A SECOND LOOK ON DEFINITION AND EQUIVALENT NORMS OF SOBOLEV SPACES

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. Sobolev's original definition of his spaces $L^{m,p}(\Omega)$ is revisited. It only assumed that $\Omega \subseteq \mathbb{R}^n$ is a domain. With elementary methods, essentially based on Poincaré's inequality for balls (or cubes), the existence of intermediate derivates of functions $u \in L^{m,p}(\Omega)$ with respect to appropriate norms, and equivalence of these norms is proved.

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1. Introduction

In 1936–38, S. L. Sobolev introduced in his pioneering works [10], [11] spaces of integrable functions having weak derivatives in L^p. These function spaces have turned out to be an appropriate framework for studying boundary value problems for partial differential equations by using methods of functional analysis. A presentation of these results obtained up to 1949–50 may be found in Sobolev's monograph [12]. On the other hand, these function spaces became a research field of independent interest, and since that time the theory of these spaces has undergone an enormous development (cf. e.g. [2], [5], [9], [13]).

The commonly used definition of these spaces in the contemporary literature is as follows. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, let $m \in \mathbb{N}$ and $1 \leqslant p \leqslant +\infty$. Then the vector space

$$W^{m,p}(\Omega) := \big\{u \in L^p(\Omega)\colon\ \exists\, \mathrm{D}^\alpha u \in L^p(\Omega)\, \forall\, |\alpha| \leqslant m\big\}$$

is nowadays called "Sobolev space". Here $\mathrm{D}^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\cdot\ldots\cdot\partial x_n^{\alpha_n}}$ denotes the weak (or generalized) derivative of u corresponding to the multi-index $\alpha = \{\alpha_1,\ldots,\alpha_n\}$ ($|\alpha| = \alpha_1+\ldots+\alpha_n$) (cf. e.g. [1], [4], [8], [14]). $W^{m,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{W^{m,p}(\Omega)} := \begin{cases} \Bigl(\sum\limits_{|\alpha|\leqslant m} \|\mathrm{D}^\alpha u\|^p_{L^p(\Omega)}\Bigr)^{1/p} & \text{if} \quad 1\leqslant p<+\infty, \\ \sum\limits_{|\alpha|\leqslant m} \|\mathrm{D}^\alpha u\|_{L^\infty(\Omega)} & \text{if} \quad p=+\infty. \end{cases}$$

The above definition of the space $W^{m,p}(\Omega)$ fits very well the weak formulation of boundary value problems for partial differential equations on *bounded* domains, but it is less convenient when *unbounded* domains are under consideration. To see this, let us consider

$$\Omega := \bigl\{x\in\mathbb{R}^n\colon\ |x|>1\bigr\},$$
$$u(x) := \begin{cases} 1-|x|^{2-n} & \text{if} \quad n\geqslant 3,\ x\in\Omega, \\ \log|x| & \text{if} \quad n=2,\ x\in\Omega. \end{cases}$$

It is readily seen that

$$\begin{aligned} &u\in L^q_{\mathrm{loc}}(\Omega) \quad \text{for any } 1\leqslant q<+\infty, \\ &u\notin L^p(\Omega), \frac{\partial u}{\partial x_i}\in L^p(\Omega) \quad (i=1,\ldots,n) \quad \text{for any } p>\tfrac{n}{n-1}, \\ &\Delta u = 0 \quad \text{in } \Omega, u=0 \quad \text{on } \partial\Omega. \end{aligned}$$

Therefore, for the function space setting of boundary value problems for PDE's in unbounded domains, the following definition seems to be more adequate:[1]

$$L^{m,p}(\Omega) := \bigl\{u\in L^1_{\mathrm{loc}}(\Omega)\colon\ \exists\,\mathrm{D}^\alpha u\in L^p(\Omega)\ \forall\,|\alpha|=m\bigr\}.$$

This is a slight generalization of Sobolev's original definition [10], [11], [12] where he used functions in $L^1(\Omega)$ in place of $L^1_{\mathrm{loc}}(\Omega)$ [notice that the letters W and L for the notation of the above spaces in [12] vary in the contemporary literature]. Our definition of $L^{m,p}(\Omega)$ coincides with that in [6], [7].

To furnish $L^{m,p}(\Omega)$ with a norm, we assume throughout this paper that $\Omega\subseteq\mathbb{R}^n$ is a *domain*, and we fix any $G\subset\subset\Omega$,[2] and define for every $u\in L^{m,p}(\Omega)$

$$\|u\|_{m,p;\Omega,G} := \|u\|_{L^1(G)} + |u|_{m,p;\Omega},$$

[1] Without any further reference, in all that follows we restrict our discussion to the case $1\leqslant p<+\infty$.
[2] That is, G is open and bounded and $\overline{G}\subset\Omega$. Clearly, we assume $G\neq\emptyset$.

where

$$|u|_{m,p;\Omega} := \Big(\sum_{|\alpha|=m} \|\mathrm{D}^\alpha u\|^p_{L^p(\Omega)} \Big)^{1/p}$$

(note that $\|u\|_{m,p;\Omega,G} = 0$ implies in particular $|u|_{m,p;\Omega} = 0$, hence $u = P$ (= polynomial of degree $\leqslant m-1$; cf. Theorem B below) a.e. on G, and $\|P\|_{L^1(G)} = 0$ gives $P \equiv 0$ on G and thus $u = P = 0$ a.e. on Ω).

