1} f(y)\,\mathrm{d}y,\ x \in I.$$

Then the maximal inequality (1) is equivalent either to

$$\|(I_k f)u\|_q \leqslant C\|fv\|_p,\ f \in P_{k-1}^{\perp} \tag{2}$$

or to

$$\|(J_k f)u\|_q \leqslant C\|fv\|_p,\ f \in P_{k-1}^{\perp}, \tag{3}$$

where P_{k-1} is the k-dimensional space of all polynomials $\varrho(t) = c_0 + c_1 t + \ldots + c_{k-1}t^{k-1}$, $t \in I$, and $P_{k-1}^{\perp} \subset L_{p,v} := \{f\colon \|fv\|_p < \infty\}$ denotes the closed subspace of $L_{p,v}$ of functions "orthogonal" to P_{k-1} in the sense that

$$\int_0^1 f(x)\varrho(x)\,\mathrm{d}x = 0 \ \text{ for all } \varrho \in P_{k-1},\ f \in P_{k-1}^{\perp}.$$

In particular, $f \in P_{k-1}^{\perp}$ if, and only if,

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 x f(x)\,\mathrm{d}x = \ldots = \int_0^1 x^{k-1} f(x)\,\mathrm{d}x = 0$$

and, obviously,

$$I_k f(x) = J_k f(x),\ \ f \in P_{k-1}^{\perp}.$$

We need the following

Lemma 1. ([14], Chapter 4, Exercise 19). *Let X be a Banach space and $Y \subset X$ the closed subspace. Let X^* be the dual space and*

$$Y^{\perp} = \{\varphi \in X^*\colon\ \varphi(y) = 0 \text{ for all } y \in Y\}.$$

Then

$$\operatorname*{dist}_X(e, Y) := \inf_{y \in Y} \|e - y\|_X = \sup_{\varphi \in Y^{\perp}} \frac{|\varphi(e)|}{\|\varphi\|_{X^*}} \tag{4}$$

for all $e \notin Y$.

Proof. Let $y \in Y$, $\varphi \in Y^{\perp}$. Then

$$\varphi(e) = \varphi(e) - \varphi(y) = \varphi(e - y)$$

and

$$|\varphi(e)| = |\varphi(e - y)| \leqslant \|\varphi\|_{X^*} \|e - y\|.$$

Consequently,

$$\sup_{\varphi \in Y^{\perp}} \frac{|\varphi(e)|}{\|\varphi\|_{X^*}} \leqslant \|e - y\|$$

and

$$\sup_{\varphi \in Y^{\perp}} \frac{|\varphi(e)|}{\|\varphi\|_{X^*}} \leqslant \operatorname*{dist}_X(e, Y). \tag{5}$$

Now suppose $e \notin Y$, $y \in Y$. Then $e - y \notin Y$ and by the Hahn-Banach theorem there exists $\varphi \in X^*$ such that $\varphi(y) = 0$ for all $y \in Y$, $\|\varphi\|_{X^*} = 1$ and $\varphi(e - y) = \|e - y\|$. This implies that $\varphi \in Y^{\perp}$ and

$$|\varphi(e)| = |\varphi(e - y)| = \|e - y\| \geqslant \operatorname*{dist}_X(e, Y).$$

Therefore,

$$\sup_{\varphi \in Y^\perp} \frac{|\varphi(e)|}{\|\varphi\|_{X^*}} \geqslant \operatorname*{dist}_X(e, Y). \tag{6}$$

Combining the estimates (5) and (6) we obtain (4). $\square$

Put

$$M_k(p,q) := \sup_{AC_p^k \ni F \neq 0} \frac{\|Fu\|_q}{\|F^{(k)}v\|_p}.$$

Because of (2) and (3) we have

$$M_k(p,q) = \sup_{f \in P_{k-1}^\perp} \frac{\|(J_k f)u\|_q}{\|fv\|_p} = \sup_{f \in P_{k-1}^\perp} \frac{\|(I_k f)u\|_q}{\|fv\|_p}. \tag{7}$$

Denote $p' = p/(p-1)$ and $q' = q/(q-1)$ for $1 < p, q < \infty$ and observe that $(L_{p,v})^* = L_{p',1/v}$ if and only if $v \in L_{p,\mathrm{loc}}$ and $1/v \in L_{p',\mathrm{loc}}$.

The following result gives an alternative version of the problems to characterize (1), (2), (3) and helps us to realise the desired solution for $p = q = k = 2$.

