ON SOME GEOMETRIC PROPERTIES OF CERTAIN KÖTHE SEQUENCE SPACES

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. It is proved that if a Köthe sequence space X is monotone complete and has the weakly convergent sequence coefficient $\mathrm{WCS}(X) > 1$, then X is order continuous. It is shown that a weakly sequentially complete Köthe sequence space X is compactly locally uniformly rotund if and only if the norm in X is equi-absolutely continuous. The dual of the product space $(\bigoplus_{i=1}^{\infty} X_i)_\Phi$ of a sequence of Banach spaces $(X_i)_{i=1}^{\infty}$, which is built by using an Orlicz function Φ satisfying the Δ_2-condition, is computed isometrically (i.e. the exact norm in the dual is calculated). It is also shown that for any Orlicz function Φ and any finite system $X_1, \ldots, X_n$ of Banach spaces, we have $\mathrm{WCS}((\bigoplus_{i=1}^{n} X_i)_\Phi) = \min\{\mathrm{WCS}(X_i)\colon i = 1, \ldots, n\}$ and that if Φ does not satisfy the Δ_2-condition, then $\mathrm{WCS}((\bigoplus_{i=1}^{\infty} X_i)_\Phi) = 1$ for any infinite sequence (X_i) of Banach spaces.

Keywords: Köthe sequence space, weakly convergent sequence coefficient, order continuity of the norm, absolute continuity of the norm, compact local uniform rotundity, Orlicz sequence space, Luxemburg norm, Orlicz norm, dual space, product space

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1. Introduction

Let X be a real Banach space and X^* be its dual. Let $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X, respectively. Let l^0 stand for the space of all real sequences and $\mathbb{N}$ and $\mathbb{R}$ stand for the set of natural numbers and the set of reals, respectively.

A Banach space $X = (X, \|\cdot\|)$ is said to be a *Köthe sequence space* if X is a subspace of l^0 such that (see [9] and [12]):
(i) If $x \in l^0$, $y \in X$ and $|x(i)| \leqslant |y(i)|$ for all $i \in \mathbb{N}$, then $x \in X$ and $\|x\| \leqslant \|y\|$.
(ii) There is an element $x \in X$ such that $x(i) > 0$ for all $i \in \mathbb{N}$.

An element $x \in X$ is said to have *absolutely continuous* norm if $\lim_{n\to\infty} \|x - x^{(n)}\| = 0$, where $x^{(n)} = (x(1), \ldots, x(n), 0, \ldots)$. We say that $x \in X$ is *order continuous* if for any sequence $\{x_n\}$ in X such that $0 \swarrow x_n(i) \leqslant |x(i)|$ for each $i \in \mathbb{N}$ ($0 \swarrow x_n \leqslant |x|$ for short) there holds $\|x_n\| \to 0$. The set of all order continuous elements in X is denoted by X_a. A Köthe sequence space X is said to be order continuous (OC for short) if $X_a = X$ (see [9]). Note that absolute continuity of the norm can be defined in any normed sequence space. A Köthe sequence space X is said to be *monotone complete* if $0 \leqslant x_n(i) \nearrow x(i)$ for any $i \in \mathbb{N}$ implies $\|x_n\| \nearrow \|x\|$.

For any sequence $\{x_n\}$ in X, we define

$$\begin{aligned} A(\{x_n\}) &= \lim_{n\to\infty} \bigl[\sup\{\|x_i - x_j\|\colon\ i, j \geqslant n, i \neq j\}\bigr] \\ A_1(\{x_n\}) &= \lim_{n\to\infty} \bigl[\inf\{\|x_i - x_j\|\colon\ i, j \geqslant n, i \neq j\}\bigr]. \end{aligned}$$

The weakly convergent sequence coefficient of X is defined by

$$\begin{aligned} \mathrm{WCS}(X) = \sup\bigl\{k > 0\colon & \text{ for each weakly convergent sequence } \{x_n\} \text{ in } X \text{ there is} \\ & y \in \mathrm{co}(\{x_n\}) \text{ such that } k \limsup_{n\to\infty} \|x_n - y\| \leqslant A(\{x_n\})\bigr\}, \end{aligned}$$

where $\mathrm{co}(\{x_n\})$ denotes the convex hull of the elements of $\{x_n\}$ (see [3]).

For any $x^* \in X^*$ and $x \in X$, the value of x^* at x is denoted by $x^*(x)$ or $\langle x, x^*\rangle$. To indicate that a sequence $\{x_n\}$ in X tends weakly to $x \in X$, we write $x_n \xrightarrow{w} x$.

The notion of normal structure was introduced by Brodskij and Milman in [2]. It is well known that Banach spaces with normal structure have the weak fixed point property (see [1], [3], [7] and [15]). It is also known (see [3]) that reflexive Banach spaces X with $\mathrm{WCS}(X) > 1$ have normal structure. Banach spaces X with $\mathrm{WCS}(X) > 1$ are said to have weakly uniformly normal structure (see [1] and [5]).

