HARDY INEQUALITIES IN FUNCTION SPACES

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$. The paper deals with inequalities of Hardy type related to the function spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$.

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1. Introduction, notation, definitions

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and let

(1) $$d(x) = \inf_{y\in\partial\Omega} |x-y| \quad \text{where} \quad x \in \Omega$$

be the distance to the boundary $\partial\Omega$ of Ω. Let $s \in \mathbb{N}$ and $1 < p < \infty$. Then there is a constant $c > 0$ with

(2) $$\int_\Omega d(x)^{-sp} \, |f(x)|^p \, \mathrm{d}x \leqslant c \sum_{|\alpha|\leqslant s} \int_\Omega |D^\alpha f(x)|^p \, \mathrm{d}x$$

for all (complex-valued) $f \in D(\Omega) = C_0^\infty(\Omega)$. This is a very well known version of Hardy's inequality. Let

$$0 < s < 1, \quad 1 < p < \infty, \quad s \neq \frac{1}{p}.$$

Then there is a constant $c > 0$ with

(3) $$\int_\Omega d^{-sp}(x) \, |f(x)|^p \, \mathrm{d}x \leqslant c \int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y + c \int_\Omega |f(x)|^p \, \mathrm{d}x$$

for all $f \in D(\Omega)$. This is the well known fractional counterpart of (2). If $s = \frac{1}{p}$ then there is no $c > 0$ with (3) for all $f \in D(\Omega)$. Both assertions are covered by [9], pp. 319–320. Of course, the righthand sides of (2) and (3) are the pth powers of the norms in the Sobolev space $H^s_p(\Omega)$ and in the Besov space $B^s_{pp}(\Omega)$, respectively. By some real and complex interpolation one can extend (2) and (3) to all Sobolev spaces $H^s_p(\Omega)$ and all (special) Besov spaces $B^s_{pp}(\Omega)$ with

$$1 < p < \infty, \quad s > 0, \quad s - \frac{1}{p} \notin \mathbb{N}_0 \tag{4}$$

(and $f \in D(\Omega)$). If one asks for inequalities of type (2) or (3) with respect to the spaces $F^s_{pq}(\Omega)$ and $B^s_{pq}(\Omega)$, then (avoiding limiting cases) there is a natural restriction of the parameters s, p, q given by

$$0 < p < \infty, \quad 0 < q < \infty, \quad s > \sigma_p = n\Big(\frac{1}{p} - 1\Big)_+. \tag{5}$$

As for $F^s_{pq}(\Omega)$, we proved in [13] the corresponding Hardy inequalities. Inequalities of this type in $B^s_{pq}(\Omega)$ with $p > 1$ and $q \geqslant 1$ have been discussed in [9], p. 319. It is the aim of this paper to complement these results and to prove a theorem which has a rather final character (with the exception of the limiting cases).

We assume that the reader is familiar with the basic notation of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ on $\mathbb{R}^n$. We refer to [8, 10, 11]. We only mention that $H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$ are the (fractional) Hardy-Sobolev spaces. Recall that Ω is a bounded C^∞ domain in $\mathbb{R}^n$. Then $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ have the usual meaning: the restriction of $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ to Ω, respectively. Let $\mathring{B}^s_{pq}(\Omega)$ and $\mathring{F}^s_{pq}(\Omega)$ be the completion of $D(\Omega)$ in $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$, respectively. Finally, let

$$\widetilde{B}^s_{pq}(\Omega) = \{f \in B^s_{pq}(\mathbb{R}^n) \colon \operatorname{supp} f \subset \overline{\Omega}\} \tag{6}$$

with the quasinorm

$$\|f|\widetilde{B}^s_{pq}(\Omega)\| = \|f|B^s_{pq}(\mathbb{R}^n)\|. \tag{7}$$

Similarly, $\widetilde{F}^s_{pq}(\Omega)$ is defined. We may assume from the very beginning that p, q, s are restricted by (5). Then without any ambiguity, one may consider $\widetilde{B}^s_{pq}(\Omega)$ and $\widetilde{F}^s_{pq}(\Omega)$ either as subspaces of $S'(\mathbb{R}^n)$ or of $D'(\Omega)$. We refer to [13] for a discussion of the matter.