Theorem 1. *Let $1 < p, q < \infty$ and the weight functions u and v be such that $(L_{p,v})^* = L_{p',1/v}$, $(L_{q,u})^* = L_{q',1/u}$. Then*

$$M_k(p,q) = \sup_{f \in L_{q',1/u}} \|f/u\|_{q'}^{-1} \operatorname*{dist}_{L_{p',1/v}} (I_k f, P_{k-1}). \tag{8}$$

P r o o f. Applying Lemma 1 and the duality of $L_{p,v}$ and $L_{p',1/v}$, $L_{q,u}$ and $L_{q',1/u}$, J_k and I_k, we write

$$\begin{aligned} M_k(p,q) &= \sup_{g \in P_{k-1}^\perp} \frac{\|(J_k g)u\|_q}{\|gv\|_p} \\ &= \sup_{g \in P_{k-1}^\perp} \sup_{f \in L_{q',1/u}} \frac{\left|\int_0^1 (J_k g) f\right|}{\|f/u\|_{q'} \|gv\|_p} \\ &= \sup_{f \in L_{q',1/u}} \|f/u\|_{q'}^{-1} \sup_{g \in P_{k-1}^\perp} \frac{\left|\int_0^1 (I_k f) g\right|}{\|gv\|_p} \\ &= \sup_{f \in L_{q',1/u}} \|f/u\|_{q'}^{-1} \operatorname*{dist}_{L_{p',1/v}} (I_k f, P_{k-1}). \end{aligned}$$

$\square$

R e m a r k. The equality (8) holds for $J_k f$ instead of $I_k f$.

THE CASE $p = 2$

The implicit formulae (8) becomes clearer when $p = 2$. Let $\mathrm{d}\mu(x) = |v(x)|^{-2}\,\mathrm{d}x$ and

$$F_k(x) = I_k(fu)(x) = \frac{1}{\Gamma(k)} \int_0^x (x-y)^{k-1} f(y)u(y)\,\mathrm{d}y.$$

Then

$$\operatorname*{dist}_{L_{2,\mu}}(F_k, P_{k-1}) = \left(\int_I \Big| F_k(x) - F_{k,0} - \sum_{i=1}^{k-1} F_{k,i}\omega_i(x) \Big|^2 \,\mathrm{d}\mu(x) \right)^{1/2},$$

where $L_{2,\mu} = \{f\colon \|f\|_{2,\mu} := \big(\int_0^1 |f|^2\,\mathrm{d}\mu\big)^{1/2} < \infty\}$ and

$$F_{k,0} = \frac{1}{\mu(I)} \int_I F_k \,\mathrm{d}\mu,$$

$$F_{k,i} = \frac{1}{\mu_i(I)} \int_I F_k \omega_i \,\mathrm{d}\mu, \ \ i = 1, \ldots, k-1$$

and polynomials $\{\omega_i(x)\}$, $i = 1, \ldots, k-1$, appear from the Gram-Schmidt orthogonalization process of $\{1, t, \ldots, t^{k-1}\}$ in $L_{2,\mu}$ (see [4], Lemma 2).

Observe, that if $p \neq 2$, $p \in (1, \infty)$ and $k = 1$, then

$$\left(\int_I |F_1 - F_{1,0}|^p \,\mathrm{d}\mu_p \right)^{1/p} \leqslant \operatorname*{dist}_{L_{p,\mu_p}}(F_1, P_0) \leqslant 2 \left(\int_I |F_1 - F_{1,0}|^p \,\mathrm{d}\mu_p \right)^{1/p},$$

(see [3]), where $\mathrm{d}\mu_p(x) = |v(x)|^{-p}\,\mathrm{d}x$.

Thus, for $p = 2$ the characterization problems of (1), (2) and (3) are equivalent to the following Poincaré-type inequality

$$\Big\| F_k - F_{k,0} - \sum_{i=1}^{k-1} F_{k,i}\omega_i \Big\|_{2,\mu} \leqslant C \|f\|_{q'}. \tag{9}$$