Zhang [21] has defined a sequence $\{x_n\}$ in X to be an asymptotic equidistant sequence if $A(\{x_n\}) = A_1(\{x_n\})$ and he has proved that

$$\begin{aligned} \mathrm{WCS}(X) = \inf\bigl\{A(\{x_n\})\colon & \ \{x_n\} \text{ is an asymptotic equidistant sequence in } S(X) \\ & \text{which is weakly convergent to zero}\bigr\}. \end{aligned}$$

Next, Prus has simplified this formula in [18], Corollary 1.4, proving that

$$\mathrm{WCS}(X) = \inf\bigl\{A(\{x_n\})\colon\ (x_n) \subset S(X) \text{ and } x_n \to 0 \text{ weakly}\bigr\}.$$

Recall (see [4] and [12]) that a Banach space X is said to be locally uniformly rotund (LUR for short) if for any $x \in S(X)$ and any $\{x_n\}$ in $S(X)$ the condition $\|x_n + x\| \to 2$ implies $\|x_n - x\| \to 0$.

We say a Banach space X is compactly locally uniformly rotund (CLUR for short) if for any $x \in S(X)$ and $\{x_n\}$ in $S(X)$ the condition $\|x_n + x\| \to 2$ implies that $\{x_n\}$ is compact in $S(X)$ (see [17]).

A Köthe sequence space X is said to have equi-absolutely continuous norm if for any $x \in S(X)$ and any $\{x_n\}$ in $S(X)$ such that $\|x_n + x\| \to 2$ we have that for any $\varepsilon > 0$ there is $j \in \mathbb{N}$ such that $\|x_n - x_n^{(j)}\| < \varepsilon$ for every $n \in \mathbb{N}$.

A sequence $\{x_n\}$ in a Banach space X is called a *Schauder basis* of X (or *basis* for short) if for each $x \in X$ there exists a unique sequence $\{a_n\}$ of scalars such that $x = \sum_{n=1}^{\infty} a_n x_n$. A basis $\{x_n\}$ of X is said to be an *unconditional basis* if every convergent series $\sum_{n=1}^{\infty} a_n x_n$ with $a_n \in \mathbb{R}$ is unconditionally convergent, i.e. for any permutation $\{\pi(n)\}$ of $\mathbb{N}$ the series $\sum_{n=1}^{\infty} a_{\pi(n)} x_{\pi(n)}$ converges (see [12]).

The basic constant of the basis $\{x_n\}$ of X is defined by $K = \sup_n \|P_n\|$, where $P_n\colon X \to X$ are the projections, i.e. $P_n(\sum_{i=1}^{\infty} a_i x_i) = \sum_{i=1}^{n} a_i x_i$ (see [12], Chapter 1).

If $\{x_n\}$ is a basis of a Banach space X such that the series $\sum_{i=1}^{\infty} a_n x_n$ converges whenever $\{a_n\}$ is a sequence of reals such that $\sup_n \| \sum_{i=1}^{n} a_i x_i\| < \infty$, then $\{x_n\}$ is called a *boundedly complete* basis of X (see [12]).

It is known (see [12], Chapter 1) that $\{x_n\}$ is a boundedly complete basis of a Banach space X if and only if $\{x_n\}$ is an unconditional basis of X and X is weakly sequentially complete.

Recall that X is said to be weakly sequentially complete if for any sequence $\{y_i\}_{i=1}^{\infty} \subset X$ such that $\lim_i x^*(y_i)$ exists for every $x^* \in X^*$ there is $y \in X$ such that $x^*(y) = \lim_i x^*(y_i)$ for every $x^* \in X^*$ (see [12], Chapter 1).

A mapping $\Phi\colon \mathbb{R} \to \mathbb{R}$ is said to be an *Orlicz function* if Φ vanishes only at zero, Φ is even and convex. An Orlicz function Φ is said to be an N-*function* if $\lim_{u\to 0}(\Phi(u)/u) = 0$ and $\lim_{u\to\infty}(\Phi(u)/u) = \infty$. For any Orlicz function Φ, we define the Orlicz sequence space

$$l^{\Phi} = \Big\{x \in l^0\colon\ I_{\Phi}(cx) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \Phi(cx(i)) < \infty \text{ for some } c > 0\Big\}.$$

We also define the subspace h^{Φ} of l^{Φ} by

$$h^{\Phi} = \{x \in l^0\colon\ I_{\Phi}(cx) < \infty \text{ for any } c > 0\}.$$

It is well known that $(l^{\Phi})_a = h^{\Phi}$ (see [6]). We consider l^{Φ} and h^{Φ} equipped with the Luxemburg norm

$$\|x\|_{\Phi} = \inf\{k > 0\colon\ I_{\Phi}(\tfrac{x}{k}) \leqslant 1\}$$

as well as with the Orlicz norm

$$\|x\|_{\Phi}^{\mathrm{O}} = \sup\Big\{\sum_{i=1}^{\infty} x_i y_i\colon\ I_{\Psi}(y) \leqslant 1\Big\},$$

where Ψ is the Orlicz function complementary to Φ in the sense of Young, i.e. $\Psi(u) = \sup\limits_{v\geqslant 0}\{|u|v - \Phi(v)\}$. The Amemiya formula for the Orlicz norm in l^Φ generated by an N-function Φ is the following

$$\|x\|_\Phi^{\mathrm{O}} = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx))$$

(see [4], [13], [14], [16] and [19]). The spaces l^Φ and h^Φ are Banach spaces under both these norms (see [4] and [13]). They are of course Köthe sequence spaces. We write for simplicity l^Φ in place of $(l^\Phi, \|\cdot\|_\Phi)$ and l^Φ_{O} in place of $(l^\Phi, \|\cdot\|^{\mathrm{O}}_\Phi)$.