The following problems occur in the study of the spaces $L^{m,p}(\Omega)$:

1. *existence of intermediate derivatives* $\mathrm{D}^\beta u \in L^p_{\mathrm{loc}}(\Omega)$ $(|\beta| \leqslant m-1)$ *for any* $u \in L^{m,p}(\Omega)$;
2. *completeness of* $L^{m,p}(\Omega)$ *with respect to the norm* $\|\cdot\|_{m,p;\Omega,G}$;
3. *equivalence of the norms* $\|\cdot\|_{m,p;\Omega,G_k}$ *for arbitrary domains* $G_k \subset\subset \Omega$ $(k = 1, 2)$.

By using the method of spherical projection operators, these problems are settled in [12] for bounded domains Ω which are finite unions of domains each of which is starshaped with respect to a ball (cf. also [3], [7], [12]).

The aim of the present paper is to solve problems 1.–3. by an entirely different and simpler method which is essentially based on Poincaré's inequality over balls. This inequality can be proved by elementary calculus arguments. Moreover, we introduce a new class of norms on $L^{m,p}(\Omega)$ which are equivalent to the norm $\|\cdot\|_{m,p;\Omega,G}$ and give a better insight into the product space structure of $L^{m,p}(\Omega)$.

A more detailed presentation of our approach will appear in a forthcoming publication.

2. Notations. Preliminaries

We introduce the notation for our following discussion:

$$\begin{aligned}\mathcal{P}(m) &:= \big\{ P = P(x)\colon\ P(x) = \sum_{|\alpha|\leqslant m} a_\alpha x^\alpha, x \in \mathbb{R}^n \big\} \\ &= \text{vector space of polynomials of degree} \leqslant m \text{ in } \mathbb{R}^n;\end{aligned}$$

for any set $G \subseteq \mathbb{R}^n$ and $x \in G$ let

$$d_x := \begin{cases} \frac{1}{4}\mathrm{dist}(x, \partial G) & \text{if } G \neq \mathbb{R}^n, \\ 1 & \text{if } G = \mathbb{R}^n; \end{cases}$$

finally,

$$B_R(x_0) := \{x \in \mathbb{R}^n\colon\ |x - x_0| < R\};$$

in particular, we denote $B_{d_x} = B_{d_x}(x)$.

2.1 We begin by proving two auxiliary results.

Theorem A. *Let $G \subset \mathbb{R}^n$ be a bounded open set. Then for any $u \in W^{m,p}(G)$ there exists a uniquely determined polynomial $P_u \in \mathcal{P}(m-1)$ such that*

$$\int_G \mathrm{D}^\alpha (u - P_u)\,\mathrm{d}x = 0 \qquad \forall |\alpha| \leqslant m-1, \tag{2.1}$$

$$\|P_u\|_{W^{m-1,p}(G)} \leqslant C\|u\|_{W^{m-1,p}(G)}, \tag{2.2}$$

where the constant C depends only on m, n, p and $\operatorname{mes} G$*.*

Proof. Let $m = 1$. Given any $u \in W^{1,p}(G)$, then

$$P_u = \frac{1}{\operatorname{mes} G} \int_G u(y)\,\mathrm{d}y$$

satisfies (2.1), (2.2) (with $C = 1$).

Suppose the claim is true for $m \geqslant 1$. Let $u \in W^{m+1,p}(G)$. Define

$$Q(x) := \sum_{|\beta|=m} b_\beta x^\beta,\ x \in \mathbb{R}^n,\ \text{where } b_\beta := \frac{1}{\beta!\,\operatorname{mes} G} \int_G \mathrm{D}^\beta u\,\mathrm{d}y.$$

Clearly,

$$\int_G \mathrm{D}^\alpha (u - Q)\,\mathrm{d}x = 0 \quad \forall |\alpha| = m,$$

$$\|Q\|_{W^{m,p}(G)} \leqslant C\|u\|_{W^{m,p}(G)} \tag{2.2′}$$

(the constant C involves bounds on $|x^\gamma|$ ($|\gamma| \leqslant m-1$) over G). The function $v := u - Q$ lies in $W^{m,p}(G)$. Hence, there exists a $P_v \in \mathcal{P}(m-1)$ such that

$$\int_G \mathrm{D}^\gamma (v - P_v)\,\mathrm{d}x = 0 \quad \forall |\gamma| \leqslant m-1,$$

$$\|P_v\|_{W^{m-1,p}(G)} \leqslant C\|v\|_{W^{m-1,p}(G)} \leqslant C'\|u\|_{W^{m,p}(G)}. \tag{2.2″}$$

Thus, $P_u := (P_v + Q) \in \mathcal{P}(m)$ and

$$\int_G \mathrm{D}^\alpha \Big(u - (P_v + Q)\Big)\,\mathrm{d}x = \begin{cases} \int_G \mathrm{D}^\alpha (u - Q)\,\mathrm{d}x = 0 & \forall |\alpha| = m, \\ \int_G \mathrm{D}^\alpha (u - Q - P_v)\,\mathrm{d}x = 0 & \forall |\alpha| \leqslant m-1, \end{cases}$$

and (2.2) (with m in place of $m-1$) is readily deduced from (2.2′) and (2.2″).

The uniqueness of P_u also follows by induction. □

Theorem B. *Let $G \subseteq \mathbb{R}^n$ be a domain. Let $u \in L^{m,p}(G)$ satisfy $\mathrm{D}^\alpha u = 0$ a.e. in G for all $|\alpha| = m$. Then there exists exactly one $P \in \mathcal{P}(m-1)$ such that*

$$u = P \quad \text{a.e. in } G.$$

P r o o f. a) Let $x \in G$ be arbitrary. For $0 < \varrho < d_x$ we consider the standard mollifier u_ϱ on the ball B_{2d_x}. Then

$$\begin{aligned} &\|u - u_\varrho\|_{L^1(B_{d_x})} \to 0 \quad \text{as } \varrho \to 0,\\ &(\mathrm{D}^\alpha u_\varrho)(\xi) = (\mathrm{D}^\alpha u)_\varrho(\xi) = 0 \quad \forall \xi \in B_{d_x}, \forall |\alpha| = m. \end{aligned}$$

Hence there exists a polynomial $P^{(\varrho)} \in \mathcal{P}(m-1)$ such that $u_\varrho|_{B_{d_x}} = P^{(\varrho)}|_{B_{d_x}}$.