The case $k = 2$

We need the following notation. Let $k > 1$, $1 < p, q < \infty$, $1/r = 1/q - 1/p$ if $1 < q < p < \infty$. Put

$$A_{k,0} = A_{k,0;(a,b),u,v} = \begin{cases} \sup\limits_{a<t<b} \left(\int_t^b (x-t)^{q(k-1)} |u(x)|^q \,\mathrm{d}x\right)^{1/q} \left(\int_a^t |v|^{-p'}\right)^{1/p'}, & p \leqslant q \\ \left(\int_a^b \left(\int_t^b (x-t)^{q(k-1)} |u(x)|^q \,\mathrm{d}x\right)^{r/q} \left(\int_a^t |v|^{-p'}\right)^{r/q'} |v(t)|^{-p'} \,\mathrm{d}t\right)^{1/r}, & p > q; \end{cases}$$

$$A_{k,1} = A_{k,1;(a,b),u,v} = \begin{cases} \sup\limits_{a<t<b} \left(\int_t^b |u|^q\right)^{1/q} \left(\int_a^t (t-x)^{p'(k-1)} |v(x)|^{-p'} \,\mathrm{d}x\right)^{1/p'}, & p \leqslant q \\ \left(\int_a^b \left(\int_t^b |u|^q\right)^{r/p} \left(\int_a^t (t-x)^{p'(k-1)} |v(x)|^{-p'} \,\mathrm{d}x\right)^{r/p'} |u(t)|^q \,\mathrm{d}t\right)^{1/r}, & p > q; \end{cases}$$

$$B_{k,0} = B_{k,0;(a,b),u,v} = \begin{cases} \sup\limits_{a<t<b} \left(\int_a^t (t-x)^{q(k-1)} |u(x)|^q \,\mathrm{d}x\right)^{1/q} \left(\int_t^b |v|^{-p'}\right)^{1/p'}, & p \leqslant q \\ \left(\int_a^b \left(\int_a^t (t-x)^{q(k-1)} |u(x)|^q \,\mathrm{d}x\right)^{r/q} \left(\int_t^b |v|^{-p'}\right)^{r/q'} |v(t)|^{-p'} \,\mathrm{d}t\right)^{1/r}, & p > q; \end{cases}$$

$$B_{k,1} = B_{k,1;(a,b),u,v} = \begin{cases} \sup\limits_{a<t<b} \left(\int_a^t |u|^q\right)^{1/q} \left(\int_t^b (x-t)^{p'(k-1)} |v(x)|^{-p'} \,\mathrm{d}x\right)^{1/p'}, & p \leqslant q \\ \left(\int_a^b \left(\int_a^t |u|^q\right)^{r/p} \left(\int_t^b (x-t)^{p'(k-1)} |v(x)|^{-p'} \,\mathrm{d}x\right)^{r/p'} |u(t)|^q \,\mathrm{d}t\right)^{1/r}, & p > q; \end{cases}$$

$$\begin{aligned} A_k &= A_{k;(a,b),u,v} = \max(A_{k,0}, A_{k,1}), \\ B_k &= B_{k;(a,b),u,v} = \max(B_{k,0}, B_{k,1}). \end{aligned}$$

The constants A_k and B_k are equivalent to the norms of the Riemann-Liouville operators I_k and J_k, respectively, from $L_{p,v}(a,b)$ into $L_{q,u}(a,b)$ [16–17].

Theorem 2. *Let $1 < p, q < \infty$, $k = 2$ and let the hypothesis of Theorem 1 be fulfilled. Then*

(10)
$$\begin{aligned} M_2(p,q) \leqslant \inf_{0<\tau<\lambda<\sigma<1} \big(&A_{2;(0,\tau),u,v} + A_{1;(\tau,\lambda),u,(x-\tau)^{-1}v(x)} + B_{1;(\tau,\lambda),(x-\tau)u(x),v} \\ &+ D^*_{\tau,\lambda} + D_{\tau,\lambda} + B_{2;(\sigma,1),u,v} + A_{1;(\lambda,\sigma),(\sigma-x)u(x),v} \\ &+ B_{1;(\lambda,\sigma),u,(\sigma-x)^{-1}v(x)} + D_{\lambda,\sigma} + D^*_{\lambda,\sigma}\big), \end{aligned}$$

where

$$D_{\tau,\lambda}=\left(\int_\tau^\lambda |u|^q\right)^{1/q}\left(\int_0^\tau (\tau-x)^{p'}|v(x)|^{-p'}\,\mathrm{d}x\right)^{1/p'},$$
$$D_{\lambda,\sigma}=\left(\int_\lambda^\sigma (\sigma-x)^q|u(x)|^q\,\mathrm{d}x\right)^{1/q}\left(\int_0^\lambda |v|^{-p'}\right)^{1/p'},$$
$$D^*_{\tau,\lambda}=\left(\int_\tau^\lambda (x-\tau)^q|u(x)|^q\,\mathrm{d}x\right)^{1/q}\left(\int_\lambda^1 |v|^{-p'}\right)^{1/p'},$$
$$D^*_{\lambda,\sigma}=\left(\int_\lambda^\sigma |u|^q\right)^{1/q}\left(\int_\sigma^1 (x-\sigma)^{p'}|v(x)|^{-p'}\,\mathrm{d}x\right)^{1/p'}.$$