We say an Orlicz function satisfies the Δ_2-condition ($\Phi \in \Delta_2$ for short) if there exist $K \geqslant 2$ and $u_0 > 0$ such that $\Phi(2u) \leqslant K\Phi(u)$ whenever $|u| \leqslant u_0$. It is known that $l^\Phi = h^\Phi$ if and only if $\Phi \in \Delta_2$ (see [4], [16] and [19]).

If $(X_i)_{i=1}^\infty$ is a sequence of Banach spaces equipped with the norms $\|\cdot\|_i$ (respectively) and Φ is any Orlicz function, we consider the Cartesian product $(\bigoplus_{i=1}^\infty X_i)_\Phi$ equipped with the norm

$$\|x\|_\Phi = \inf\left\{k > 0\colon \sum_{i=1}^\infty \Phi(\|x_i\|_i/k) \leqslant 1\right\}$$

for any $x = (x_i)_{i=1}^\infty$ with $x_i \in X_i$, $i \in \mathbb{N}$. The finite product $(\bigoplus_{i=1}^n X_i)_\Phi$ of Banach spaces $X_1, \ldots, X_n$ is defined analogously.

2. Results

Remark 1. A Köthe sequence space X is order continuous if and only if it is absolutely continuous.

Proof. Assume that X is order continuous and $x \in X$. Then $|x| \geqslant |x - x^{(n)}| \searrow 0$, whence $\|x - x^{(n)}\| \searrow 0$ as $n \to \infty$, i.e. X is absolutely continuous.

Assume now that X is absolutely continuous and $0 \swarrow x_n \leqslant |x|$ where $x_n, x \in X$ for $n = 1, 2, \ldots$. We need to show that $\|x_n\| \to 0$. Take an arbitrary $\varepsilon > 0$. Let $j \in \mathbb{N}$ be such that $\|x - x^{(j)}\| < \frac{\varepsilon}{2}$. Hence, the inequality $x_n \leqslant x$ yields $\|\sum_{i=j+1}^\infty x_n(i)e_i\| < \frac{\varepsilon}{2}$ for any $n \in \mathbb{N}$. Since $x_n \to 0$ coordinatewise, there is $n_\varepsilon \in \mathbb{N}$ such that $\|\sum_{i=1}^j x_n(i)e_i\| < \frac{\varepsilon}{2}$ for all $n > n_\varepsilon$. Consequently, we obtain for $n > n_\varepsilon$,

$$\|x_n\| = \left\|\sum_{i=1}^\infty x_n(i)e_i\right\| \leqslant \left\|\sum_{i=1}^j x_n(i)e_i\right\| + \left\|\sum_{i=j+1}^\infty x_n(i)e_i\right\| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

This finishes the proof. $\square$

Theorem 1. *If X is a monotone complete Köthe sequence space and the inequality* $\mathrm{WCS}(X) > 1$ *holds, then X is order continuous.*

P r o o f. If X is not order continuous, then X_a is a closed proper subspace of X. By Riesz's Lemma (see [20], p. 64), for any $\theta \in (0,1)$ there is $x_0 \in S(X)$ such that $\|x_0 - x\| \geqslant \theta$ for any $x \in X_a$. Hence, in virtue of the monotone completeness of X there is a sequence $\{n_i\}$ of natural numbers such that $n_i \nearrow \infty$ and

$$\Big\| \sum_{n_i+1}^{n_{i+1}} x_0(j)e(j) \Big\| \geqslant \Big(1 - \tfrac{1}{i+1}\Big)\,\theta.$$

Define

$$x_i = \sum_{n_i+1}^{n_{i+1}} x_0(j)e(j) \quad (i = 1, 2, \ldots).$$

It is obvious that

$$\Big(1 - \tfrac{1}{i+1}\Big)\,\theta \leqslant \|x_i\| \leqslant 1 \quad (i = 1, 2, \ldots). \tag{1}$$

We will show that

$$x_i \xrightarrow{w} 0. \tag{2}$$

We may assume without loss of generality that $x_0 \geqslant 0$. Take any nonnegative functional $x^* \in X^*$. Then we have for any $n \in \mathbb{N}$

$$\sum_{i=1}^{n} x^*(x_i) = x^*\Big(\sum_{i=1}^{n} x_i\Big) \leqslant x^*(x_0) < \infty,$$

whence the series $\sum_{i=1}^{\infty} x^*(x_i)$ converges and consequently $x^*(x_i) \to 0$ as $i \to \infty$. Therefore, by the fact that every $x^* \in X^*$ can be written as a difference of two positive functionals, condition (2) is proved.