The space $\mathcal{P}(m-1)$ being complete with respect to the norm $\big\| \cdot |_{B_{d_x}} \big\|_{L^1(B_{d_x})}$, there exists a $P^{(x)} \in \mathcal{P}(m-1)$ such that $\|P^{(x)} - u_\varrho\|_{L^1(B_{d_x})} \to 0$ as $\varrho \to 0$. Thus, $u = P^{(x)}$ a.e. in B_{d_x}.

b) Let $x, y \in G$ with $B_{d_x} \cap B_{d_y} \neq \emptyset$. Then $B_{d_x} \cap B_{d_y}$ is open and therefore $P^{(x)} = P^{(y)}$ on $B_{d_x} \cap B_{d_y}$, i.e. $P^{(x)} \equiv P^{(y)}$.

c) Let $x_0 \in G$ be arbitrary, but fixed. Define

$$M := \big\{x \in G\colon\ P^{(x)} \equiv P^{(x_0)}\big\}.$$

Clearly, $x_0 \in M$. The set M is open, for $x \in M$ and $y \in B_{d_x}$ imply $B_{d_x} \cap B_{d_y} \neq \emptyset$, and by b) we have $P^{(y)} \equiv P^{(x)} \equiv P^{(x_0)}$, i.e. $B_{d_x} \subset M$. On the other hand, M is relatively closed. Indeed, let $x_k \in M$ and $x_k \to x \in G$. Then $x_k \in B_{d_x}$ for all $k \geqslant k_0$, and again using b) gives $P^{(x)} \equiv P^{(x_{k_0})} \equiv P^{(x_0)}$, i.e. $x \in M$. Thus, $M = G$.

d) Set $P := P^{(x_0)}$. Let G_k ($k \in \mathbb{N}$) be bounded open sets such that $G_k \subset\subset G_{k+1} \subset\subset G$ and $G = \bigcup\limits_{k=1}^{\infty} G_k$. Given any $k \in \mathbb{N}$ there exist $x_i \in \overline{G}_k$ ($i = 1, \dots, s$) such that $\overline{G}_k \subset \bigcup\limits_{i=1}^{s} B_{d_{x_i}}$. By a) and c),

$$\|u - P\|_{L^1(G_k)} \leqslant \sum_{i=1}^{s} \|u - P\|_{L^1(B_{d_{x_i}})} = 0.$$

Hence $u = P$ a.e. in G_k, and therefore $u = P$ a.e. in G. $\square$

2.2 The following result is fundamental to our subsequent discussion.

Theorem C. (Poincaré's inequality). *Let $B_R = B_R(x_0)$ be any fixed ball. Then there exists a constant $C(R) > 0$ (depending also on m, n and p) such that*

$$\left\{ \begin{aligned} &\|u\|_{W^{m-1,p}(B_R)} \leqslant C(R)|u|_{m,p;B_R},\\ &\forall u \in W^{m,p}(B_R) \text{ with } \int_{B_R} \mathrm{D}^\beta u \,\mathrm{d}x = 0 \quad \forall |\beta| \leqslant m-1. \end{aligned} \right. \tag{2.3}$$

An elementary proof of this theorem [3] for $m = 1$ which is based on potential estimates, may be found in [4]. The proof for $m \geqslant 2$ follows by induction.

3. Existence of intermediate derivatives

Without any further reference, throughout the following discussion we denote by Ω a domain in $\mathbb{R}^n$. Our first result in solving problem 1 is

Theorem 3.1. *Let $u \in L^{m,p}(\Omega)$. Then there exist the weak derivatives*

$$\mathrm{D}^\beta u \in L^p_{\mathrm{loc}}(\Omega) \quad \forall |\beta| \leqslant m-1.$$

Proof. a) Let $x \in \Omega$ be arbitrary. As above in the proof of Theorem B, for $0 < \varrho < d_x$ we consider the standard mollifier u_ϱ on B_{2d_x}. For any sequence (ϱ_k) such that $0 < \varrho_k < d_x$ and $\varrho_k \to 0$, we have

$$\|u - u_{\varrho_k}\|_{L^1(B_{d_x})} \to 0, \quad |u - u_{\varrho_k}|_{m,p;B_{d_x}} \to 0 \text{ as } k \to \infty. \tag{3.1}$$

By Theorem A (with $G = B_{d_x}$), there exists a $P_{u_{\varrho_k}} \in \mathcal{P}(m-1)$ such that

$$\int\limits_{B_{d_x}} \mathrm{D}^\beta (u_{\varrho_k} - P_{u_{\varrho_k}})\,\mathrm{d}x = 0 \quad \forall |\beta| \leqslant m-1. \tag{3.2}$$

Define $v_k := u_{\varrho_k} - P_{u_{\varrho_k}}$. Then $\mathrm{D}^\alpha v_k = \mathrm{D}^\alpha u_{\varrho_k}$ on B_{d_x} for all $|\alpha| = m$. Using (3.1) and (3.2), Theorem C gives

$$\|v_k - v_l\|_{W^{m-1,p}(B_{d_x})} \leqslant C(d_x)|u_{\varrho_k} - u_{\varrho_l}|_{m,p;B_{d_x}} \to 0$$

and thus $\|v_k - v_l\|_{W^{m,p}(B_{d_x})} \to 0$ as $k, l \to \infty$. Hence, there exists $v \in W^{m,p}(B_{d_x})$ such that $\|v - v_k\|_{W^{m,p}(B_{d_x})} \to 0$ as $k \to \infty$.

