POSITIVE SOLUTIONS OF CRITICAL QUASILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^N$

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. We consider the existence of positive solutions of

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u + \alpha h(x)|u|^{q-2}u + f(x)|u|^{p^*-2}u \tag{1}$$

in $\mathbb{R}^N$, where $\lambda, \alpha \in \mathbb{R}$, $1 < p < N$, $p^* = Np/(N-p)$, the critical Sobolev exponent, and $1 < q < p^*$, $q \neq p$. Let $\lambda_1^+ > 0$ be the principal eigenvalue of

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} g(x)|u|^p > 0, \tag{2}$$

with $u_1^+ > 0$ the associated eigenfunction. We prove that, if $\int_{\mathbb{R}^N} f|u_1^+|^{p^*} < 0$, $\int_{\mathbb{R}^N} h|u_1^+|^q > 0$ if $1 < q < p$ and $\int_{\mathbb{R}^N} h|u_1^+|^q < 0$ if $p < q < p^*$, then there exist $\lambda^* > \lambda_1^+$ and $\alpha^* > 0$, such that for $\lambda \in [\lambda_1^+, \lambda^*)$ and $\alpha \in [0, \alpha^*)$, (1) has at least one positive solution.

Keywords: the p-Laplacian, positive solutions, critical exponent

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1. Introduction

We study the existence of positive solutions to the following problem in $\mathbb{R}^N$

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u + \alpha h(x)|u|^{q-2}u + f(x)|u|^{p^*-2}u, \tag{1.1$_\lambda$}$$

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where $\lambda, \alpha \in \mathbb{R}$, $1 < p < N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $p^* = Np/(N-p)$, $1 < q < p^*$, $q \neq p$, f, g and h satisfy $g^+ \not\equiv 0$, $f^\pm \not\equiv 0$, $h^+ \not\equiv 0$, and other conditions. The problem is closely related to the following eigenvalue problem,

$$(1.2)_\lambda \qquad -\Delta_p u = \lambda g(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} g(x)|u|^p > 0.$$

It is known that $(1.2)_\lambda$ has an eigenvalue $\lambda_1^+ > 0$ associated with a positive eigenfunction u_1^+ (see [6]).

Equations involving critical Sobolev exponents have been studied extensively, and there exists a large body of literature. We refer to [5] and the references therein. Specifically, Swanson and Yu [12] studied $(1.1)_\lambda$ for the case $\lambda \in (0, \lambda_1^+)$ and $p < q < p^*$. It is shown in [12] that if $g \geqslant 0$, $g \in L^{N/p}(\mathbb{R}^N)$, $f \geqslant 0$, and $h \geqslant h_0 > 0$ in $\mathbb{R}^N$, then $(1.1)_\lambda$ has a positive solution if $\lambda \in (0, \lambda_1^+)$. Noussair, Swanson and Yang [11] investigated the problem

$$-\Delta_m u = p(x)u^\tau + q(x)u^\gamma$$

on an open connected smooth domain, where $2 \leqslant m < N$, $m - 1 < \gamma < \tau$, and $\tau + 1 = Nm/(N-m)$. The existence of at least one positive solution was obtained for both p and q nonnegative and satisfying other local conditions. More recently Noussair and Swanson [10] considered

$$(1.3) \qquad -\Delta u = p|u|^{\tau-2}u + q|u|^{\gamma-2}u \quad \text{in } \mathbb{R}^N,$$

where $2 < \gamma < \tau = 2N/(N-2)$, and showed, under suitable assumptions, including nonnegativity of p and q, that (1.3) has two positive decaying solutions. The existence of two positive solutions of $(1.1)_\lambda$ was studied for the case $p < q < p^*$ and $f \equiv 0$ in [4], and for the case $h(x) \equiv 0$ in [5]. Various forms of the equation

$$(1.4) \qquad -\Delta_p u + a(x)|u|^{p-2}u = \beta h(x)|u|^{q-2}u + k(x)|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N$$

are treated by Alves, Gonçalves and Miyagaki in [1], [2] and [7], where a, h and k are nonnegative, $1 < q \neq p$, $q < p^*$, and $\beta \geqslant 0$. The existence of nonnegative solutions was obtained via Mountain Pass arguments. Specifically, [1] deals with the case $1 < q < p$, $a \equiv 0$, $k \equiv 1$; [2] the case $1 < q < p$ and $\beta = 1$; and [7] the case $a \equiv 0$, $\beta = 1$, $k \equiv 1$, and $1 < q < p^*$, $q \neq p$, $p \geqslant 2$.

In this paper we are mainly concerned with the situation where $\lambda \geqslant \lambda_1^+$. We note that, for $\lambda \in (0, \lambda_1^+)$, the functional $\int_{\mathbb{R}^N}(|\nabla u|^p - \lambda g|u|^p)$ is always positive for $u \neq 0$, so one can use a Mountain Pass type argument to show that $(1.1)_\lambda$ has a positive solution. Assuming $h > 0$ in some open set in $\mathbb{R}^N$, one can even prove the existence of two positive solutions by first finding a local nonzero minimizer of the associated functional and then using the Mountain Pass Theorem to find a saddle point. This is the approach used in [1], [2] and [7]. For $\lambda \geqslant \lambda_1^+$, the situation is different. The problem is that in this case, the functional $\int_{\mathbb{R}^N}(|\nabla u|^p - \lambda g|u|^p)$ is no longer positive definite. Even a local minimizer is difficult to find. Specifically, for $\lambda > \lambda_1^+$, $\int_{\mathbb{R}^N}(|\nabla u|^p - \lambda g|u|^p)$ will always approach $-\infty$ as $\|u\| \to \infty$ in the direction of u_1^+, while it can achieve positive values in other directions. For $\lambda = \lambda_1^+$, $\int_{\mathbb{R}^N}(|\nabla u|^p - \lambda g|u|^p)$ will always be zero in the direction of u_1^+. This destroys the Mountain Pass structure. Here we use a procedure devised by Tarantello [13] and further utilized in [4] and [5]. The conditions

$$\int_{\mathbb{R}^N} f(u_1^+)^{p^*} < 0, \quad \int_{\mathbb{R}^N} h(u_1^+)^q > 0, \tag{1.5}$$

$$\int_{\mathbb{R}^N} f(u_1^+)^{p^*} < 0, \quad \int_{\mathbb{R}^N} h(u_1^+)^q < 0, \tag{1.6}$$

are essential in our presentation. Under further related local conditions on g, h and f, we can prove the existence of positive solutions of $(1.1)_\lambda$.