It is obvious that $\|x_i - x_k\| \leqslant \|x_0\| = 1$. Define $y_i = x_i/\|x_i\|$ for $i = 1, 2, \ldots$. Take any $i, k \in \mathbb{N}$ and assume without loss of generality that $\|x_i\| \leqslant \|x_k\|$. Then

$$\begin{aligned} 1 \leqslant \|y_i - y_k\| &= \Big\| \frac{x_i}{\|x_i\|} - \frac{x_k}{\|x_k\|} \Big\| = \frac{1}{\|x_i\|} \Big\| x_i - \frac{\|x_i\|}{\|x_k\|} x_k \Big\| \\ &\leqslant \frac{1}{\|x_i\|}\|x_i - x_k\| \leqslant \frac{\|x_0\|}{\|x_i\|} \leqslant \frac{1}{(1 - \frac{1}{i+1})\theta} \to \frac{1}{\theta}. \end{aligned}$$

By the arbitrariness of θ in $(0,1)$ and the Prus formula for $\mathrm{WCS}(X)$, this yields $\mathrm{WCS}(X) = 1$. □

Theorem 2. *If X is a weakly sequentially complete Köthe sequence space, then X is CLUR if and only if the norm in X is equi-absolutely continuous.*

Proof. *Necessity.* We will show first that X is not CLUR whenever X is not OC. If X is not OC, there exists in X_+ a sequence (x_n) with pairwise disjoint supports and with $\|x_n\| = 1$ for all $n \in \mathbb{N}$, and $x \in X_+$ such that $x_n \leqslant x$ for all $n \in \mathbb{N}$. Let $x^* \in X^*$ and $x^* \geqslant 0$. Then we have for any $k \in \mathbb{N}$,

$$\sum_{n=1}^{k} x^*(x_n) = x^*\left(\sum_{n=1}^{k} x_n\right) \leqslant x^*(x) < \infty,$$

whence the series $\sum_{n=1}^{\infty} x^*(x_n)$ converges. Thus $x^*(x_n) \to 0$. Therefore $x_n \xrightarrow{w} 0$ (see the argumentation after (2) in the proof of Theorem 1). Define

$$y = \sup_n x_n = \sum_{n=1}^{\infty} x_n, \quad y_n = y - x_n.$$

Then $0 \leqslant y_n \leqslant y$ and $y_n \xrightarrow{w} y$. By the lower semicontinuity of the norm $\|\cdot\|$ with respect to the weak convergence, we have $\|y_n\| \to \|y\|$. Thus, defining

$$z_n = \frac{y_n}{\|y_n\|}, \quad z = \frac{y}{\|y\|},$$

we have $\|z_n\| = \|z\| = 1$ and

$$\begin{aligned} 0 \leqslant \left| \|z + z_n\| - \left\| \frac{y_n + y}{\|y\|} \right\| \right| &\leqslant \left\| z_n - \frac{y_n}{\|y\|} \right\| + \left\| z - \frac{y}{\|y\|} \right\| \\ = \left\| \frac{y_n}{\|y_n\|} - \frac{y_n}{\|y\|} \right\| &= \left| \frac{1}{\|y_n\|} - \frac{1}{\|y\|} \right| \|y_n\| \to 0. \end{aligned} \tag{3}$$

Moreover, $(y_n + y)/\|y\| \xrightarrow{w} 2y/\|y\|$ and $(y_n + y)/\|y\| \leqslant 2y/\|y\|$, whence

$$\frac{\|y_n + y\|}{\|y\|} \to 2 \left\| \frac{y}{\|y\|} \right\| = 2. \tag{4}$$

Combining (3) and (4), we get $\|z + z_n\| \to 2$. On the other hand, since the elements of the sequence (z_n) are pairwise orthogonal,

$$\|z_m - z_n\| \geqslant \max(\|z_m\|, \|z_n\|) = 1$$

for every $m, n \in \mathbb{N}$, $m \neq n$. Therefore, (z_n) has no norm-convergent subsequence in X, i.e. X is not CLUR.

Finally, we will prove that the norm in X is equi-absolutely continuous whenever X is CLUR. Otherwise, there are $x, x_n \in S(X)$ $(n = 1, 2 \ldots)$ and $\varepsilon_0 > 0$ such that $\|x + x_n\| \to 2$ and for any $j \in \mathbb{N}$ there is $n_j > j$ such that

$$\Big\| \sum_{i>j} x_{n_j}(i) e_i \Big\| \geqslant \varepsilon_0. \tag{5}$$

Since $\|x_{n_j} + x\| \to 2$ as $j \to \infty$ and X is CLUR, there is a subsequence $\{x'_{n_j}\}$ of $\{x_{n_j}\}$ and $x' \in S(X)$ such that

$$\|x'_{n_j} - x'\| \to 0 \quad \text{as } j \to \infty. \tag{6}$$

Since X is OC (as it has been shown at the beginning of the proof), there is $i_0 \in \mathbb{N}$ such that

$$\Big\| \sum_{i>i_0} x'(i) e_i \Big\| < \tfrac{\varepsilon_0}{3}.$$

By (6) there is $j_1 \in \mathbb{N}$ such that $\|x'_{n_j} - x'\| < \frac{\varepsilon_0}{3}$ for any $j > j_1$. Also by (6), $x'_{n_j} \to x'$ coordinatewise. Therefore, there is $j_2 > j_1$ such that

$$\Big\| \sum_{i=1}^{i_0} (x'_{n_j} - x') e_i \Big\| < \tfrac{\varepsilon_0}{3} \quad \text{for any } j > j_2.$$

