WEIGHTED MULTIDIMENSIONAL INEQUALITIES FOR MONOTONE FUNCTIONS

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. We discuss the characterization of the inequality

$$\left(\int_{\mathbb{R}_+^N} f^q u\right)^{1/q} \leqslant C\left(\int_{\mathbb{R}_+^N} f^p v\right)^{1/p}, \quad 0 < q,\ p < \infty,$$

for monotone functions $f \geqslant 0$ and nonnegative weights u and v and $N \geqslant 1$. We prove a new multidimensional integral modular inequality for monotone functions. This inequality generalizes and unifies some recent results in one and several dimensions.

Keywords: integral inequalities, monotone functions, several variables, weighted L^p spaces, modular functions, convex functions, weakly convex functions

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1. Introduction

Let $\mathbb{R}_+^N := \{(x_1, \ldots, x_N);\ x_i \geqslant 0, i = 1, 2, \ldots, N\}$ and $\mathbb{R}_+ := \mathbb{R}_+^1$. Assume that $f\colon \mathbb{R}_+^N \to \mathbb{R}_+$ is monotone which means that it is monotone with respect to each variable. We denote $f\downarrow$, when f is decreasing (= nonincreasing) and $f\uparrow$ when f is increasing (= nondecreasing). Throughout this paper ω, u, v are positive measurable functions defined on $\mathbb{R}_+^N$, $N \geqslant 1$.

A function P on $[0, \infty)$ is called a modular function if it is strictly increasing, with the values 0 at 0 and ∞ at ∞. For the definition of an N-function we refer to [7]. We say that a modular function P is weakly convex if $2P(t) \leqslant P(Mt)$, for all $t > 0$ and some constant $M > 1$. All convex modular functions are obviously weakly convex. The function $P_1(t) = t^p$, $0 < p < 1$ and the function $P_2(t) = \exp(\sqrt{t}) - 1$ are weakly convex, but not convex. See also [6].

In order to motivate this investigation and put it into a frame we use Section 2 to present the characterization of the inequality

$$\left(\int_{\mathbb{R}_+^N} f^q u\right)^{1/q} \leqslant C\left(\int_{\mathbb{R}_+^N} f^p v\right)^{1/p}, \quad 0 < p, q < \infty, \tag{1}$$

for all $f\downarrow$ or $f\uparrow$.

In Section 3 we will characterize the weights ω, u and v such that

$$Q^{-1}\left(\int_{\mathbb{R}_+^N} Q\left(\omega(x)f(x)\right)u(x)\,\mathrm{d}x\right) \leqslant P^{-1}\left(\int_{\mathbb{R}_+^N} P\left(Cf(x)\right)v(x)\,\mathrm{d}x\right) \tag{2}$$

holds for modular functions P and Q, where P is weakly convex and $0 \leqslant f\downarrow$. Here and in the sequel $C > 0$ denotes a constant independent of f.

Conventions and notation. Products and quotients of the form $0\cdot\infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$ are taken to be 0. $\mathbb{Z}$ stands for the set of all integers and χ_E denotes the characteristic function of a set E.

2. Weighted L^p inequalities for monotone functions

In the one-dimensional case the inequality (1) was characterized in [8, Proposition 1] for both alternative cases $0 < p \leqslant q < \infty$ and $0 < q < p < \infty$ as follows:

(a) If $N = 1$, $0 < p \leqslant q < \infty$, then (1) is valid for all $f\downarrow$ if and only if

$$A_0 := \sup_{t>0}\left(\int_0^t u\right)^{1/q}\left(\int_0^t v\right)^{-1/p} < \infty$$

and the constant $C = A_0$ is sharp.

(b) If $N = 1$, $0 < q < p < \infty$, $1/r = 1/q - 1/p$, then (1) is true for all $f\downarrow$ if and only if

$$B_0 := \left(\int_0^\infty \left(\int_0^t u\right)^{r/p}\left(\int_0^t v\right)^{-r/p} u(t)\,\mathrm{d}t\right)^{1/r} < \infty.$$

Moreover,

$$\left(\frac{q^2}{pr}\right)^{1/p} B_0 \leqslant C \leqslant \left(\frac{r}{q}\right)^{1/r} B_0$$

and

$$B_0^r = \frac{q}{r}\frac{\left(\int_0^\infty u\right)^{r/q}}{\left(\int_0^\infty v\right)^{r/p}} + \frac{q}{p}\int_0^\infty \left(\int_0^t u\right)^{r/q}\left(\int_0^t v\right)^{-r/q} v(t)\,\mathrm{d}t.$$

(c) Similar characterizations are valid when $f\uparrow$, with the only change that the integrals over $[0,t]$ are replaced by integrals over $[t,\infty]$.