Main Result. *Assume* (1.5) *if* $1 < q < p$ *and* (1.6) *if* $p < q < p^*$. *Then there exist* $\lambda^* > \lambda_1^+$ *and* $\alpha^* > 0$, *such that for any* $\lambda \in [\lambda_1^+, \lambda^*)$ *and* $\alpha \in [0, \alpha^*)$, $(1.1)_\lambda$ *has a positive solution (see Theorems* 3.8 *and* 4.7 *for precise assumptions on* f, g *and* h*).*

In our setting, g and h are allowed more flexibility than in [1], [2] and [7], e.g., they may or may not change sign. But (1.5) forces f to change sign, and (1.6) forces both f and h to change sign. We note that here we need an additional condition that α is small enough. While this is the case for $1 < q < p$ in [1], [2] and [7], no such smallness restriction is postulated to h in [7] and [12], for the case $p < q < p^*$.

This paper is organized as follows: In Section 2 we study the geometric structure of certain solution manifolds of the associated functional. Section 3 provides the proof of the existence result for the case $1 < q < p$. The case $p < q < p^*$ is discussed in Section 4.

2. Geometry of the solution manifolds for $1 < q < p$

We collect our basic assumptions and recall some known results. We assume throughout this paper that $1 < p < N$, $p^* = Np/(N-p)$, $1 < q < p^*$ and $q \neq p$. We also assume

(g0) $g(x) = g^+(x) - g^-(x)$, $\quad g^+, g^- \geqslant 0$, $\quad$ and $\quad g^+ \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$, $g^- \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, (h0) $h \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N) \cap L^Q(\mathbb{R}^N)$, where $Q = Np[Np - q(N-p)]^{-1}$.

Let

$$\omega(x) = \frac{1}{(1+|x|)^p}, \ x \in \mathbb{R}^N,$$
$$w(x) = \max\{g^-(x), \omega(x)\} > 0, \ x \in \mathbb{R}^N.$$

Let V be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|$ defined by

$$\|u\| = \left(\int |\nabla u|^p + \int w(x)|u|^p \right)^{1/p}.$$

Here and henceforth the integrals are taken on $\mathbb{R}^N$ unless otherwise stated. Then V is a uniformly convex Banach space. In this paper $\|\cdot\|_p$ will denote the usual L^p norm, and $D^{1,p}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_D = \left(\int |\nabla u|^p \right)^{1/p}.$$

Note that since $V \subset D^{1,p}(\mathbb{R}^N)$, a weakly convergent sequence in V is also weakly convergent in $D^{1,p}(\mathbb{R}^N)$. By Hardy's inequality, $D^{1,p}(\mathbb{R}^N)$ is embedded continuously in $L^p(\mathbb{R}^N, \omega(x))$, so a strongly convergent sequence in $D^{1,p}(\mathbb{R}^N)$ is also strongly convergent in $L^p(\mathbb{R}^N, \omega(x))$.

Throughout this paper the function f is always assumed to satisfy

(f0) $f^\pm \not\equiv 0$ and $f(x) \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$.

We have (from Lemma 2.3 of [6]):

Proposition 2.1. *Assume the above conditions are satisfied. Then there exists a unique, simple isolated eigenvalue $\lambda_1^+ > 0$, such that the eigenvalue problem* $(1.2)_\lambda$ *has a positive eigenfunction $u_1^+ \in V$ associated with λ_1^+.*

Next we introduce the following functional

$$I_\lambda(u) = \frac{1}{p} \int (|\nabla u|^p - \lambda g|u|^p) - \frac{\alpha}{q} \int h|u|^q - \frac{1}{p^*} \int f|u|^{p^*}. \tag{2.1}$$

It is clear that the functional I_λ is well defined on V. Obviously a critical point of I_λ in V is a (weak) solution of $(1.1)_\lambda$. We can always assume that critical points of I_λ are nonnegative functions since I_λ is an even functional. For simplicity, we will assume in the sequel that $\alpha > 0$, for the case $\alpha = 0$ has been covered in [5].

Define

$$
\begin{aligned}
J_\lambda(u) &= \int (|\nabla u|^p - \lambda g|u|^p),\\
\Lambda_\lambda &= \{u \in V\colon\ \Psi_\lambda(u) := \langle I'_\lambda(u), u\rangle = 0\}\\
&= \Big\{u \in V\colon\ J_\lambda(u) = \alpha \int h|u|^q + \int f|u|^{p^*}\Big\},
\end{aligned}
$$

and

$$
\Lambda_\lambda^- = \{u \in \Lambda_\lambda\colon\ \langle \Psi'_\lambda(u), u\rangle < 0\}. \tag{2.2}
$$

We list the following equivalent expressions of this set.

$$
\begin{aligned}
\Lambda_\lambda^- &= \Big\{u \in \Lambda_\lambda\colon\ (p-q)J_\lambda(u) < (p^*-q)\int f|u|^{p^*}\Big\}\\
&= \Big\{u \in \Lambda_\lambda\colon\ (p^*-p)J_\lambda(u) > \alpha(p^*-q)\int h|u|^q\Big\}\\
&= \Big\{u \in \Lambda_\lambda\colon\ \alpha(p-q)\int h|u|^q < (p^*-p)\int f|u|^{p^*}\Big\}.
\end{aligned} \tag{2.3}
$$

We note that it is not entirely clear whether Λ_λ^- is nonempty for general g, h and f. To show that $\Lambda_\lambda^- \neq \emptyset$, we introduce other conditions on g, f and h.

(f1) $f(0) = \|f\|_\infty$ and for some $r > 0$, $f(x) > 0$ for $x \in B(0, 2r)$,
(h1) $h(x) \geqslant h_0 > 0$ in $B(0, 2r)$,
(g1) $g(x) \geqslant g_0 > 0$ in $B(0, 2r)$.