Hence in virtue of (5), we get

$$\begin{aligned}
\tfrac{1}{3}\varepsilon_0 > \|x'_{n_j} - x'\| &= \Big\| \sum_{i=1}^{i_0} (x'_{n_j}(i) - x'(i)) e_i + \sum_{i>i_0} x'_{n_j}(i) e_i - \sum_{i>i_0} x'(i) e_i \Big\| \\
&\geqslant \Big\| \sum_{i>i_0} x'_{n_j}(i) e_i \Big\| - \Big\| \sum_{i=1}^{i_0} (x'_{n_j}(i) - x'(i)) e_i \Big\| - \Big\| \sum_{i>i_0} x'(i) e_i \Big\| \\
&\geqslant \Big\| \sum_{i>i_0} x'_{n_j}(i) e_i \Big\| - \tfrac{2}{3}\varepsilon_0 \geqslant \tfrac{1}{3}\varepsilon_0
\end{aligned}$$

when j is large enough. This contradiction finishes the proof of the necessity.

Sufficiency. We will show first that if X is OC, then $\{e_i\}$ is an unconditional basis in X. For any $x \in X$ let $x^{(n)} = \sum_{i=1}^{n} x(i) e_i$ as above. Since X is OC, Remark 1 yields that X is absolutely continuous. So, given any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\Big\| \sum_{i>n_0} x(i) e_i \Big\| < \varepsilon.$$

Hence

$$\|x - x^{(n)}\| = \left\| \sum_{i=n+1}^{\infty} x(i)e_i \right\| \leqslant \left\| \sum_{i>n_0} x(i)e_i \right\| < \varepsilon$$

for any $n > n_0$, which means that $x = \lim_{n\to\infty} x^{(n)}$, i.e. $\{e_i\}$ is a basis of X.

Moreover, given any $\varepsilon > 0$, there is $i(\varepsilon) \in \mathbb{N}$ such that $\| \sum_{i=i_1}^{i_2} x(i)e_i\| < \varepsilon$ for all $i_1, i_2 \geqslant i(\varepsilon)$. So, for any permutation $(\theta_i)_{i=1}^{\infty}$ of natural numbers, the inequality $\theta_i \geqslant i$ for all $i \in \mathbb{N}$ yields $\| \sum_{i=i_1}^{i_2} x(\theta_i)e_{\theta_i}\| < \varepsilon$ for all $i_1, i_2 \geqslant i(\varepsilon)$, which means that the sequence $s_j = \sum_{i=1}^{j} x(\theta_i)e_{\theta_i}$ is a Cauchy sequence, whence it follows that the series $\sum_{i=1}^{\infty} x(\theta_i)e_{\theta_i}$ is convergent. Therefore $\{e_i\}$ is an unconditional basis of X.

Let the norm in X be equi-absolutely continuous, $x \in S(X)$, (x_n) be a sequence in S(X) and $\|x_n + x\| \to 2$. Since equi-absolute continuity of X implies OC of X, $\{e_i\}$ is an unconditional basis of X. If K is the basic constant, we have for any $i \in \mathbb{N}$,

$$|x_n(i)|\|e_i\| = \left\| \sum_{k=1}^{i} x_n(k)e_k - \sum_{k=1}^{i-1} x_n(k)e_k \right\| \leqslant K\|x_n\| + K\|x_n\| = 2K.$$

Therefore, using the diagonal method, one can find a sequence $a = a(i)_{i=1}^{\infty} \in l^0$ and a subsequence $\{x'_n\}$ of $\{x_n\}$ such that $x'_n(i) \to a(i)$ as $n \to \infty$ for every $i \in \mathbb{N}$ and $\sup_n \| \sum_{i=1}^{n} a(i)e_i\| < \infty$. Since $\{e_i\}$ is an unconditional basis of X, by the assumption that X is weakly sequentially complete, $\{e_i\}$ is a boundedly complete basis of X. So the series $\sum_{i=1}^{\infty} a(i)e_i$ converges in X. Denote $a = \sum_{i=1}^{\infty} a(i)e_i$. Given any $\varepsilon > 0$, there is $i' \in \mathbb{N}$ such that $\| \sum_{i>i'} a(i)e_i\| < \frac{\varepsilon}{3}$. Next, by the equi-absolute continuity of the norm, there exists $i_0 \in \mathbb{N}$, $i_0 > i'$, such that $\| \sum_{i>i_0} x'_n(i)e_i\| < \frac{\varepsilon}{3}$. Since $x'_n \to a$ coordinatewise, for $n \in \mathbb{N}$ large enough there holds