Let $\varphi \in C_c^\infty(\Omega)$. For any multiindex α with $|\alpha| = m$ we have

$$\begin{aligned}
\int\limits_{B_{d_x}} (u - v)\mathrm{D}^\alpha \varphi\,\mathrm{d}x &= \int\limits_{B_{d_x}} u\mathrm{D}^\alpha \varphi\,\mathrm{d}x - \lim_{k\to\infty} \int\limits_{B_{d_x}} v_k \mathrm{D}^\alpha \varphi\,\mathrm{d}x \\
&= (-1)^m \int\limits_{B_{d_x}} (\mathrm{D}^\alpha u)\varphi\,\mathrm{d}x - (-1)^m \lim_{k\to\infty} \int\limits_{B_{d_x}} (\mathrm{D}^\alpha u_{\varrho_k})\varphi \mathrm{d}x \\
&= 0.
\end{aligned}$$

[3] If we replace for $x \in \mathbb{R}^n$ the Euclidean norm $|x| = |x|_2 = \big(\sum\limits_{i=1}^n x_i^2\big)^{1/2}$ by the equivalent norm $|x|_\infty := \max\{|x_i|,\ i = 1, \dots, n\}$, then a "ball" $B_R(x_0)$ with respect to $|\cdot|_\infty$-norm is the cube $W_R(x_0) := \{x \in \mathbb{R}^n \colon |x_i - x_{0\,i}| < R,\ i = 1, \dots, n\}$. In this case Poincaré's inequality for $m = 1$ admits a very simple proof by induction on n.

By Theorem B, there exists a polynomial $P \in \mathcal{P}(m-1)$ satisfying $u - v = P$ a.e. in B_{d_x}. Thus, $u = (v + P) \in W^{m,p}(B_{d_x})$.

b) Let $\Omega' \subset\subset \Omega$ be arbitrary. There exist $x_i \in \Omega'$ $(i = 1, \ldots, s)$ such that $\overline{\Omega'} \subset \bigcup_{i=1}^{s} B_{d_{x_i}}$. From a) we have $u|_{B_{d_{x_i}}} \in W^{m,p}(B_{d_{x_i}})$ $(i = 1, \ldots, s)$. Then, by a standard argument, $u \in W^{m,p}(\Omega')$ (cf. e.g. [8], [12]).

c) Let Ω_j be subsets of Ω satisfying $\Omega_j \subset\subset \Omega_{j+1}$ $(j = 1, 2, \ldots)$ and $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. Let $u_j^{(\beta)} \in L^p(\Omega_j)$ denote the weak D^β-derivative $(|\beta| \leqslant m-1)$ of u in Ω_j. It follows that

$$\int_{\Omega_j} u_{j+1}^{(\beta)} \varphi \, \mathrm{d}x = \int_{\Omega} u_j^{(\beta)} \varphi \, \mathrm{d}x \quad \forall \varphi \in C_c^\infty(\Omega_j),$$

i.e. $u_{j+1}^{(\beta)} = u_j^{(\beta)}$ a.e. in Ω_j. Therefore we can choose appropriate representatives in $u_j^{(\beta)}$ (not relabelled) such that $u_{j+1}^{(\beta)}(x) = u_j^{(\beta)}(x)$ for all $x \in \Omega_j$. Then the function

$$u^{(\beta)}(x) := u_j^{(\beta)}(x) \quad \text{for } x \in \Omega_j \quad (j = 1, 2, \ldots)$$

is well defined on Ω, $u^{(\beta)} \in L^p_{\mathrm{loc}}(\Omega)$ and $u^{(\beta)} = \mathrm{D}^\beta u$ a.e. in Ω. $\square$

R e m a r k. Let $\Omega \in C^0$ (i.e. Ω is bounded and the boundary of Ω is locally the graph of a continuous function). Let $u \in L^{m,p}(\Omega)$. Using a method from [8] for proving the compactness of the imbedding $W^{1,p}(\Omega) \subset L^p(\Omega)$, we obtain:

$$\mathrm{D}^\beta u \in L^p(\Omega) \quad \forall |\beta| \leqslant m-1$$

(cf. also [3]).

4. Norms on $L^{m,p}(\Omega)$. Completeness of $L^{m,p}(\Omega)$

Let $G \subset\subset \Omega$. By Theorem 3.1 we may define, for any $u \in L^{m,p}(\Omega)$,

$$\begin{aligned} |\!|u|\!|_{m-1;G} &:= \sum_{|\beta| \leqslant m-1} \left| \int_G \mathrm{D}^\beta u \, \mathrm{d}x \right|, \\ |\!|u|\!|_{m,p;\Omega,G} &:= |\!|u|\!|_{m-1;G} + |u|_{m,p;\Omega}. \end{aligned}$$

The norm $|\!| \cdot |\!|_{m,p;\Omega,G}$ seems to be better adapted to the study of completeness and equivalence of various norms on the space $L^{m,p}(\Omega)$ than the norm $\| \cdot \|_{m,p;\Omega,G}$ considered in the introduction.

To begin with, we prove

Lemma 4.1. *There holds:*

1. $\|\cdot\|_{m-1;G}$ *is a norm on* $\mathcal{P}(m-1)$.
2. $\|\cdot\|_{m,p;\Omega,G}$ *is a norm on* $L^{m,p}(\Omega)$.