Lemma 2.2. *Suppose* (f0), (f1), (g0), (g1), (h0) *and* (h1) *hold. Then for $\lambda > 0$ in any bounded interval, there exists $\alpha_1 > 0$ such that $\Lambda_\lambda^- \neq \emptyset$ provided $\alpha \in (0, \alpha_1)$.*

Proof. Define, for $\varepsilon > 0$,

$$
u_\varepsilon(x) = \frac{\psi(x)}{(\varepsilon + |x|^{p/(p-1)})^{(N-p)/p}}, \quad v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon(x)\|_{p^*}},
$$

where $\psi \in C_0^\infty(B(0, 2r))$ is such that $0 \leqslant \psi(x) \leqslant 1$ and $\psi(x) \equiv 1$ on $B(0, r)$.
Consider for $t > 0$,

$$
\Psi_\lambda(tv_\varepsilon) = t^p J_\lambda(v_\varepsilon) - \alpha t^q \int h|v_\varepsilon|^q - t^{p^*} \int f|v_\varepsilon|^{p^*}.
$$

Let $s_\alpha(t) = at^p - \alpha bt^q - ct^{p^*}$, with $a = J_\lambda(v_\varepsilon)$, $b = \int h|v_\varepsilon|^q$ and $c = \int f|v_\varepsilon|^{p^*}$. It is clear that $b > 0$, $c > 0$. By continuous dependence of the principal eigenvalue on the domain, $a > 0$ for $\varepsilon > 0$ small enough. Fix this ε so a, b, c are fixed and let α vary. One easily sees that $s_\alpha(t) \to -\infty$ as $t \to \infty$. Moreover, as $b \to 0$, $s_\alpha(t) \to s_0(t) := at^p - ct^{p^*}$ in C^1 with respect to t. Let t_0 be such that $s_0(t_0) = 0$ and $s_0(t) > 0$ for $t < t_0$. Then $s_0'(t_0) < 0$. By C^1 convergence of s_α to s_0, we easily conclude that there exist $\alpha_1 > 0$ and $\tau > 0$, such that if $0 < \alpha < \alpha_1$, $s_\alpha(t_\alpha) = 0$ and $s_\alpha'(t_\alpha) < 0$ for some $t_\alpha \in (t_0 - \tau, t_0 + \tau)$, that is, $t_\alpha v_\varepsilon \in \Lambda_\lambda^-$. This completes the proof. $\square$

Next we study the geometry of the set Λ_λ^- for $\lambda > 0$. We will seek a critical point of I_λ on Λ_λ^-. Observe that for any $u \in \Lambda_\lambda$,

$$\begin{aligned} I_\lambda(u) &= \frac{1}{N}\int f|u|^{p^*} + \alpha\Big(\frac{1}{p} - \frac{1}{q}\Big)\int h|u|^q \\ &= \frac{1}{N}J_\lambda(u) - \alpha\Big(\frac{1}{q} - \frac{1}{p^*}\Big)\int h|u|^q. \end{aligned} \tag{2.4}$$

We also assume that $\|u_1^+\| = 1$.

The next lemma requires the following conditions.

(f2) $\int f(u_1^+)^{p^*} < 0$.

(h2) $\int h(u_1^+)^q > 0$.

Lemma 2.3. *Assume $p > q$, (f0), (f1), (f2), (g0), (g1), (h0), (h1) and (h2) hold. Then there exist $\lambda^* > \lambda_1^+$ and $\alpha_2 > 0$ with $\alpha_2 \leqslant \alpha_1$, such that for any $\bar\lambda \in (0, \lambda_1^+)$, there exists $\sigma > 0$, such that for any $\lambda \in [\bar\lambda, \lambda^*)$ and $\alpha \in (0, \alpha_2)$, we have $J_\lambda(u) \geqslant \sigma\|u\|^p$ for any $u \in \Lambda_\lambda^-$.*

Proof. We argue by contradiction. Suppose there exist λ_n, α_n and $u_n \in \Lambda_{\lambda_n}^-$ such that

$$\alpha_n \to 0, \quad \lambda_n \to \widehat\lambda \in [\bar\lambda, \lambda_1^+], \quad J_{\lambda_n}(u_n) < \frac{1}{n}\|u_n\|^p. \tag{2.5}$$

We explicitly note that here $\Lambda_{\lambda_n}^-$ also depends on α_n. Let $v_n = u_n/\|u_n\|$. Without loss of generality we may assume $v_n \to v_0$ weakly in V. Then we have $\int g^+|v_n|^p \to \int g^+|v_0|^p$ by compactness. We then derive by weak lower semicontinuity of the norm that

$$\begin{aligned} 0 &\leqslant \int |\nabla v_0|^p - \widehat\lambda\int g^+|v_0|^p + \widehat\lambda\int g^-|v_0|^p \\ &\leqslant \liminf_{n\to\infty}\Big(\int |\nabla v_n|^p - \lambda_n\int g^+|v_n|^p + \lambda_n\int g^-|v_n|^p\Big) \\ &= \liminf_{n\to\infty} J_{\lambda_n}(v_n) \leqslant \lim_{n\to\infty}\frac{1}{n} = 0. \end{aligned} \tag{2.6}$$

There are two possibilities: (1) $v_0 = 0$, and (2) $v_0 = ku_1^+$ for some $k \neq 0$, and $\widehat{\lambda} = \lambda_1^+$. If $v_0 = 0$, it follows from (2.6) that $\int |\nabla v_n|^p \to 0$ and $\int g^-|v_n|^p \to 0$. Thus $v_n \to 0$ in V, contradicting $\|v_n\| = 1$. If $v_0 = ku_1^+$ for some $k \neq 0$, and $\widehat{\lambda} = \lambda_1^+$, then we have, by the weak convergence of v_n to ku_1^+ and (2.6),

$$\begin{aligned}
\lambda_1^+ \int g^+|ku_1^+|^p &= \int (|\nabla ku_1^+|^p + \lambda_1^+ g^-|ku_1^+|^p) \\
&\leqslant \liminf_{n\to\infty} \int |\nabla v_n|^p + \liminf_{n\to\infty} \lambda_n \int g^-|v_n|^p \\
&\leqslant \liminf_{n\to\infty} \int (|\nabla v_n|^p + \lambda_n g^-|v_n|^p) \\
&= \lim_{n\to\infty} \lambda_n \int g^+|v_n|^p = \lambda_1^+ \int g^+|ku_1^+|^p.
\end{aligned}$$

It then follows that

$$\liminf_{n\to\infty} \int |\nabla v_n|^p = \int |\nabla ku_1^+|^p, \quad \liminf_{n\to\infty} \int g^-|v_n|^p = \int g^-|ku_1^+|^p.$$