$$\begin{aligned}\|x'_n - a\| &= \left\| \sum_{i=1}^{\infty}(x'_n(i) - a(i))e_i \right\| = \left\| \sum_{i\leqslant i_0}(x'_n(i) - a(i))e_i + \sum_{i>i_0}(x'_n(i) - a(i))e_i \right\| \\ &\leqslant \left\| \sum_{i\leqslant i_0}(x'_n(i) - a(i))e_i \right\| + \left\| \sum_{i>i_0} x'_n(i)e_i \right\| + \left\| \sum_{i>i_0} a(i)e_i \right\| < \varepsilon,\end{aligned}$$

which means that $\|x'_n - a\| \to 0$, finishing the proof. □

The next theorem characterizes isometrically the dual of $(\bigoplus_{n+1}^{\infty} X_i)_\Phi$ in the case when $\Phi \in \Delta_2$.

Theorem 3. *If the Orlicz function Φ satisfies the Δ_2-condition, then for any sequence $(X_i)_{i=1}^{\infty}$ of Banach spaces we have $[(\bigoplus_{n=1}^{\infty} X_i)_\Phi]^* = (\bigoplus_{n=1}^{\infty} X_i^*)_\Psi$, where the space on the right side is equipped with the Orlicz norm and Ψ denotes the Orlicz function complementary to Φ in the sense of Young.*

Proof. Denote $X = (\bigoplus_{n=1}^{\infty} X_i)_\Phi$ and define $f(x) = \sum_{i=1}^{\infty} \langle x(i), v(i)\rangle$ for $x = (x(i))_{i=1}^{\infty} \in X$, where $v = (v(i)) \in (\bigoplus_{n=1}^{\infty} X_i^*)_\Psi$. Then f is a linear functional and

$$\begin{aligned}\|f\| &= \sup_{\|x\|=1} f(x) = \sup_{\|x\|=1} \sum_{i=1}^{\infty} \langle x(i), v(i)\rangle \\ &\leqslant \sup_{\|x\|=1} \sum_{i=1}^{\infty} \|x(i)\|_{X_i} \|v(i)\|_{X_i^*} \leqslant \|x\|_\Phi \|v\|_\Psi^{\mathrm{O}} = \|v\|_\Psi^{\mathrm{O}},\end{aligned}$$

whence it follows that $f \in X^*$ and $\|f\| \leqslant \|v\|_\Psi^{\mathrm{O}}$. Now, we will show the converse inequality. Let $a = (a(i))_{i=1}^{\infty} \in l^\Psi$ and $I_\Psi(a) \leqslant 1$. By the definition of $\|v(i)\|$, for any $i \in \mathbb{N}$ there exists $\overline{x}(i) \in X_i$ such that $\|\overline{x}(i)\| = a(i)$ and

$$\langle \overline{x}(i), v(i)\rangle \geqslant \|v(i)\| a(i) - \varepsilon/2^i.$$

Therefore

$$\begin{aligned}\|f\| &= \sup\Big\{\sum_{i=1}^{\infty} \langle x(i), v(i)\rangle \colon \sum_{i=1}^{\infty} \Psi(\|x_i\|) \leqslant 1\Big\} \\ &\geqslant \sup\Big\{\sum_{i=1}^{\infty} \|v(i)\| a(i) \colon \sum_{i=1}^{\infty} \Psi(a(i)) \leqslant 1\Big\} - \varepsilon = \|(\|v(i)\|)_{i=1}^{\infty}\|_\Psi^{\mathrm{O}} - \varepsilon,\end{aligned}$$

whence by the arbitrariness of $\varepsilon > 0$, the inequality $\|f\| \geqslant \|v\|_\Psi^{\mathrm{O}} = \|(\|v(i)\|_{X^*})_{i=1}^{\infty}\|_\Psi^{\mathrm{O}}$ follows. Therefore $\|f\| = \|v\|_\Psi^{\mathrm{O}}$.

To finish the proof, we need only to show that every $f \in X^*$ is of the above form. Define $P_n \colon X \to X$ by

$$P_n x = (\underbrace{0, \ldots, 0}_{n-1}, x(n), 0, \ldots)$$

for any $x = x(i)_{i=1}^{\infty} \in X$ and let $A_n \colon X_n \to X$ be defined by

$$A_n(x(n)) = (\underbrace{0, \ldots, 0}_{n-1}, x(n), 0, \ldots).$$

We have $A_n(x(n)) = P_n x$ for any $x \in X$ with the n-th coordinate equal to $x(n)$. Let $f \in X^*$ and define $P_n f(x) = f(P_n x)$. Since $\|P_n x\|_\Phi < \varepsilon$ implies $\|x(n)\|_{X_n} < \varepsilon \Phi^{-1}(1)$ and $\|x(n)\|_{X_n} < \varepsilon$ implies $\|P_n x\|_\Phi < \varepsilon/a$, where $a > 0$ satisfies $\Phi(a) = 1$, it follows that X_n is isomorphic to $P_n X$ and P_n is an isomorphism. Define $f_n \colon X_n \to \mathbb{R}$, by $\langle x(n), f_n\rangle = \langle A_n(x(n)), f\rangle$. Then $\langle x(n), f_n\rangle = \langle x, P_n f\rangle \stackrel{\mathrm{def}}{=} \langle P_n x, f\rangle$ for $x \in X$ satisfying $P_n x = x(n)$. Therefore $f_n \in X_n^*$ and by $\Phi \in \Delta_2$, we have