Since the one-dimensional inequality (1) expresses the embedding of classical Lorentz spaces, further generalizations and references in this directions can be found in [3].

The multidimensional case was recently treated in [1, Theorem 2.2], for the case $0 < p \leqslant q < \infty$ and in [2, Theorem 4.1], for the case $0 < q < p < \infty$ as follows:

(a) If $0 < p \leqslant q < \infty$, then (1) is valid for all $f \downarrow$ if and only if

$$A_N := \sup_{D \in \mathcal{D}_d} \frac{\left(\int_D u\right)^{1/q}}{\left(\int_D v\right)^{1/p}} < \infty$$

and the constant $C = A_N$ is sharp. Here the supremum is taken over the set $\mathcal{D}_d$ of all "decreasing" domains, i.e., for which the characteristic function is a decreasing function in each variable.

(b) If $0 < q < p < \infty$, then (1) is valid for all $f \downarrow$ if and only if

$$B_N^r := \sup_{0 \leqslant h \downarrow} \int_0^\infty \left(\int_{D_{h,t}} v \right)^{-r/p} \mathrm{d}\left(- \left(\int_{D_{h,t}} u \right)^{r/q} \right) < \infty,$$

where

$$D_{h,t} = \{x \in \mathbb{R}_+^N ;\ h(x) > t\}.$$

Moreover,

$$\frac{1}{2^{1/q}(2^{r/q} + 2^{r/p})^{1/r}} B_N \leqslant C \leqslant 4^{1/q} B_N.$$

If $N = 1$, P and Q are N-functions and $Q \circ P^{-1}$ is convex, then some weight characterizations of the inequality (2) have been obtained in [4] and [5].

For $N > 1$, P and Q N-functions and $Q \circ P^{-1}$ convex, (2) holds for all $0 \leqslant f \downarrow$ if and only if there exists a constant $A = A(\Phi_1, \Phi_2, u, v, \omega)$ such that, for all $\varepsilon > 0$ and $D \in \mathcal{D}_d$,

$$Q^{-1}\left(\int_D Q\left(\varepsilon \omega(x)\right) u(x)\,\mathrm{d}x \right) \leqslant P^{-1}\left(P\left(A\varepsilon\right) \int_D v(x)\,\mathrm{d}x \right).$$

This characterization can be found in [2, Theorem 2.1].

However, if Q and P are not N-functions (hence not convex) and $Q \circ P^{-1}$ is not convex, then the problem of characterizing weights for which (2) holds seems to be to a large extent open. For $N = 1$ the first characterization of this type was given in [6].

In the next section we characterize the weights for which (2) holds when P is weakly convex. This result generalizes both the corresponding one-dimensional result obtained in [6] and the multidimensional case obtained in [2]. Some particular cases of (2) will also be pointed out.

Let $0 \leqslant h(x) \downarrow$ and $t > 0$. Denote

$$D_{h,t} := \{x \in \mathbb{R}^N_+ ;\ h(x) > t\},$$

and

$$\mathcal{D}_d := \bigcup_{0 \leqslant h\downarrow} \bigcup_{t>0} D_{h,t}.$$

The set $\mathcal{D}_d$ consists of all "decreasing" domains $D_{h,t}$. In particular, $\chi_{D_{h,t}}$ is decreasing in each variable. For a strictly decreasing, positive sequence $\{t_k\}$, such that $t_k \to 0$ as $k \to \infty$ we put