We deduce that (passing to a subsequence if necessary) $v_n \to ku_1^+$ strongly in V. We then derive from (2.3) that

$$\|u_n\|^{p-p^*} J_{\lambda_n}(v_n) < \frac{p^*-q}{p-q} \int f|v_n|^{p^*} \to \frac{p^*-q}{p-q} \int f|ku_1^+|^{p^*} < 0. \tag{2.7}$$

This contradicts (2.6) if $\|u_n\| \not\to 0$ or $J_{\lambda_n}(u_n) \geqslant 0$. Suppose $\|u_n\| \to 0$ and $J_{\lambda_n}(u_n) < 0$. It follows from (2.3) that $\int h|u_n|^q < 0$. That is, $\int h|v_n|^q \leqslant 0$, which contradicts (h2). This proves the lemma. $\square$

Remark 2.4. For $\lambda \in (0, \lambda_1^+)$, conditions (f2) and (h2) are not needed because $J_\lambda(u) \geqslant 0$ for all u. Assumptions (f2) and (h2) are introduced to compensate for the possibility that $J_\lambda(u)$ is negative.

Lemma 2.5. *Assume $p > q$, (f0), (f1), (f2), (g0), (g1), (h0), (h1) and (h2) hold. For any $\bar{\lambda} \in (0, \lambda_1^+)$, there exist $\varrho > 0$ and $\alpha^* > 0$ with $\alpha^* \leqslant \alpha_2$, such that for any $\lambda \in [\bar{\lambda}, \lambda^*)$, $\alpha \in (0, \alpha^*)$ and $u \in \Lambda_\lambda^-$, we have $-\langle \Psi_\lambda'(u), u\rangle \geqslant \varrho$.*

Proof. We first claim that there exists $\zeta > 0$, independent of λ, such that $\|u\| > \zeta$ for all $u \in \Lambda_\lambda^-$. If this were not true, then for some $u_n \in \Lambda_{\lambda_n}^-$, $\lambda_n \in [\bar{\lambda}, \lambda^*)$, $u_n \to 0$. Dividing (2.3) by $\|u_n\|^p$ we obtain, using Lemma 2.3,

$$0 < \sigma \leqslant J_{\lambda_n}(v_n) < \frac{p^*-q}{p-q} \int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-p} \to 0, \tag{2.8}$$

a contradiction, where $v_n = u_n/\|u_n\|$.

Now, by Young's inequality and Lemma 2.3, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}-\langle \Psi'_\lambda(u), u\rangle &= (p^* - p)J_\lambda(u) - \alpha(p^* - q)\int h|u|^q \\ &\geqslant (p^* - p)\sigma\|u\|^p - \alpha(p^* - q)\|h\|_Q \cdot \|u\|^q \\ &\geqslant ((p^* - p)\sigma - \varepsilon)\zeta^p - \alpha C_\varepsilon \|h\|_Q^{p/(p-q)}.\end{aligned}$$

The proof is complete. $\square$

Corollary 2.6. *Under the conditions of Lemma 2.5, for any $\bar\lambda \in (0, \lambda_1^+)$, there exists $\alpha^* > 0$ such that Λ_λ^- is a closed set for $\lambda \in [\bar\lambda, \lambda^*)$ provided $\alpha \in (0, \alpha^*)$.*

3. Proof of existence of solutions for $1 < q < p$

Lemma 3.1. *Assume* (f0), (f1), (f2), (g0), (g1), (h0), (h1) *and* (h2) *hold. Then I_λ is bounded below on Λ_λ^- for $\lambda \in (0, \lambda^*)$ and $\alpha \in (0, \alpha^*)$, where α^* and λ^* are given in Lemma* 2.5.

Proof. Suppose for some $u_n \in \Lambda_\lambda^-$, $I_\lambda(u_n) \to -\infty$. Then $\|u_n\| \to \infty$. Since $u_n \in \Lambda_\lambda^-$, $\int f|u_n|^{p^*} > 0$ by (2.3) and Lemma 2.3. Dividing $I_\lambda(u_n)$ by $\|u_n\|^p$ we obtain from (2.4) that

$$\frac{I_\lambda(u_n)}{\|u_n\|^p} = \frac{1}{N}\int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-p} - \alpha\Big(\frac{1}{q} - \frac{1}{p}\Big)\int h|v_n|^q \cdot \|u_n\|^{q-p} \to \ell \leqslant 0,$$

with $v_n = u_n/\|u_n\|$. It then follows that $\int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-p} \to N\ell \leqslant 0$. On the other hand, dividing

$$J_\lambda(u_n) = \int f|u_n|^{p^*} + \alpha\int h|u_n|^q$$

by $\|u_n\|^p$ we obtain, using Lemma 2.3,

$$0 < \sigma \leqslant J_\lambda(v_n) = \int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-p} + \alpha\int h|v_n|^q \cdot \|u_n\|^{q-p} \to N\ell \leqslant 0,$$

a contradiction. So I_λ is bounded below on Λ_λ^-. $\square$

Thus we can define $c_0 = \inf\limits_{\Lambda_\lambda^-} I_\lambda(u)$.

Lemma 3.2. *Assume* (f0), (f1), (f2), (g0), (g1), (h0), (h1) *and* (h2) *hold. Then for any $\lambda \in (0, \lambda^*)$, $\alpha \in (0, \alpha^*)$, there exists a minimizing sequence $\{u_n\} \subset \Lambda_\lambda^-$ of I_λ on Λ_λ^- which converges weakly to a solution u of* $(1.1)_\lambda$.

P r o o f. We first show that any minimizing sequence of I_λ on Λ_λ^- is bounded. Suppose $\{u_n\}$ is an unbounded minimizing sequence of I_λ on Λ_λ^-. Dividing $I_\lambda(u_n)$ by $\|u_n\|^q$, we conclude that, since $I_\lambda(u_n)$ is bounded, $\int f|u_n|^{p^*}\cdot\|u_n\|^{-q}$ is bounded by (2.4). Thus $J_\lambda(u_n)\cdot\|u_n\|^{-q}$ is also bounded by (2.3). Let $v_n = u_n/\|u_n\|$. Then $\sigma \leqslant J_\lambda(v_n) \to 0$, a contradiction. Thus any minimizing sequence $\{u_n\}$ in Λ_λ^- is bounded.