If

$$0 < p < \infty, \quad 0 < q < \infty, \quad s > \sigma_p, \quad \text{and} \quad s - \frac{1}{p} \notin \mathbb{N}_0, \tag{8}$$

then

$$\widetilde{F}^s_{pq}(\Omega) = \overset{\circ}{F}{}^s_{pq}(\Omega) \quad \text{and} \quad \widetilde{B}^s_{pq}(\Omega) = \overset{\circ}{B}{}^s_{pq}(\Omega). \tag{9}$$

The first assertion coincides with [13], Theorem 2.4.2. As for the B-spaces with $1 < p < \infty$, $1 \leqslant q < \infty$ we refer to [9], Theorem 4.3.2/1, pp. 317–318. However, the extension of the technique used there to all p, q, s with (8) is covered by the F-case and the real interpolation formula (27) below. (First one proves this assertion for $n = 1$, then one can reduce the case $n > 1$ to the onedimensional case by using [10], Theorem 2.5.13, p. 115). Hence we take (8), (9) for granted. An extension of (9) to values $s - \frac{1}{p} \in \mathbb{N}_0$ is not possible in general. We refer to [9], p. 319, formula (10), for the B-case (restricted to $1 < p < \infty$, $1 < q < \infty$) and to [13], 2.4.4, formula (2.74), for the F-case (restricted to $1 < p < \infty$, $0 < q < \infty$). We will not need (8), (9) in the sequel, but it illustrates the well known exceptional role of the parameters s, p with $s - \frac{1}{p} \in \mathbb{N}_0$ in inequalities of Hardy type.

Finally, let

$$\Omega^t = \{x \in \Omega \colon\ d(x) < t\} \quad \text{where} \quad t > 0. \tag{10}$$

2. Results and comments

2.1. Theorem. *Let p, q, s be given by* (5).

(i) *There is a constant $c > 0$ such that*

$$\int_\Omega d^{-sp}(x)|f(x)|^p \,\mathrm{d}x \leqslant c\|f\,|\widetilde{F}^s_{pq}(\Omega)\|^p \tag{11}$$

for all $f \in \widetilde{F}^s_{pq}(\Omega)$.

(ii) *There is a constant $c > 0$ such that*

$$\int_0^\infty t^{-sq}\left(\int_{\Omega^t} |f(x)|^p \,\mathrm{d}x\right)^{\frac{q}{p}} \frac{\mathrm{d}t}{t} \leqslant c\|f\,|\widetilde{B}^s_{pq}(\Omega)\|^q \tag{12}$$

for all $f \in \widetilde{B}^s_{pq}(\Omega)$.

2.2. Remark. Part (i) coincides essentially with [13], Proposition 2.2.5. Part (ii), restricted to $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, may be found in [9], p. 319. In other words, compared with what is known, the theorem extends (12) to the full scale of parameters given by (5). Maybe one can do this by following the rather tricky arguments in [13] now armed with the characterization of all the spaces $B^s_{pq}(\mathbb{R}^n)$ by differences as in [10], Theorem 2.5.12, p. 110. However, our intention is different. We prove part (ii) for all parameters admitted by real interpolation of part (i).

2.3. Special cases. If $p=q$ then (12) coincides with

$$\int_\Omega d^{-sp}(x)\,|f(x)|^p\,\mathrm{d}x \leqslant c\|f\,|\widetilde{B}^s_{pp}(\Omega)\|^p \tag{13}$$

for all $f \in \widetilde{B}^s_{pp}(\Omega)$. Recall that $H^s_p = F^s_{p,2}$ are the Hardy-Sobolev spaces. If p, q, s are given by (8), then we have (9) and, hence, there is a number $c>0$ such that

$$\int_\Omega d^{-sp}(x)\,|f(x)|^p\,\mathrm{d}x \leqslant c\|f\,|H^s_p(\Omega)\|^p \tag{14}$$

and

$$\int_\Omega d^{-sp}(x)\,|f(x)|^p\,\mathrm{d}x \leqslant c\|f\,|B^s_{pp}(\Omega)\|^p \tag{15}$$

for all $f \in D(\Omega)$. This generalizes (2) and (3).