$$\begin{aligned}\sum_{i=n+1}^{\infty} f_i(x(i)) &= \sum_{i=n+1}^{\infty} P_i f(x) = f((\underbrace{0, \ldots, 0}_{n}, x(n+1), x(n+2), \ldots)) \\ &= f(x - x^{(n)}) \leqslant \|f\| \|x - x^{(n)}\| \to 0.\end{aligned}$$

Hence $f(x) = \sum_{n=1}^{\infty} \langle x(n), f_n \rangle$, where $f_n \in X_n^*$ $(n = 1, 2, \ldots)$. We will show that $(\|f_n\|_{X_n^*}) \in l^\Psi$. Let $(a(n))_{n=1}^{\infty} \in l^\Phi$ and $\sum_{n=1}^{\infty} \Phi_n(a(n)) \leqslant 1$. By the definition of $\|f_n\|_{X_n^*}$ there exists $\overline{x}(n) \in X_n$ such that $\|\overline{x}(n)\|_{X_n} = a(n)$ and

$$\|f_n\| a(n) \leqslant \langle \overline{x}(n), f_n \rangle + \varepsilon / 2^n,$$

whence for any $m \in \mathbb{N}$

$$\begin{aligned} \sum_{n=1}^{m} \|f_n\| a(n) &\leqslant \sum_{n=1}^{m} \langle \overline{x}(n), f_n \rangle + \varepsilon = \langle \overline{x}^{(m)}, f \rangle + \varepsilon \\ &\leqslant \|f\| \|\overline{x}^{(m)}\|_\Psi^{\mathrm{O}} + \varepsilon \leqslant \|f\| \|\overline{x}\|_\Psi^{\mathrm{O}} + \varepsilon, \end{aligned}$$

which means that $\sum_{n=1}^{\infty} \|f_n\|_{X_n^*} a(n) < \infty$ and consequently, by the arbitrariness of $(a(n))_{n=1}^{\infty}$ satisfying $\sum_{n=1}^{\infty} \Phi_n(a(n)) \leqslant 1$, we obtain that $(\|f_n\|_{X_n^*}) \in l^\Psi$. This finishes the proof. □

Theorem 4. *If Φ is an Orlicz function which does not satisfy the Δ_2-condition, then* $\mathrm{WCS}((\bigoplus_{i=1}^{\infty} X_i)_\Phi) = 1$ *for any infinite sequence (X_i) of Banach spaces.*

P r o o f. If $\Phi \notin \Delta_2$, there is $a = (a(n))_{n=1}^{\infty} \in S(l^\Phi)$ such that $I_\Phi((1+\lambda)a) = \infty$ for any $\lambda > 0$ (see [4], [8]). For any $i \in \mathbb{N}$ one can find $x(i) \in X_i$ such that $\|x(i)\| = a(i)$. Define $x = (x(1), x(2), \ldots, x(i), \ldots)$. Then

$$I_\Phi((1+\lambda)(x - x^{(n)})) = \sum_{i=n+1}^{\infty} \Phi((1+\lambda)a_i) = \infty$$

for each $n \in \mathbb{N}$. So, there is a sequence $\{n_i\}_{i=1}^{\infty}$ of natural numbers such that for $x_i = \sum_{i=n_i+1}^{n_{i+1}} x(i) e_i$ there holds $(1 - \frac{1}{i+1}) \leqslant \|x_i\| \leqslant 1$ $(i = 1, 2, \ldots)$. Since $[(\bigoplus_{i=1}^{\infty} X_i)_\Phi]^* = (\bigoplus_{i=1}^{\infty} X_i^*)_\Psi + S$, where S is the space of those functionals from the dual space which vanish on sequences with finite number of coordinates different from zero, we easily get that $x_i \xrightarrow{w} 0$. Now, we can get the equality from the thesis in the same way as in the proof of Theorem 1.

□

Theorem 5. *For any Orlicz function Φ, any $n \in \mathbb{N}$ and any system of Banach spaces $X_1, \ldots, X_n$, we have* $\mathrm{WCS}((\bigoplus_{i=1}^{n} X_i)_\Phi) = \min\{\mathrm{WCS}(X_i)\colon\ i = 1, \ldots, n\}$.