Since Λ_λ^- is a closed set by Corollary 2.6, it follows from Theorem 4.1 and Remark 4.1 of [9] that we can replace $\{u_n\}$ by another minimizing sequence $\{z_n\}\subset\Lambda_\lambda^-$ such that $\|u_n - z_n\| < 1/n$, and for any $y\in\Lambda_\lambda^-$,

$$I_\lambda(y) > I_\lambda(z_n) - \tfrac{1}{n}\|y - z_n\|. \tag{3.1}$$

We want to show that $I'_\lambda(z_n)\to 0$. Choose w_n of unit norm so that

$$\langle I'_\lambda(z_n), w_n\rangle \geqslant \|I'_\lambda(z_n)\| - o(1)$$

as $n\to\infty$. It will suffice to show that

$$\langle I'_\lambda(z_n), w_n\rangle \to 0. \tag{3.2}$$

For each n, let $g_n(t,s) = \Psi_\lambda(tz_n - sw_n)$. Then $g_n(1,0)=0$ and

$$\tfrac{\partial g_n}{\partial t} = \langle \Psi'_\lambda(z_n), z_n\rangle \neq 0 \text{ at } t=1, s=0.$$

It follows from the C^1 Implicit Function Theorem that for each n, for small enough s, there exists $t_n\in C^1$ so that $\Psi_\lambda(t_n(s)z_n - sw_n)=0$, i.e. $t_n(s)z_n - sw_n\in\Lambda_\lambda$ and

$$\langle \Psi'_\lambda(z_n), z_n\rangle t'_n(0) - \langle \Psi'_\lambda(z_n), w_n\rangle = 0. \tag{3.3}$$

Since z_n is a bounded sequence, so is $\|\Psi'_\lambda(z_n)\|$, and we then conclude from (3.3) and Lemma 2.5 that

$$t'_n(0) \text{ is uniformly bounded in } n. \tag{3.4}$$

We fix n, and consider $v_n(s) = t_n(s)z_n - sw_n - z_n$. Since $\|w_n\|=1$, we have

$$\|v_n(s)\| \leqslant |s|(1 + (|t'_n(0) + o(1)|)\|z_n\|) \tag{3.5}$$

as $s\to 0$. Moreover $z_n\in\Lambda_\lambda$ gives $\langle I'_\lambda(z_n), z_n\rangle = 0$, so

$$I_\lambda(z_n) - I_\lambda(t_n(s)z_n - sw_n) = \langle I'_\lambda(z_n), -v_n(s)\rangle + o(v_n(s)) = \langle I'_\lambda(z_n), sw_n\rangle + o(s) \tag{3.6}$$

follows from (3.5). By continuity of $\langle \Psi'_\lambda(u), u\rangle$, we have

$$\langle \Psi'_\lambda(t_n(s)z_n - sw_n), t_n(s)z_n - sw_n\rangle - \langle \Psi'_\lambda(z_n), z_n\rangle \to 0$$

as $s \to 0$. We then conclude from this and Lemma 3.4 that

$$\langle \Psi'_\lambda(t_n(s)z_n - sw_n), t_n(s)z_n - sw_n\rangle < 0$$

for s small enough, so $t_n(s)z_n - sw_n \in \Lambda_\lambda^-$.

Dividing (3.6) by s and using (3.1) with $y = t_n(s)z_n - sw_n$ and (3.5), we obtain

$$|\langle I'_\lambda(z_n), w_n\rangle| \leqslant n^{-1}(1 + (|t'_n(0)|)\|z_n\|) + o(1).$$

Letting $n \to \infty$ we conclude that $\langle I'_\lambda(z_n), w_n\rangle$ tends to zero by boundedness of z_n and (3.4). This establishes (3.2).

Assume now that $z_n \to u$ weakly in V. We have, then, as in the proof of Lemma 3.1 of [5], since $I'_\lambda(z_n) \to 0$, that u is a weak solution of $(1.1)_\lambda$, i.e.,

$$-\Delta_p u = \lambda g|u|^{p-2}u + \alpha h|u|^{q-2}u + f|u|^{p^*-2}u$$

in V. This proves the lemma. $\square$

Thus we have obtained a weak solution of $(1.1)_\lambda$. To show that this solution is nontrivial, we need some preparation. Let S be the best Sobolev constant, i.e.,

$$S = \inf\left\{\frac{\|\nabla u\|_p^p}{\|u\|_{p^*}^p} : u \in W_0^{1,p}(\mathbb{R}^N) \setminus \{0\}\right\},$$

and $S_0 = S^{N/p}\|f\|_\infty^{(p-N)/p}/N$. Recall the concentration-compactness principle of P. L. Lions ([8]).

Proposition 3.3. *Let $\{u_n\}$ converge weakly to u in $D^{1,p}(\mathbb{R}^N)$ such that $|u_n|^{p^*}$ and $|\nabla u_n|^p$ converge weakly to nonnegative measures ν and μ on $\mathbb{R}^N$ respectively. Then, for some at most countable set J, we have*

(i) $\nu = |u|^{p^*} + \sum_{j\in J} \nu_j\delta_{x_j}$;

(ii) $\mu \geqslant |\nabla u|^p + \sum_{j\in J} \mu_j\delta_{x_j}$;

(iii) $S\nu_j^{p/p^*} \leqslant \mu_j$,

where $x_j \in \mathbb{R}^N$, δ_{x_j} is the Dirac measure at x_j, and ν_j and μ_j are nonnegative constants.

Lemma 3.4. *Assume* (f0), (f1), (f2), (g0), (g1), (h0), (h1) *and* (h2) *hold. For* $\lambda \in [0, \lambda^*)$ *and* $\alpha \in (0, \alpha^*)$, *any minimizing sequence* $\{u_n\}$ *of* I_λ *on* Λ_λ^- *satisfying* $I_\lambda(u_n) < S_0$ *either converges strongly to a solution* $u \in \Lambda_\lambda^-$, *hence* $u \neq 0$, *or converges weakly to a nontrivial solution* $u \in \Lambda_\lambda$.

P r o o f. Let $\{u_n\}$ be such a minimizing sequence. We can assume without loss of generality that $\{u_n\}$ is bounded (cf. Lemma 3.2).