2.4. Exceptional cases. It is well known in the theory of function spaces that the spaces B^s_{pq}, and F^s_{pq}, with

$$0<p<\infty, \quad 0<q<\infty, \quad s>\sigma_p, \quad s-\frac{1}{p} \in \mathbb{N}_0\,, \tag{16}$$

play a special role. Restricted to $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ in case of the B-spaces and to $H^s_p = F^s_{p,2}$ with $1<p<\infty$ we refer to [5, 6, 9] and, more recently, [7]. In [13] we treated F-spaces with (16). Also the boundary behaviour of functions belonging to the B^s_{pq}-spaces or F^s_{pq}-spaces with (16) is somewhat delicate. Final assertions may be found in [8], p. 83, due to J. Franke, [4]; see also [13]. We mention only a rather special case which will be useful later on. Let $I=(0,1)$ be the unit interval. Then, for $1<p<\infty$,

$$\overset{\circ}{B}{}^{\frac{1}{p}}_{pp}(I) = B^{\frac{1}{p}}_{pp}(I) \quad \text{and} \quad \overset{\circ}{H}{}^{\frac{1}{p}}_{p}(I) = H^{\frac{1}{p}}_{p}(I), \tag{17}$$

whereas $\widetilde{B}^{\frac{1}{p}}_{pp}(I)$ does not coincide with $B^{\frac{1}{p}}_{pp}(I)$ and $\widetilde{H}^{\frac{1}{p}}_{p}(I)$ does not coincide with $H^{\frac{1}{p}}_{p}(I)$. One may ask for substitutes of (14), (15) in these exceptional cases. Let $A_p(I)$ be either $B^{\frac{1}{p}}_{pp}(I)$ or $H^{\frac{1}{p}}_{p}(I)$, and let $A^1_p(I)$ be either $B^{1+\frac{1}{p}}_{pp}(I)$ or $H^{1+\frac{1}{p}}_{p}(I)$. Let log be taken with respect to base 2.

2.5. Proposition. *Let $1<p<\infty$ and $0<\delta<\frac{1}{4}$.*
(i) *If $\varkappa>1$, then there is a constant $c>0$ such that*

$$\int_0^\delta \frac{|f(x)|^p}{x|\log x|^p\,\log^\varkappa|\log x|}\,\mathrm{d}x \leqslant c\|f\,|A_p(I)\|^p \tag{18}$$

and

$$\int_0^\delta \frac{|f(x)|^p}{x^{p+1}|\log x|^p \log^\varkappa |\log x|}\,\mathrm{d}x \leqslant c\|f|A^1_p(I)\|^p \tag{19}$$

for all $f \in D(I)$.

(ii) *Let* $\varkappa < 0$. *There exists a function*

$$f_0 \in \overset{\circ}{B}{}^{\frac{1}{p}}_{pp}(I) \cap \overset{\circ}{H}{}^{\frac{1}{p}}_{p}(I) \tag{20}$$

such that the lefthand side of (18) *(with* f_0 *in place of* f*) is infinite (divergent integral) and there exists a function*

$$f_1 \in \overset{\circ}{B}{}^{1+\frac{1}{p}}_{pp}(I) \cap \overset{\circ}{H}{}^{1+\frac{1}{p}}_{p}(I) \tag{21}$$

such that the lefthand side of (19) *(with* f_1 *in place of* f*) is infinite (divergent integral).*

2.6. Remark. There is a gap between the sufficient condition $\varkappa > 1$ in (i) and the necessary condition $\varkappa \geqslant 0$ in (ii). Otherwise $x|\log x|^p$ and $x^{1+p}|\log x|^p$ is the appropriate replacement of $d^{sp}(x)$ in (14), (15) in the exceptional cases $s = \frac{1}{p}$ and $s = 1 + \frac{1}{p}$, respectively.