Proof. 1. Let be $P \in \mathcal{P}(m-1)$ such that $\|P\|_{m-1;G} = 0$. We may write $P(x) = \sum\limits_{|\beta|\leqslant m-1} a_\beta x^\beta$, $x \in \mathbb{R}^n$. In particular, it follows that

$$0 = \sum_{|\beta|=m-1} \left| \int_G \mathrm{D}^\beta P \,\mathrm{d}x \right| = \sum_{|\beta|=m-1} |a_\beta|\, \beta! \operatorname{mes} G,$$

i.e., $a_\beta = 0$ for all $|\beta| = m-1$. Repeating this argument gives $P \equiv 0$.

2. Assume $u \in L^{m,p}(\Omega)$ satisfies $\|u\|_{m,p;\,\Omega,G} = 0$. By Theorem B, $u = P$ a.e. in Ω, where $P \in \mathcal{P}(m-1)$. Then 1. implies $u = 0$ a.e. in Ω.

All the other properties of both norms are readily seen. $\square$

We now define

$$L^{m,p}_G(\Omega) := \left\{ u \in L^{m,p}(\Omega) \colon \int_G \mathrm{D}^\beta u \,\mathrm{d}x = 0 \quad \forall |\beta| \leqslant m-1 \right\}.$$

Then

$$L^{m,p}(\Omega) = L^{m,p}_G(\Omega) \oplus \mathcal{P}(m-1)|_\Omega \qquad \text{(direct decomposition).} \tag{4.1}$$

Indeed, by Theorem A there exists a uniquely determined $P_u \in \mathcal{P}(m-1)$ such that

$$\int_G \mathrm{D}^\beta (u - P_u) \,\mathrm{d}x = 0 \quad \forall |\beta| \leqslant m-1.$$

Then $u_0 := (u - P_u) \in L^{m,p}_G(\Omega)$ and $u = u_0 + P_u$.

If $v \in L^{m,p}_G(\Omega) \cap \mathcal{P}(m-1)|_\Omega$, i.e. $v \in \mathcal{P}(m-1)$ and $\int_G \mathrm{D}^\beta v \,\mathrm{d}x = 0$ for all $|\beta| \leqslant m-1$, it follows that $v \equiv 0$.

With the decomposition $u = u_0 + P_u$ just introduced, we have

$$\|u\|_{m,p;\,\Omega,G} = \|P_u\|_{m-1;\,G} + |u_0|_{m,p;\,\Omega}.$$

Furthermore, if G is a domain with $G \subset\subset \Omega$, then $|\cdot|_{m,p;\,\Omega}$ is a norm on $L^{m,p}_G(\Omega)$ (cf. Theorem B).

Our principal result in this section is

Theorem 4.2. *Let (u_k) be a sequence of functions in $L^{m,p}(\Omega)$ such that*

$$|u_k - u_l|_{m,p;\,\Omega} \to 0 \quad \text{as } k,l \to \infty.$$

Let $x_0 \in \Omega$ be arbitrary, but fixed, and let $P_{u_k} = P_{u_k}^{(x_0)} \in \mathcal{P}(m-1)$ be the polynomial from Theorem A*:*

$$\int_{B_{d_{x_0}}} \mathrm{D}^\beta (u_k - P_{u_k})\,\mathrm{d}x = 0 \quad \forall |\beta| \leqslant m-1 \quad (k = 1, 2, \ldots).$$

Then there exists a $u \in L^{m,p}(\Omega)$ such that

$$\big\|u - (u_k - P_{u_k})\big\|_{W^{m-1,p}(\Omega')} \to 0 \quad \text{as } k \to \infty,\ \forall \Omega' \subset\subset \Omega, \tag{4.2}$$

$$|u - u_k|_{m,p;\,\Omega} \to 0 \quad \text{as } k \to \infty. \tag{4.3}$$

P r o o f. a) From Theorem C (Poincaré's inequality) it follows that

$$\begin{aligned}
&\|(u_k - P_{u_k}) - (u_l - P_{u_l})\|^p_{W^{m,p}(B_{d_{x_0}})} \\
&\qquad = \|(u_k - u_l) - (P_{u_k} - P_{u_l})\|^p_{W^{m-1,p}(B_{d_{x_0}})} + |u_k - u_l|^p_{m,p;\,B_{d_{x_0}}} \\
&\qquad \leqslant (1 + (C(d_{x_0}))^p)|u_k - u_l|^p_{m,p;\,B_{d_{x_0}}} \to 0 \quad \text{as } k,l \to \infty,
\end{aligned}$$

i.e., $(u_k - P_{u_k})$ is Cauchy in $W^{m,p}(B_{d_{x_0}})$.

b) Define

$$M := \big\{x \in \Omega\colon\ (u_k - P_{u_k}) \text{ is Cauchy in } W^{m,p}(B_{d_x})\big\}.$$

By a), $x_0 \in M$. We shall show that M is open. To this end, let $x \in M$. Given any $y \in B_{d_x}$ we have to show that $(u_k - P_{u_k})$ is Cauchy in $W^{m,p}(B_{d_y})$. Indeed, let $P_{u_k}^{(y)} \in \mathcal{P}(m-1)$ satisfy

$$\int_{B_{d_y}} \mathrm{D}^\beta (u_k - P_{u_k}^{(y)})\,\mathrm{d}x = 0 \quad \forall\, |\beta| \leqslant m-1$$