P r o o f. If $f\in P_1^\perp$, then for all $x\in[0,1]$ we have

$$I_2f(x)=J_2f(x). \tag{11}$$

Let $\lambda\in(0,1)$ and for any $\tau\in(0,\lambda)$ and $x\in(\tau,\lambda)$ we find

$$\begin{aligned}I_2f(x)&=\int_0^x\left(\int_0^s f\right)\mathrm{d}s=\int_0^\tau\left(\int_0^s f\right)\mathrm{d}s+\int_\tau^x\left(\int_0^s f\right)\mathrm{d}s\\
&=\int_0^\tau(\tau-y)f(y)\,\mathrm{d}y-\int_\tau^x\left(\int_s^1 f\right)\mathrm{d}s\\
&=\int_0^\tau(\tau-y)f(y)\,\mathrm{d}y-\int_\tau^x f(y)\left(\int_\tau^y \mathrm{d}s\right)\mathrm{d}y\\
&\quad-\int_x^\lambda f(y)\left(\int_\tau^x \mathrm{d}s\right)\mathrm{d}y-\int_\lambda^1 f(y)\left(\int_\tau^x \mathrm{d}s\right)\mathrm{d}y\\
&=\int_0^\tau(\tau-y)f(y)\,\mathrm{d}y-\int_\tau^x(y-\tau)f(y)\,\mathrm{d}y\\
&\quad-(x-\tau)\int_x^\lambda f-(x-\tau)\int_\lambda^1 f.\end{aligned}$$

Analogously, with $\sigma\in(\lambda,1)$ for $x\in(\lambda,\sigma)$ we write

$$\begin{aligned}I_2f(x)=J_2f(x)&=\int_x^1\left(\int_s^1 f\right)\mathrm{d}s\\
&=\int_\sigma^1\left(\int_s^1 f\right)\mathrm{d}s+\int_x^\sigma\left(\int_s^1 f\right)\mathrm{d}s\\
&=\int_\sigma^1(y-\sigma)f(y)\,\mathrm{d}y-\int_x^\sigma(\sigma-y)f(y)\,\mathrm{d}y\\
&\quad-(\sigma-x)\int_\lambda^x f-(\sigma-x)\int_0^\lambda f.\end{aligned}$$

Now we estimate the norm of each term on the right hand side. Using [16–17] we obtain

$$\left\|\chi_{[0,\tau]}\,(I_2 f)\,u\right\|_q \leqslant A_{2;(0,\tau),u,v}\left\|\chi_{[0,\tau]} f v\right\|_p \leqslant A_{2;(0,\tau),u,v}\left\|f v\right\|_p.$$

Plainly

$$\begin{aligned}\left\|\chi_{[\tau,\lambda]}\,(I_2 f)\,u\right\|_q &\leqslant \left\|\chi_{[\tau,\lambda]}(x)u(x)\int_0^\tau (\tau-y)f(y)\,\mathrm{d}y\right\|_q \\ &\quad+\left\|\chi_{[\tau,\lambda]}(x)u(x)\int_\tau^x (y-\tau)f(y)\,\mathrm{d}y\right\|_q+\left\|\chi_{[\tau,\lambda]}(x)u(x)(x-\tau)\int_x^\lambda f\right\|_q \\ &\quad+\left\|\chi_{[\tau,\lambda]}(x)u(x)(x-\tau)\int_\lambda^1 f\right\|_q\end{aligned}$$

(we use the Hölder inequality for the first and the fourth term and the upper estimates which follow from the weighted Hardy inequalities [13] for the second and the third term)

$$\leqslant \left(D_{\tau,\lambda}+A_{1;(\tau,\lambda),u,(x-\tau)^{-1}v(x)}+B_{1;(\tau,\lambda),(x-\tau)u(x),v}+D^*_{\tau,\lambda}\right)\left\|f v\right\|_p.$$