2.7. Comment. Both (11) and (12) are sharp in the following sense. If

$$0 < p < \infty, \quad 0 < q < \infty, \quad s > n\left(\frac{1}{\min(p,q)} - 1\right)_+ \tag{22}$$

then

$$\|f|\widetilde{F}^s_{pq}(\Omega)\|^p \sim \|f|F^s_{pq}(\Omega)\|^p + \int_\Omega d^{-sp}(x)\,|f(x)|^p\,\mathrm{d}x \tag{23}$$

(equivalent quasinorms), [13], Theorem 2.2.8. If

$$1 < p < \infty, \quad 1 \leqslant q < \infty, \quad s > 0, \tag{24}$$

then

$$\|f|\widetilde{B}^s_{pq}(\Omega)\|^q \sim \|f|B^s_{pq}(\Omega)\|^q + \int_0^\infty t^{-sq}\left(\int_{\Omega^t} |f(x)|^p\,\mathrm{d}x\right)^{\frac{q}{p}} \frac{\mathrm{d}t}{t}, \tag{25}$$

(equivalent quasinorms), [9], p. 319. There is hardly any doubt that (25) holds for all p, q, s with (5).

3. Proofs

3.1. Proof of Theorem 2.1. As said above, part (i) is covered by [13], Proposition 2.2.5. We use this assertion to prove part (ii). Let $0 < \theta < 1$ and

$$0 < p < \infty, \quad 0 < q < \infty, \quad s_1 > s_0 > \sigma_p, \quad s = (1-\theta)s_0 + \theta s_1. \tag{26}$$

Then we have by real interpolation

$$\Big(\widetilde{F}^{s_0}_{pp}(\Omega),\ \widetilde{F}^{s_1}_{pp}(\Omega)\Big)_{\theta,q} = \widetilde{B}^{s}_{pq}(\Omega). \tag{27}$$

Without $\sim$ this is a well known interpolation formula, [10], 2.4.2, p. 64. Recall $F^s_{pp} = B^s_{pp}$. Then (27), restricted to $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, is covered by [9], Theorem 4.3.2/2, p. 318. Using the techniques developed in [10], one can extend the proof in [9] from $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ to $0 < p < \infty$, $0 < q < \infty$. In particular, we have (27) with (26). Let $0 < p < \infty$ and let $L_p(\Omega, d^{-s})$ be the quasi-Banach space, quasinormed by

$$\Big(\int_\Omega |f(x)|^p\, d^{-sp}(x)\,\mathrm{d}x\Big)^{\frac{1}{p}}. \tag{28}$$

Let p, q, s_1, s_0, s be given by (26). Then

$$\big(L_p(\Omega, d^{-s_0}),\ L_p(\Omega, d^{-s_1})\big)_{\theta,q} \tag{29}$$

is a quasi-Banach space with the quasinorm

$$\Bigg[\int_0^\infty \lambda^{\theta\frac{q}{p}} \Big(\int_{d(x)\leqslant \lambda^{\frac{1}{(s_0-s_1)p}}} |f(x)|^p\, d^{-s_0p}(x)\,\mathrm{d}x\Big)^{\frac{q}{p}} \frac{\mathrm{d}\lambda}{\lambda}\Bigg]^{\frac{1}{q}}. \tag{30}$$

We refer to [1], p. 127. With $t = \lambda^{\frac{1}{(s_0-s_1)p}}$ the quasinorm (30) is equivalent to

$$\Bigg[\int_0^\infty t^{-\theta(s_1-s_0)q} \Big(\int_{d(x)\leqslant t} |f(x)|^p\, d^{-s_0p}(x)\,\mathrm{d}x\Big)^{\frac{q}{p}} \frac{\mathrm{d}t}{t}\Bigg]^{\frac{1}{q}}. \tag{31}$$