Assume that $u_n \to u$ weakly in V. We conclude as in the proof of Proposition 2.3 of [5] that

$$-\Delta_p u = \lambda g|u|^{p-2}u + f|u|^{p^*-2}u + \alpha h|u|^{q-2}u$$

in V, that is, $I'_\lambda(u) = 0$ and hence $u \in \Lambda_\lambda$.

Suppose that $u_n \not\to u$ strongly in V and $u = 0$. Then for some j, ν_j given by Proposition 3.3 is not zero. We obtain, using the fact that $\int h|u_n|^q \to 0$ (cf. Proposition 2.3 of [5]),

$$\begin{aligned} S_0 > I_\lambda(u_n) &= \frac{1}{N}\int f|u_n|^{p^*} + \alpha\Big(\frac{1}{p} - \frac{1}{q}\Big)\int h|u_n|^q \\ &\geqslant \frac{1}{N}\sum_{j\in J} f(x_j)\nu_j \geqslant \frac{1}{N}\sum_{j\in J}\frac{S^{N/p}}{f(x_j)^{(N-p)/p}} \geqslant S_0, \end{aligned}$$

a contradiction. Here we used the facts that $f(x_j)\nu_j = \mu_j$ and $\nu_j \geqslant (S/f(x_j))^{N/P}$, which follow from the proof of Proposition 2.4 of [5]. This proves the lemma. $\square$

We need more conditions on f. Assume

(f3) for $x \in B(0, 2r)$,

$$f(x) = f(0) + o(|x|^k), \quad k = \frac{N}{q} \text{ if } q \geqslant \frac{N(p-1)}{N-p}, \quad k = \frac{N-p}{p-1} \text{ if } q < \frac{N(p-1)}{N-p},$$

or

(f3)′ for $x \in B(0, 2r)$,

$$f(x) = f(0) + o(|x|^\delta), \quad \delta = \frac{N-p}{p-1}.$$

Lemma 3.5. *Assume* (f0), (f1), (f2), (f3) or (f3)′, (g0), (g1), (h0) *and* (h1) *hold. Then for* $\lambda > 0$, *and* $\varepsilon > 0$ *small enough, we have*

$$\sup_{t\geqslant 0} I_\lambda(tv_\varepsilon) < \frac{1}{N}S^{N/p}\|f\|_\infty^{(p-N)/p} = S_0, \tag{3.7}$$

where v_ε *is given in the proof of Lemma 2.2.*

P r o o f. Our proof is similar to that of Lemma 4.1 of [12]. Recall for $\varepsilon > 0$,

$$u_\varepsilon(x) = \frac{\psi(x)}{(\varepsilon + |x|^{p/(p-1)})^{(N-p)/p}}, \quad v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon(x)\|_{p^*}},$$

where $\psi \in C_0^\infty(B(0,2r))$ is such that $0 \leqslant \psi(x) \leqslant 1$ and $\psi(x) \equiv 1$ on $B(0,r)$.

Calculations show that (cf. the proof of Lemma 5.6 in [5])

$$\int |v_\varepsilon|^t = \begin{cases} K\varepsilon^{(N(p-t)+tp)(p-1)/p^2}, & \text{if } t > \frac{p^*}{p'}, \\ K\varepsilon^{N(p-1)/p^2}|\ln\varepsilon|, & \text{if } t = \frac{p^*}{p'}, \\ K\varepsilon^{t(N-p)/p^2}, & \text{if } t < \frac{p^*}{p'}, \end{cases} \tag{3.8}$$

and

$$\int |\nabla v_\varepsilon|^t = \begin{cases} K'\varepsilon^{N(p-t)(p-1)/p^2}, & \text{if } t > \frac{N(p-1)}{N-1}, \\ K'\varepsilon^{t(N-p)/p^2}|\ln\varepsilon|, & \text{if } t = \frac{N(p-1)}{N-1}, \\ K'\varepsilon^{t(N-p)/p^2}, & \text{if } t < \frac{N(p-1)}{N-1}. \end{cases} \tag{3.9}$$

In particular, we have

$$\int |v_\varepsilon| = \begin{cases} K\varepsilon^{(N-p)/p^2}, & \text{if } p > \frac{2N}{N+1}, \\ K\varepsilon^{(N-p)/p^2}|\ln\varepsilon|, & \text{if } p = \frac{2N}{N+1}, \\ K\varepsilon^{(N(p-1)+p)(p-1)/p^2}, & \text{if } p < \frac{2N}{N+1}, \end{cases} \tag{3.10}$$

$$\int |v_\varepsilon|^p = \begin{cases} K\varepsilon^{p-1} & p^2 < N, \\ K\varepsilon^{p-1}|\ln\varepsilon| & p^2 = N, \\ K\varepsilon^{(N-p)/p}, & p^2 > N. \end{cases} \tag{3.11}$$

$$\int v_\varepsilon^{p^*} = 1,$$

$$\int |\nabla v_\varepsilon| = \begin{cases} K'\varepsilon^{(N-p)/p^2}, & \text{if } p > \frac{2N-1}{N}, \\ K'\varepsilon^{(N-p)/p^2}|\ln\varepsilon|, & \text{if } p = \frac{2N-1}{N}, \\ K'\varepsilon^{N(p-1)^2/p^2}, & \text{if } p < \frac{2N-1}{N}, \end{cases} \tag{3.12}$$

$$\int |\nabla v_\varepsilon|^{p-1} = K'\varepsilon^{(N-p)(p-1)/p^2}, \tag{3.13}$$

and

$$\int |\nabla v_\varepsilon|^p = \frac{\int |\nabla u_\varepsilon|^p}{\|u_\varepsilon\|_{p^*}^p} = \frac{K_1}{K_2} + O(\varepsilon^{(N-p)/p}), \tag{3.14}$$

where $K_1/K_2 = S$.