(cf. Theorem A). The same reasoning as in a) gives: $(u_k - P_{u_k}^{(y)})$ is Cauchy in $W^{m,p}(B_{d_y})$. Denoting $E := B_{d_x} \cap B_{d_y}$ we find

$$\begin{aligned}
&\big\|(P_{u_k} - P_{u_k}^{(y)}) - (P_{u_l} - P_{u_l}^{(y)})\big\|_{W^{m-1,p}(E)} \\
&\quad \leqslant \big\|(P_{u_k} - u_k) - (P_{u_l} - u_l)\big\|_{W^{m-1,p}(B_{d_x})} + \big\|(u_k - P_{u_k}^{(y)}) - (u_l - P_{u_l}^{(y)})\big\|_{W^{m-1,p}(B_{d_y})} \\
&\quad \to 0 \quad \text{as } k,l \to \infty.
\end{aligned}$$

Thus, $(P_{u_k} - P^{(y)}_{u_k})$ is Cauchy in $W^{m-1,p}(E)$ and therefore also in $W^{m-1,p}(B_{d_y})$ (since all norms on the finite dimensional space $\mathcal{P}(m-1)$ are equivalent). Hence, $(u_k - P_{u_k})$ is Cauchy in $W^{m,p}(B_{d_y})$.

Next, we shall show that M is relatively closed, i.e., $x_s \in M$, $x_s \to x \in \Omega$ implies $x \in M$. Indeed, we have $x_{s_0} \in B_{d_x}$ for a sufficiently large s_0, and $(u_k - P_{u_k})$ is Cauchy in $W^{m,p}(B_{d_{x_{s_0}}})$. Since $B_{d_{x_{s_0}}} \cap B_{d_x} \neq \emptyset$, we obtain as above that $(u_k - P_{u_k})$ is Cauchy in $W^{m,p}(B_{d_x})$. Thus, $M = \Omega$.

c) Let $\Omega' \subset\subset \Omega$ be arbitrary. Then there exist $x_i \in \Omega'$ $(i = 1, \ldots, t)$ with $\overline{\Omega'} \subset \bigcup_{i=1}^{t} B_{d_{x_i}}$. From b) we have: the sequence $(u_k - P_{u_k})$ is Cauchy in $W^{m,p}(B_{d_{x_i}})$ $(i = 1, \ldots, t)$, and therefore also in $W^{m,p}(\Omega')$. Hence, there exists a $u \in W^{m,p}(\Omega')$ (possibly depending on Ω') such that

$$\big\| u - (u_k - P_{u_k}) \big\|_{W^{m,p}(\Omega')} \to 0 \quad \text{as } k \to \infty.$$

d) Let Ω_j be subsets of Ω satisfying $\Omega_j \subset\subset \Omega_{j+1}$ $(j = 1, 2, \ldots)$ and $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. Let $v_j \in W^{m,p}(\Omega_j)$ be the limit function of $(u_k - P_{u_k})$ in $W^{m,p}(\Omega_j)$ (cf. c)). Then $v_{j+1} = v_j$ a.e. in Ω_j, and by choosing appropriate representatives of v_j, we may define a measurable function u on Ω, such that $u = v_j$ a.e. in Ω_j $(j = 1, 2, \ldots)$. An analogous construction for the weak derivatives $\mathrm{D}^\alpha v_j$ gives $\mathrm{D}^\alpha u = \mathrm{D}^\alpha v_j$ a.e. in Ω_j $(j = 1, 2, \ldots)$. We obtain $u \in W^{m,p}_{\mathrm{loc}}(\Omega)$ and (4.2).

e) It remains to show that $u \in L^{m,p}(\Omega)$ and (4.3). To see this, we note that our assumptions imply for any $|\alpha| = m$ the existence of functions $w_\alpha \in L^p(\Omega)$ such that $\|w_\alpha - \mathrm{D}^\alpha u_k\|_{L^p(\Omega)} \to 0$ as $k \to \infty$. Hence, $w_\alpha = \mathrm{D}^\alpha u$ a.e. in Ω_j (Ω_j is from c) and the claim follows. ☐

Corollary 4.3. Let $(u_k) \subset L^{m,p}_{B_{d_{x_0}}}(\Omega)$ be Cauchy with respect to the norm $|\cdot|_{m,p;\,\Omega}$.[4] Then there exists $u \in L^{m,p}_{B_{d_{x_0}}}(\Omega)$ such that

$$\|u - u_k\|_{W^{m-1,p}(\Omega')} \to 0 \quad \text{as } k \to \infty, \quad \forall \Omega' \subset\subset \Omega,$$
$$|u - u_k|_{m,p;\,\Omega} \to 0 \quad \text{as } k \to \infty.$$

Indeed, the proof of Theorem 4.2 remains true with $P_{u_k} \equiv 0$. If we choose $\Omega' = B_{d_{x_0}}$ then $\int_{B_{d_{x_0}}} \mathrm{D}^\beta u \, dx = \lim\limits_{k\to\infty} \int_{B_{d_{x_0}}} \mathrm{D}^\beta u_k \, dx = 0$ for $|\beta| \leqslant m-1$ and therefore $u \in L^{m,p}_{B_{d_{x_0}}}(\Omega)$.

The following two results give the solution of problem 2 (cf. Introduction).

[4] That is, we consider the norm $|\!|\cdot|\!|_{m,p;\,\Omega,G}$ with $G = B_{d_{x_0}}$ on the subspace $L^{m,p}_{B_{d_{x_0}}}(\Omega)$ of $L^{m,p}(\Omega)$ (cf. Lemma 4.1).