Similarly, applying (11),

$$\begin{aligned}&\left\|\chi_{[\lambda,\sigma]}\,(I_2 f)\,u\right\|_q \\ &\qquad\leqslant \left(D^*_{\lambda,\sigma}+B_{1;(\lambda,\sigma),u,(\sigma-x)^{-1}v(x)}+A_{1;(\lambda,\sigma),(\sigma-x)u(x),v}+D_{\lambda,\sigma}\right)\left\|f v\right\|_p. \\ &\left\|\chi_{[\sigma,1]}\,(I_2 f)\,u\right\|_q=\left\|\chi_{[\sigma,1]}\,(J_2 f)\,u\right\|_q\leqslant B_{2;(\sigma,1),u,v}\left\|f v\right\|_p.\end{aligned}$$

Finally we obtain

$$\begin{aligned}\left\|\,(I_2 f)\,u\right\|_q &\leqslant \left\|\chi_{[0,\tau]}\,(I_2 f)\,u\right\|_q+\left\|\chi_{[\tau,\lambda]}\,(I_2 f)\,u\right\|_q \\ &\quad+\left\|\chi_{[\lambda,\sigma]}\,(I_2 f)\,u\right\|_q+\left\|\chi_{[\sigma,1]}\,(I_2 f)\,u\right\|_q \\ &\leqslant \big(A_{2;(0,\tau),u,v}+D_{\tau,\lambda}+A_{1;(\tau,\lambda),u,(x-\tau)^{-1}v(x)}+B_{1;(\tau,\lambda),(x-\tau)u(x),v} \\ &\quad+D^*_{\tau,\lambda}+D^*_{\lambda,\sigma}+B_{1;(\lambda,\sigma),u,(\sigma-x)^{-1}v(x)} \\ &\quad+A_{1;(\lambda,\sigma),(\sigma-x)u(x),v}+D_{\lambda,\sigma}+B_{2;(\sigma,1),u,v}\big)\left\|f v\right\|_p.\end{aligned}$$

Since τ, λ and σ were arbitrary the upper bound (10) of $M_2(p,q)$ follows. □

R e m a r k . Theorem 2 gives the upper bound for $M_k(p,q)$, when $k=2$. Obviously the similar upper estimates can be proved by the same method for $k>2$. We omit the details.

Denote $\mathcal{E}$ the right hand side of (10) when $p=q=2$. The following result brings the characterization of (1) for $p=q=k=2$.

Theorem 3. *Let the hypothesis of Theorem* 1 *be fulfilled for* $p = q = 2$. *Then*

$$\frac{1}{40}\kappa\mathcal{E} \leqslant M_2(2,2) \leqslant \mathcal{E}, \tag{12}$$

where $\kappa = \kappa(v)$.

P r o o f. The upper bound is an immediate corollary of Theorem 2. To prove the lower bound we use Theorem 1 and the arguments from Lemma 7 [4]. Let

$$\mathrm{d}\mu(x) = |v(x)|^{-2}\,\mathrm{d}x; \qquad \mu(I) = \int_I \mathrm{d}\mu(y);$$

$$\omega(x) = \int_I (x-y)\,\mathrm{d}\mu(y);\ \ \mathrm{d}\mu_1(x) = |\omega(x)|^2\,\mathrm{d}\mu(x);\ \ \mu_1(I) = \int_I \mathrm{d}\mu_1(y).$$

If we take the point $\lambda \in I$ such that $\omega(\lambda) = 0$ and choose τ, σ so that

$$0 < \tau < \lambda < \sigma < 1,\ \mu(0,\tau) = \mu(\tau,\lambda) \text{ and } \mu(\lambda,\sigma) = \mu(\sigma,b),$$

then there exist positive numbers $\delta_i = \delta_i(v) \in (0,1)$, $i = 1,\dots,5$ for which

$$\mu(0,\lambda) = \delta_1\mu(I),\ \mu_1(\tau,\lambda) = \delta_2\mu_1(I),\ \mu_1(\lambda,\sigma) = \delta_3\mu_1(I),$$

$$\int_0^\tau (\tau-s)^2\,\mathrm{d}\mu(s) = \delta_4\frac{\mu_1(I)}{\mu(I)^2},$$

$$\int_\sigma^1 (s-\sigma)^2\,\mathrm{d}\mu(s) = \delta_5\frac{\mu_1(I)}{\mu(I)^2}.$$

Set $\delta = \min\limits_i \delta_i$ and $\kappa = (\delta)^{3/2}$. Then Lemma 7 [4] gives us the required lower bound $M_2(2,2) \geqslant \frac{1}{40}\kappa\mathcal{E}$. □

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