$$D_k = D_{h,t_k} := \{x \in \mathbb{R}^N_+ ;\ h(x) > t_k\}, k \in \mathbb{Z}.$$

Obviously, $D_{k+1} \supset D_k$ and we define

$$\Delta_k = \Delta_{h,t_k} := D_{k+1} \setminus D_k.$$

Hence, $\Delta_k \bigcap \Delta_n = \emptyset$, $k \neq n$ and $\mathbb{R}^N_+ = \bigcup\limits_k \Delta_k$. For simplicity we also assume in the sequel that

$$\int_{\mathbb{R}^N_+} v(x)\,\mathrm{d}x = \infty. \tag{3}$$

Theorem 3.1. *Let Q and P be modular functions and P weakly convex. Then* (2) *holds for all $0 \leqslant f \downarrow$ if and only if there exists a constant $B > 0$ such that*

$$Q^{-1}\Bigl(\sum_{k\in\mathbb{Z}} \int_{\Delta_k} Q\Bigl(\frac{\varepsilon_k}{B}\omega(x)\Bigr)\, u(x)\,\mathrm{d}x\Bigr) \leqslant P^{-1}\Bigl(\sum_{k\in\mathbb{Z}} P(\varepsilon_k) \int_{\Delta_k} v(x)\,\mathrm{d}x\Bigr) \tag{4}$$

is satisfied for all positive decreasing sequences $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ and all increasing sequences of decreasing sets $\{D_k\}_{k\in\mathbb{Z}}$ such that $\int_{D_k} v(x)\,\mathrm{d}x = 2^k$.

Proof. The necessity follows, if we replace f in (2) by the decreasing function $f = \sum\limits_{k\in\mathbb{Z}} \varepsilon_k \chi_{\Delta_k}$, $\{\varepsilon_k\}_k$ being a decreasing sequence.

Next we consider the sufficiency. Fix $f \downarrow$ and set $\varepsilon_k = Bt_k$, $D_k = D_{f,t_k}$ and $\Delta_k = \Delta_{f,t_k}$. Because $\mathbb{R}^N_+ = \bigcup\limits_k \Delta_k$ we obtain, using also (4) and the facts that Q, P,

Q^{-1}, P^{-1} are increasing and f is decreasing,

$$\begin{aligned}
Q^{-1}\Big(\int_{\mathbb{R}_+^N} Q\left(\omega(x)f(x)\right)u(x)\,\mathrm{d}x\Big) &= Q^{-1}\Big(\sum_{k\in\mathbb{Z}}\int_{\Delta_k} Q\left(\omega(x)f(x)\right)u(x)\,\mathrm{d}x\Big)\\
&\leqslant Q^{-1}\Big(\sum_{k\in\mathbb{Z}}\int_{\Delta_k} Q\left(\omega(x)t_k\right)u(x)\,\mathrm{d}x\Big)\\
&\leqslant P^{-1}\Big(\sum_{k\in\mathbb{Z}} P(Bt_k)\int_{\Delta_k} v(x)\,\mathrm{d}x\Big)\\
&= P^{-1}\Big(\sum_{k\in\mathbb{Z}} 2P(Bt_k)\int_{\Delta_{k-1}} v(x)\,\mathrm{d}x\Big)\\
&\leqslant P^{-1}\Big(\sum_{k\in\mathbb{Z}}\int_{\Delta_{k-1}} 2P(Bf(x))v(x)\,\mathrm{d}x\Big).
\end{aligned}$$

Therefore, by using the assumption that P is weakly convex, we find that

$$\begin{aligned}
Q^{-1}\Big(\int_{\mathbb{R}_+^N} Q\left(\omega(x)f(x)\right)u(x)\,\mathrm{d}x\Big) &\leqslant P^{-1}\Big(\sum_{k\in\mathbb{Z}}\int_{\Delta_{k-1}} P(MBf(x))v(x)\,\mathrm{d}x\Big)\\
&= P^{-1}\Big(\int_{\mathbb{R}_+^N} P\left(MBf(x)\right)v(x)\,\mathrm{d}x\Big),
\end{aligned}$$

i.e., (2) holds with $C = MB$. The proof is complete. $\square$

We will give now two important corollaries of Theorem 3.1.