Theorem 4.4. *Let $G \subset\subset \Omega$. Let $(u_k) \subset L_G^{m,p}(\Omega)$ be Cauchy with respect to the norm $|\cdot|_{m,p;\,\Omega}$. Then there exists $u \in L_G^{m,p}(\Omega)$ such that*

$$\|u-u_k\|_{W^{m-1,p}(\Omega')} \to 0 \quad \text{as } k \to \infty, \quad \forall \Omega' \subset\subset \Omega,$$
$$|u-u_k|_{m,p;\,\Omega} \to 0 \quad \text{as } k \to \infty.$$

Proof. a) By Theorem A, (and Theorem 3.1), there exists $P_{u_k} \in \mathcal{P}(m-1)$ so that $v_k := u_k - P_{u_k} \in L^{m,p}_{B_{d_{x_0}}}(\Omega)$ $(k=1,2,\ldots)$, where $x_0 \in \Omega$ is an arbitrary, fixed point. Then (v_k) is Cauchy with respect to the norm $|\cdot|_{m,p;\,\Omega}$, and Corollary 4.3 guarantees the existence (and uniqueness) of a $v \in L^{m,p}_{B_{d_{x_0}}}(\Omega)$ such that

$$\begin{aligned}\|v-v_k\|_{W^{m-1,p}(\Omega')} &\to 0 \quad \text{as } k \to \infty, \quad \forall \Omega' \subset\subset \Omega,\\ |v-v_k|_{m,p;\,\Omega} &\to 0 \quad \text{as } k \to \infty.\end{aligned} \tag{4.4}$$

b) Again, by Theorem A we find a polynomial $Q \in \mathcal{P}(m-1)$ such that $u := (v+Q) \in L_G^{m,p}(\Omega)$. It follows that

$$|u-u_k|_{m,p;\,\Omega} = |v+Q-v_k-P_{u_k}|_{m,p;\,\Omega} = |v-v_k|_{m,p;\,\Omega} \to 0$$

as $k \to \infty$.

c) Observing that $\|u\|_{m-1;\,G} = \|u_k\|_{m-1;\,G} = 0$, we obtain from (4.4) with $\Omega' = G$

$$\begin{aligned}\|Q-P_{u_k}\|_{m-1;\,G} &= \|Q-u+(u-u_k)+(u_k-P_{u_k})\|_{m-1;\,G}\\ &\leqslant \|v-v_k\|_{m-1;\,G}\\ &\leqslant c_0(\operatorname{mes} G)^{1/p'}\|v-v_k\|_{W^{m-1,p}(G)} \to 0\end{aligned}$$

as $k \to \infty$, where c_0 is a constant which depends on m, n and p only. Since the space $\mathcal{P}(m-1)$ is finite dimensional, we have

$$\|Q-P_{u_k}\|_{m-1;\,G} \to 0 \quad \text{iff } \|Q-P_{u_k}\|_{W^{m-1,p}(\Omega')} \to 0 \quad \text{for any } \Omega' \subset\subset \Omega$$

(cf. Lemma 4.1). Whence

$$\begin{aligned}\|u-u_k\|_{W^{m-1,p}(\Omega')} &= \big\|v+Q-(v_k+P_{u_k})\big\|_{W^{m-1,p}(\Omega')}\\ &\leqslant \|v-v_k\|_{W^{m-1,p}(\Omega')} + \|Q-P_{u_k}\|_{W^{m-1,p}(\Omega')} \to 0\end{aligned}$$

as $k \to \infty$. □

Theorem 4.5. *Let $G \subset\subset \Omega$. Then $L^{m,p}(\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{m,p;\,\Omega,G}$.*

Proof. The claim follows by combining the direct decomposition (4.1) with Lemma 4.1 and Theorem 4.4. □

We begin by proving a Poincaré type inequality.

Theorem 5.1. *Let $G \subset\subset \Omega$. Then for every $\Omega' \subset\subset \Omega$ there exists a constant $C_{\Omega'} > 0$, such that*

$$\|u\|_{W^{m-1,p}(\Omega')} \leqslant C_{\Omega'} |u|_{m,p;\,\Omega} \quad \forall u \in L^{m,p}_G(\Omega). \tag{5.1}$$

P r o o f. Assume (5.1) fails. Then we find a sequence $(u_k) \subset L^{m,p}_G(\Omega)$ satisfying

$$\|u_k\|_{W^{m-1,p}(\Omega')} = 1 \quad (k = 1, 2, \ldots), \quad |u_k|_{m,p;\,\Omega} \to 0 \text{ as } k \to \infty.$$

However, Theorem 4.4 implies $\|u_k\|_{W^{m-1,p}(\Omega')} \to 0$ as $k \to \infty$, a contradiction. □

The following theorem plays the key role for our discussion of equivalent norms on $L^{m,p}(\Omega)$.

Theorem 5.2. *Let $G_i \subset\subset \Omega$ $(i = 1, 2)$. Then there exists a constant $K = K_{G_1,G_2} > 0$ such that*

$$|\!|u|\!|_{m,p;\,\Omega,G_1} \leqslant K |\!|u|\!|_{m,p;\,\Omega,G_2} \qquad \forall u \in L^{m,p}(\Omega). \tag{5.2}$$