Using the (apparently crude) estimate

$$t^{-s_0p}\int_{\Omega^t} |f(x)|^p\,\mathrm{d}x \leqslant \int_{\Omega^t} |f(x)|^p\, d^{-s_0p}(x)\,\mathrm{d}x \tag{32}$$

it follows that (31) can be estimated from below by

$$\Bigg[\int_0^\infty t^{-sq} \Big(\int_{\Omega^t} |f(x)|^p\,\mathrm{d}x\Big)^{\frac{q}{p}} \frac{\mathrm{d}t}{t}\Bigg]^{\frac{1}{q}}. \tag{33}$$

This observation together with (11), (27) and (29), (30), and the interpolation property prove (12). $\square$

3.2. Remark. As noted above, the estimate (32) seems to be rather crude. But this is not the case in the above context. If one discretizes (31) via $t = 2^{-j}$, say, with $j \in \mathbb{N}_0$, some calculations prove that (33) is even an equivalent quasinorm in the space in (29). By (25) this is not a surprise.

3.3. Proof of Proposition 2.5.

Step 1. We prove (18). Let $1 < p < q < \infty$. Based on [12] we proved in [3], p. 90,

$$\left(\int_0^1 |f(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leqslant c\, q^{(1-\frac{1}{p})} \, \|f \big| A_p(I)\|, \tag{34}$$

where $c > 0$ is independent of q and, of course, of $f \in A_p(I)$ (but may depend on p). We may assume $\delta = 2^{-J}$ with $J \in \mathbb{N}$ large. Then

$$\begin{aligned} &\int_0^\delta \frac{|f(x)|^p}{x|\log x|^p \,\log^{\varkappa}|\log x|} \,\mathrm{d}x \\ &\qquad \leqslant c_1 \sum_{j=J}^\infty 2^j j^{-p} (\log j)^{-\varkappa} \int_{2^{-j-1}}^{2^{-j}} |f(x)|^p \,\mathrm{d}x \\ &\qquad \leqslant c_2 \sum_{j=J}^\infty 2^j j^{-p} (\log j)^{-\varkappa} \left(\int_{2^{-j-1}}^{2^{-j}} |f(x)|^j \,\mathrm{d}x\right)^{\frac{p}{j}} 2^{-j(1-\frac{p}{j})} \\ &\qquad \leqslant c_3 \sum_{j=J}^\infty j^{-p} (\log j)^{-\varkappa} j^{(1-\frac{1}{p})p} \|f \big| A_p(I)\|^p. \end{aligned} \tag{35}$$

(18) follows from (35) and $\varkappa > 1$.

Step 2. We prove (19). By [2] we have for some $c > 0$,

$$|f(x)| \leqslant c\,|x|\,|\log x|^{1-\frac{1}{p}} \, \|f \big| A_p^1(I)\| \quad \text{for all } f \in D(I). \tag{36}$$

Inserting this estimate in the lefthand side of (19) and using again $\varkappa > 1$ we arrive at (19).

Step 3. We prove part (ii). Let $\sigma p = 1 - \varkappa > 1$. By [12] or [3], 2.7.1, p. 82, and (17) we may choose for f_0 in (20) the function

$$f_0(x) = |\log x|^{1-\frac{1}{p}} \,\log^{-\sigma} |\log x| \,\psi(x), \quad 0 < x < \delta, \tag{37}$$

where $\psi(x)$ is a C^∞ function with support in $(-\delta, \delta)$ and $\psi(0) = 1$. As for f_1 in (21) we may choose

$$f_1(x) = \psi(x) \int_0^x f_0(y) \,\mathrm{d}y \geqslant cx \,|\log x|^{1-\frac{1}{p}} \log^{-\sigma} |\log x| \,\psi(x), \tag{38}$$

$0 < x < \delta$, again with $\sigma p = 1 - \varkappa > 1$. Here $c > 0$ is a suitable constant. As for the last estimate we refer to [2]. Otherwise, $f_1 \in A^1_p(I)$ is clear in both versions of $A^1_p(I)$. Finally, the boundary value $f_1(0) = 0$ is sufficient for (21). This is covered by [4] or [8], p. 83. $\square$

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