Corollary 3.2. *If P and Q are as in Theorem 3.1 and $Q\circ P^{-1}$ is convex, then (2) holds if and only if, for all $\varepsilon > 0$ and decreasing sets D, there exists a $C > 0$ such that*

$$Q^{-1}\left(\int_D Q\Big(\frac{\omega(x)}{C}P^{-1}\Big(\frac{\varepsilon}{\int_D v}\Big)\Big)u(x)\,\mathrm{d}x\right) \leqslant P^{-1}(\varepsilon). \tag{5}$$

Proof. For the necessity we just have to substitute f in (2) with the function

$$f_0(x) = \frac{P^{-1}\left(\frac{\varepsilon}{\int_D v}\right)}{C}\chi_D(x).$$

Next we prove the sufficiency, i.e., that (5) implies (2). According to Theorem 3.1 it is sufficient to prove that (5) implies (4). By applying (5) with $\varepsilon = P(C\varepsilon_k)\int_{D_{k+1}} v$ for each decreasing set D_{k+1} and using the convexity of $Q\circ P^{-1}$ and the weak convexity of P we find that

$$\begin{aligned}\Bigl(\sum_{k\in\mathbb{Z}}\int_{\Delta_k} Q\left(\varepsilon_k\omega(x)\right)u(x)\,\mathrm{d}x\Bigr) &\leqslant \Bigl(\sum_{k\in\mathbb{Z}}\int_{D_{k+1}} Q\left(\varepsilon_k\omega(x)\right)u(x)\,\mathrm{d}x\Bigr)\\ &\leqslant \sum_{k\in\mathbb{Z}} Q\circ P^{-1}\Bigl(P(C\varepsilon_k)\int_{D_{k+1}} v\Bigr)\\ &\leqslant Q\circ P^{-1}\Bigl(\sum_{k\in\mathbb{Z}} 2P(C\varepsilon_k)\int_{D_k} v\Bigr)\\ &\leqslant Q\circ P^{-1}\Bigl(\sum_{k\in\mathbb{Z}} P(MC\varepsilon_k)2^k\Bigr)\\ &= Q\circ P^{-1}\Bigl(\sum_{k\in\mathbb{Z}} P(MC\varepsilon_k)\int_{\Delta_k} v\Bigr).\end{aligned}$$

Hence (4) follows with $B = MC$ and the corollary is proved. □

Remark. If $Q(x) = x^q$ and $P(x) = x^p$, $0 < p \leqslant q < \infty$, then $Q\circ P^{-1}$ is convex and the condition (5) coincides with condition (3). Hence, Corollary 3.2 generalizes Theorem 2.2(d) in [1].

Remark. For $N = 1$ the condition (5) reads

$$Q^{-1}\Bigl(\int_0^r Q\Bigl(\frac{\omega(x)}{B}P^{-1}\Bigl(\frac{\varepsilon}{\int_0^r v}\Bigr)\Bigr)u(x)\,\mathrm{d}x\Bigr) \leqslant P^{-1}\left(\varepsilon\right), \quad \forall r > 0.$$

Thus, if $N = 1$, then Corollary 3.2 coincides with Corollary 1 in [6].

Finally we apply Theorem 3.1 with $P(x) = x^p$ and $Q(x) = x^q$, $0 < p, q < \infty$, and obtain the following result:

Corollary 3.3. *The inequality* (1) *holds for all* $0 < f \downarrow$ *if and only if there exists a constant* $K = K(p,q)$ *such that*

$$\Bigl(\sum_{k\in\mathbb{Z}}\varepsilon_k^q\int_{\Delta_k} u(x)\,\mathrm{d}x\Bigr)^{1/q} \leqslant K\Bigl(\sum_{k\in\mathbb{Z}}\varepsilon_k^p\int_{\Delta_k} v(x)\,\mathrm{d}x\Bigr)^{1/p}$$

for all positive decreasing sequences $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ *and such that* $\int_{D_k} v(x)\,\mathrm{d}x = 2^k$.

Remark. For $N = 1$ a similar characterization is given in [6]. For other multidimensional characterizations of (1) in the case $0 < p \leqslant q < \infty$ see [1] and in the case $0 < q < p < \infty$ see [2] (cf. Section 2).

F i n a l   r e m a r k s . (i) The results in this paper can also be formulated when we remove the technical assumption (3) (cf. [2], [8]).

(ii) Similar results to all results in this paper can be formulated also for increasing functions of several variables.

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