P r o o f. We make use of the direct decomposition (4.1) with $G = G_2$: given any $u \in L^{m,p}(\Omega)$, we have $u = u_{02} + P_u$ where $u_{02} \in L^{m,p}_{G_2}(\Omega)$, $P_u \in \mathcal{P}(m-1)$. By (5.1) with $\Omega' = G_1$,

$$\begin{aligned} |\!|u|\!|_{m,p;\,\Omega,G_1} &= |\!|u_{02} + P_u|\!|_{m-1,G_1} + |u_{02}|_{m,p;\,\Omega} \\ &\leqslant |\!|P_u|\!|_{m-1;G_1} + c_0 (\operatorname{mes} G_1)^{1/p'} \|u_{02}\|_{W^{m-1,p}(G_1)} + |u_{02}|_{m,p;\,\Omega} \\ &\leqslant |\!|P_u|\!|_{m-1;\,G_1} + (1 + c_0 (\operatorname{mes} G_1)^{1/p'} C_{G_1}) |u_{02}|_{m,p;\,\Omega}. \end{aligned} \tag{5.3}$$

On the other hand, there exists a constant $K_0 = K_{0;\,G_1,G_2} > 0$ such that

$$|\!|P|\!|_{m-1;G_1} \leqslant K_0 |\!|P|\!|_{m-1;\,G_2} \qquad \forall P \in \mathcal{P}(m-1).$$

Inserting this inequality with $P = P_u$ into (5.3) gives

$$|\!|u|\!|_{m,p;\,\Omega,G_1} \leqslant K (|\!|P_u|\!|_{m-1;\,G_2} + |u_{02}|_{m,p;\,\Omega})$$

where $K = \max\{K_0, 1 + c_0 (\operatorname{mes} G_1)^{1/p'} C_{G_1}\}$. Finally, observing that $|\!|P_u|\!|_{m-1;\,G_2} = |\!|u_{02} + P_u|\!|_{m-1;\,G_2}$, $|u_{02}|_{m,p;\,\Omega} = |u_{02} + P_u|_{m,p;\,\Omega'}$, the claim (5.2) follows. □

Theorem 5.3. *Let $G \subset\subset \Omega$. Then there exist constants $K_i > 0$ $(i = 1, 2)$ such that*[5]

$$K_1 \|u\|_{m,p;\,\Omega,G} \leqslant \boldsymbol{|} u \boldsymbol{|}_{m,p;\,\Omega,G} \leqslant K_2 \|u\|_{m,p;\,\Omega,G} \quad \forall u \in L^{m,p}(\Omega). \tag{5.4}$$

P r o o f. First of all, we note that

$$\widetilde{K}_1 \|P\|_{L^1(G)} \leqslant \boldsymbol{|} P \boldsymbol{|}_{m-1;\,G} \leqslant \widetilde{K}_2 \|P\|_{L^1(G)} \quad \forall P \in \mathcal{P}(m-1), \tag{5.5}$$

where the constants $\widetilde{K}_i > 0$ $(i = 1, 2)$ depend on m, n and $\operatorname{mes} G$ only.

Let $u \in L^{m,p}(\Omega)$. We have the decomposition $u = u_0 + P_u$ with $u_0 \in L^{m,p}_G(\Omega)$, $P_u \in \mathcal{P}(m-1)$ (cf. (4.1)). From (5.1) (with $\Omega' = G$) we obtain $\|u_0\|_{L^1(G)} \leqslant C_G(\operatorname{mes} G)^{1/p'} |u_0|_{m,p;\,\Omega}$. Thus by (5.5)

$$\begin{aligned}
\|u\|_{m,p;\,\Omega,G} &= \|u_0 + P_u\|_{L^1(G)} + |u_0|_{m,p;\,\Omega} \\
&\leqslant \max\bigl\{\widetilde{K}_1^{-1}, 1 + C_G(\operatorname{mes} G)^{1/p'}\bigr\}\bigl(\boldsymbol{|} P_u \boldsymbol{|}_{m-1;G} + |u_0|_{m,p;\,\Omega}\bigr),
\end{aligned}$$

i.e., the first inequality in (5.4) with $K_1 = \bigl(\max\{\widetilde{K}_1^{-1}, 1 + C_G(\operatorname{mes} G)^{1/p'}\}\bigr)^{-1}$.

To prove the second inequality in (5.4), we use once more (5.1) (with $\Omega' = G$) to obtain

$$\begin{aligned}
\boldsymbol{|} u \boldsymbol{|}_{m,p;\,\Omega,G} &= \boldsymbol{|} P_u \boldsymbol{|}_{m-1;G} + |u_0|_{m,p;\,\Omega} \\
&\leqslant \widetilde{K}_2 \|u_0 + P_u\|_{L^1(G)} + \widetilde{K}_2 (\operatorname{mes} G)^{1/p'} \|u_0\|_{L^p(G)} + |u_0|_{m,p;\,\Omega} \\
&\leqslant K_2 (\|u_0 + P_u\|_{L^1(G)} + |u_0|_{m,p;\,\Omega}) \\
&= K_2 \|u\|_{m,p;\,\Omega,G},
\end{aligned}$$

where $K_2 = \max\bigl\{\widetilde{K}_2, 1 + C_G \widetilde{K}_2 (\operatorname{mes} G)^{1/p'}\bigr\}$. □

Combining Theorem 5.2 and 5.3 gives: *Let $G_i \subset\subset \Omega$ $(i = 1, 2)$. Then the norms $\|\cdot\|_{m,p;\Omega,G_1}$ and $\|\cdot\|_{m,p;\,\Omega,G_2}$ are equivalent.* From another point of view our choice of the norm $\boldsymbol{|} \cdot \boldsymbol{|}_{m,p;\Omega,G}$ (for $G \subset\subset \Omega$) seems to be very natural, too. If we consider the factor space $L^{m,p}(\Omega)/\mathcal{P}(m-1)$ equipped with the usual norm then it is isometrically isomorphic to $L^{m,p}_G(\Omega)$.

[5] Cf. the introduction to the definition of the norm $\|\cdot\|_{m,p;\,\Omega,G}$.

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