Note that for $\varepsilon > 0$ small enough, $J_\lambda(v_\varepsilon) > 0$, so $I_\lambda(tv_\varepsilon)$ attains its maximum at some $t_\varepsilon \in (0,\infty)$ with $s'(t_\varepsilon) = 0$, where $s(t) = I_\lambda(tv_\varepsilon)$. That is,

$$0 = s'(t_\varepsilon) = t_\varepsilon^{p-1}\Big(\int (|\nabla v_\varepsilon|^p - \lambda g|v_\varepsilon|^p) - \alpha t_\varepsilon^{q-p}\int h|v_\varepsilon|^q - t_\varepsilon^{p^*-p}\int f|v_\varepsilon|^{p^*}\Big).$$

Thus, by (g1) and (h1),

$$t_\varepsilon^{p^*-p} \leqslant \frac{\int |\nabla v_\varepsilon|^p}{f(x^*)\int |v_\varepsilon|^{p^*}},$$

where $f(x^*) = \inf_{x\in B(0,r)} f(x) > 0$. It then follows that t_ε is bounded from above. We may also assume that t_ε is bounded from below, otherwise $I_\lambda(t_\varepsilon v_\varepsilon) \to 0$ as $\varepsilon \to 0$. Now,

$$I_\lambda(t_\varepsilon v_\varepsilon) = \sup_{t\geqslant 0} I_\lambda(tv_\varepsilon) = E(\varepsilon) - F(\varepsilon) + V(\varepsilon), \tag{3.15}$$

where

$$\begin{aligned} E(\varepsilon) &= \frac{t_\varepsilon^p}{p}\int |\nabla v_\varepsilon|^p - \frac{f(0)t_\varepsilon^{p^*}}{p^*}\int v_\varepsilon^{p^*},\\ F(\varepsilon) &= \frac{\lambda t_\varepsilon^p}{p}\int g v_\varepsilon^p + \alpha\frac{t_\varepsilon^q}{q}\int h v_\varepsilon^q,\\ V(\varepsilon) &= \frac{t_\varepsilon^{p^*}}{p^*}\int (f(0) - f(x))v_\varepsilon^{p^*}. \end{aligned}$$

The maximum of $ap^{-1}t^p - b(p^*)^{-1}t^{p^*}$ is achieved at $t = (a/b)^{(N-p)/p^2}$ for positive a, b, so

$$E(\varepsilon) \leqslant \Big(\frac{1}{p} - \frac{1}{p^*}\Big)[f(0)]^{(p-N)/p}\Big[\int |\nabla v_\varepsilon|^p\Big]^{N/p}\Big[\int v_\varepsilon^{p^*}\Big]^{-N/p^*} = \frac{1}{N}S^{N/p}\|f\|_\infty^{(p-N)/p}.$$

We also have, for k and δ given in (f3) and (f3)$'$ respectively,

$$V_\varepsilon = O(\varepsilon^{k(p-1)/p}),\ O(\varepsilon^{\delta(p-1)/p}).$$

Assuming (f3) holds, we estimate, using the fact that t_ε is bounded from below,

$$F(\varepsilon) \geqslant \alpha h_0 \int v_\varepsilon^q = \begin{cases} K\varepsilon^{[N(p-q)+qp](p-1)/p^2}, & \text{if } q > \frac{N(p-1)}{N-p},\\ K\varepsilon^{N(p-1)/p^2}|\ln\varepsilon|, & \text{if } q = \frac{N(p-1)}{N-p},\\ K\varepsilon^{q(N-p)/p^2}, & \text{if } q < \frac{N(p-1)}{N-p}. \end{cases}$$

From (f3) we derive that for $\varepsilon > 0$ small enough, $F(\varepsilon)$ dominates $V(\varepsilon)$. Thus we conclude from the above that, for $\varepsilon > 0$ small enough and $K > 0$,

$$(3.16) \qquad I_\lambda(t_\varepsilon v_\varepsilon) \leqslant \begin{cases} S_0 - K\varepsilon^{[N(p-q)+qp](p-1)/p^2}, & \text{if } q > \frac{N(p-1)}{N-p}, \\ S_0 - K\varepsilon^{N(p-1)/p^2}|\ln\varepsilon|, & \text{if } q = \frac{N(p-1)}{N-p}, \\ S_0 - K\varepsilon^{q(N-p)/p^2}, & \text{if } q < \frac{N(p-1)}{N-p}. \end{cases}$$

On the other hand, assume (f3)$'$ holds. We have

$$F(\varepsilon) \geqslant g_0 \int v_\varepsilon^p = \begin{cases} K\varepsilon^{p-1}, & \text{if } p^2 < N, \\ K\varepsilon^{p-1}|\ln\varepsilon|, & \text{if } p^2 = N, \\ K\varepsilon^{(N-p)/p}, & \text{if } p^2 > N. \end{cases}$$

Since $p-1 \leqslant (N-p)/p$ for $p^2 \leqslant N$ and $\delta(p-1)/p > (N-p)/p$ for $p^2 > N$ by (f3)$'$, $F(\varepsilon)$ dominates $V(\varepsilon)$. Again we have

$$(3.16)' \qquad I_\lambda(t_\varepsilon v_\varepsilon) \leqslant \begin{cases} S_0 - K\varepsilon^{p-1}, & \text{if } p^2 < N, \\ S_0 - K\varepsilon^{p-1}|\ln\varepsilon|, & \text{if } p^2 = N, \\ S_0 - K\varepsilon^{(N-p)/p}, & \text{if } p^2 > N. \end{cases}$$

The lemma then follows. $\square$

Lemma 3.6. *Assume* (f0), (f1), (f2), (f3) or (f3)$'$, (g0), (g1), (h0), (h1), (h2), *and λ and α as in Lemma* 2.5. *Then* $c_0 = \inf_{\Lambda_\lambda^-} I_\lambda(u) < S_0$.

This lemma follows from Lemma 3.5 and the fact that $t_\varepsilon v_\varepsilon \in \Lambda_\lambda^-$ for some $t_\varepsilon > 0$ (cf. the proof of Lemma 2.2). Thus we have proved, via Lemmas 3.4 and 3.6, the existence of a nonnegative solution. The next result shows that the solution is actually positive.

Proposition 3.7. *Let u be a nonnegative solution of* $(1.1)_\lambda$ *with* $q \leqslant p^*$. *Then* $u > 0$ *in* $\mathbb{R}^N$.

The proof is essentially as that of Lemma 4.3 of [12] and is omitted.

Now we can state our main result.

Theorem 3.8. *Assume that* (f0), (f1), (f2), (f3) *or* (f3)$'$, (g0), (g1), (h0), (h1), *and* (h2) *hold. Then there exist $\lambda^* > \lambda_1^+$ and $\alpha^* > 0$, so that the problem*

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u + \alpha h(x)|u|^{q-2}u + f(x)|u|^{p^*-2}u$$

has at least one positive solution in V for any $\lambda \in [\lambda_1^+, \lambda^*)$ and $\alpha \in (0, \alpha^*)$.