P r o o f. For $n = 1$ the result is obvious. Assume without loss of generality that $n = 2$. Let $X = (\bigoplus_{i=1}^{2} X_i)_\Phi$. Note that X_1 and X_2 are isomorphically isometrically embedded into X. In order to see this, define $P\colon X_1 \to X$ by $Px = (ax, 0)$ for any

$x \in X_1$, where $a > 0$ satisfies $\Phi(a) = 1$. It is obvious that P is a linear *one-to-one* operator between X_1 and X. Moreover, $I_\Phi(Px/\|x\|_{X_1}) = \Phi(\|ax\|_{X_1}/\|x\|_{X_1}) = \Phi(a) = 1$, whence $\|Px/\|x\|_{X_1}\|_\Phi = 1$ i.e. $\|Px\|_\Phi = \|x\|_{X_1}$, which means that P is an isometry. Analogously, the operator $Q\colon X_2 \to X$, defined by $Qx = (0, ay)$ for any $y \in X_2$, is an isomorphic isometry. Therefore $\mathrm{WCS}(X) \leqslant \min(\mathrm{WCS}(X_1), \mathrm{WCS}(X_2))$.

To prove the converse inequality take any asymptotic equidistant and weakly convergent sequence $\{x_n\}$ in $S(X)$. Then $x_n = (x_n^1, x_n^2)$, where $x_n^1 \in X_1$, $x_n^2 \in X_2$. By [21], Proposition 2, there is a sequence $\{n_k\}$ in $\mathbb{N}$ such that both $\{x_{n_k}^1\}$ and $\{x_{n_k}^2\}$ are asymptotic equidistant sequences and both of them are convergent, say

$$\lim_{k\to\infty} \|x_{n_k}^1\|_{X_1} = a, \quad \lim_{k\to\infty} \|x_{n_k}^2\|_{X_2} = b.$$

We have

$$1 = \|(x_n^1, x_n^2)\|_\Phi = \|(\|x_n^1\|_{X_1}, \|x_n^2\|_{X_2})\|_\Phi = \Phi(\|x_n^1\|_{X_1}) + \Phi(\|x_n^2\|_{X_2}).$$

for all $n \in \mathbb{N}$, whence $\Phi(a) + \Phi(b) = 1$. Moreover,

$$\begin{aligned}
&\lim_{i,j\to\infty} \|x_{n_i} - x_{n_j}\|_\Phi = \lim_{i,j\to\infty} \|(x_{n_i}^1 - x_{n_j}^1, x_{n_i}^2 - x_{n_j}^2)\|_\Phi \\
&= \lim_{i,j\to\infty} \bigl\{ \inf\{k > 0\colon \Phi(\|x_{n_i}^1 - x_{n_j}^1\|_{X_1}/k) + \Phi(\|x_{n_i}^2 - x_{n_j}^2\|_{X_2}/k) \leqslant 1\}\bigr\}.
\end{aligned}$$

Taking into account that $\|x_{n_i}^1 - x_{n_j}^1\| \geqslant a\,\mathrm{WCS}(X_1)$ and $\|x_{n_i}^2 - x_{n_j}^2\| \geqslant b\,\mathrm{WCS}(X_2)$, we have

$$\begin{aligned}
\lim_{i,j\to\infty} \|x_{n_i} - x_{n_j}\| &\geqslant \lim_{i,j\to\infty} \{\inf\{k > 0\colon \Phi(a\,\mathrm{WCS}(X_1)/k) + \Phi(b\,\mathrm{WCS}(X_2)/k) \leqslant 1\}\} \\
&\geqslant \lim_{i,j\to\infty} \{\inf\{k > 0\colon \Phi(a \min(\mathrm{WCS}(X_1), \mathrm{WCS}(X_2))/k) \\
&\quad + \Phi(b \min(\mathrm{WCS}(X_1), \mathrm{WCS}(X_2))/k) \leqslant 1\}\} \\
&= \min\{\mathrm{WCS}(X_1), \mathrm{WCS}(X_2)\}.
\end{aligned}$$

Hence $\mathrm{WCS}(X) \geqslant \min\{\mathrm{WCS}(X_1), \mathrm{WCS}(X_2)\}$ and consequently

$$\mathrm{WCS}(X) = \min\{\mathrm{WCS}(X_1), \mathrm{WCS}(X_2)\}.$$

□

R e m a r k 2. Theorem 1 in [5] says that if $X_1, \ldots, X_n$ are reflexive Banach spaces and X is the space $\mathbb{R}^k$ with a monotone norm, then

$$\mathrm{WCS}\biggl(\Bigl(\bigoplus_{i=1}^n X_i\Bigr)_X\biggr) = \min\{\mathrm{WCS}(X_i)\colon i = 1, \ldots, n\}.$$

Although Theorem 5 concerns only $X = \mathbb{R}^n$ equipped with the Luxemburg norm generated by an Orlicz function Φ, the reflexivity of X_i was not assumed.

R e m a r k 3. Landes [10] discussing permanence properties of normal structure for a finite product of Banach spaces has shown that if both X and Y are reflexive with $\mathrm{WCS}(X) > 1$ and $\mathrm{WCS}(Y) > 1$, then $Z = (X \times Y)_{l_1}$ has normal structure. It follows from Theorem 5 that without reflexivity of X and Y, the space Z has even weak uniform normal structure.

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