For this case, the set Λ_λ^- is defined as in (2.3). We first have the following result.

Lemma 4.1. *Assume* $\operatorname{supp} f^+ \cap \operatorname{supp} h^+$ *contains an open set. Then* $\Lambda_\lambda^- \neq \emptyset$ *for* $\lambda > 0$.

Proof. Suppose $\operatorname{supp} f^+ \cap \operatorname{supp} h^+$ contains an open set B and let $\varphi > 0$ be such that $\operatorname{supp} \varphi \subset B$ with $J_\lambda(\varphi) > 0$. Such φ exists as explained in the proof of Lemma 2.2. For $t > 0$, we have

$$\Psi(t\varphi) = t^p J_\lambda(\varphi) - \alpha t^q \int h\varphi^q - t^{p^*} \int f\varphi^{p^*}.$$

Let again $s(t) = at^p - \alpha bt^q - ct^{p^*}$, with $a = J_\lambda(\varphi) > 0$, $b = \int h\varphi^q > 0$ and $c = \int f\varphi^{p^*} > 0$. Obviously $s(t) > 0$ for $t > 0$ small and $s(t) \to -\infty$ as $t \to \infty$. Suppose $s(t_0) = 0$. Then

$$\begin{aligned} s'(t_0) &= t_0^{q-1}(pat_0^{p-q} - \alpha qb - p^* ct_0^{p^*-q}) \\ &= t_0^{q-1}[\alpha(p-q)b - (p^*-p)ct_0^{p^*-q}] < 0 \end{aligned}$$

since $p < q$. That is, $t_0\varphi \in \Lambda_\lambda^-$. This concludes the proof. $\square$

Remark 4.2. Note that (f1) and (h1) imply that $\operatorname{supp} f^+ \cap \operatorname{supp} h^+$ contains an open set. So we will assume for simplicity in the sequel that (f1) and (h1) hold. We also note that Lemma 4.1 holds if $h \equiv 0$.

Instead of (h2), we need
(h2)′ $\int h|u_1^+|^q < 0$.

Lemma 4.3. *Assume* (f0), (f1), (f2), (g0), (g1), (h0), (h1) *and* (h2)′ *hold. Then there exist* $\lambda_1^* > \lambda_1^+$ *and* $\alpha_1 > 0$ *such that for any* $\bar\lambda \in (0, \lambda_1^+)$, *there exists* $\sigma > 0$, *such that for any* $\lambda \in [\bar\lambda, \lambda_1^*)$ *and* $\alpha \in (0, \alpha_1)$, $J_\lambda(u) \geqslant \sigma\|u\|^p$ *for any* $u \in \Lambda_\lambda^-$.

Proof. If the conclusion were false, there would exist λ_n, α_n and $u_n \in \Lambda_{\lambda_n}^-$ such that

$$\alpha_n \to 0, \quad \lambda_n \to \widehat{\lambda} \in [\bar\lambda, \lambda_1^+], \quad J_{\lambda_n}(u_n) < \frac{1}{n}\|u_n\|^p.$$

As in the proof of Lemma 2.3, we conclude that $v_n = u_n/\|u_n\| \to ku_1^+$ for some $k \neq 0$. Then instead of (2.7) we have

$$\|u_n\|^{p-p^*} J_{\lambda_n}(v_n) > \frac{p^*-q}{p-q} \int f|v_n|^{p^*} \to \frac{p^*-q}{p-q} \int f|ku_1^+|^{p^*} > 0, \tag{4.1}$$

since $p < q$ and $\int f|ku_1^+|^{p^*} < 0$ by (f2). If $\|u_n\| \not\to 0$, then (4.1) contradicts the fact that $J_{\lambda_n}(v_n) \to 0$. If $\|u_n\| \to 0$, (4.1) implies that $J_{\lambda_n}(v_n) > 0$. We then have

$$\alpha_n \int h|v_n|^q + \int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-q} = J_{\lambda_n}(v_n) \cdot \|u_n\|^{p-q} > 0. \tag{4.2}$$

Note that $\int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-q} \to 0$ since $\|u_n\| \to 0$. Inequality (4.2) then implies $\int h|v_n|^q > 0$, contradicting (h2)$'$. Thus the lemma is proved. □

Remark 4.4. We point out that Lemma 4.3 holds if $h \equiv 0$.

The reason is that instead of (4.2), we now have

$$\int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-p} = J_{\lambda_n}(v_n) > 0. \tag{4.2$'$}$$

This leads to a contradiction again.

Lemma 4.5. *Assume that* (f0), (f1), (f2), (g0), (g1), (h0), (h1) *and* (h2)$'$ *hold. Then for* $\lambda \in (0, \lambda^*)$ *and* $\alpha \in (0, \alpha_1)$,
(i) $I_\lambda(u) > 0$ *for any* $u \in \Lambda_\lambda^-$,
(ii) *any minimizing sequence of* I_λ *on* Λ_λ^- *is uniformly bounded.*

Proof. We observe that, for $u \in \Lambda_\lambda^-$, from (2.4) and Lemma 4.3,

$$\begin{aligned} I_\lambda(u) &= \frac{1}{N} J_\lambda(u) - \alpha \frac{p^* - q}{p^* q} \int h|u|^q \\ &> \Big[\frac{1}{N} - \frac{p^* - p}{qp^*}\Big] J_\lambda(u) = \frac{q-p}{Nq} J_\lambda(u) \geqslant \frac{q-p}{Nq} \sigma \|u\|^p. \end{aligned} \tag{4.3}$$

Since σ only depends on α_1 and λ^*, the conclusions then follow directly. This completes the proof. □

Now, by Lemma 4.5, there exists $R > 0$, so that for any $\alpha \in (0, \alpha_1)$ and $\lambda \in (0, \lambda^*)$, for any minimizing sequence $\{u_n\} \subset \Lambda_\lambda^-$ of I_λ (here I_λ also depends on α), we can assume that, by taking a subsequence if necessary, $\|u_n\| \leqslant R$. Define $U_R = \{u \in V \colon \|u\| \leqslant R\}$.

Lemma 4.6. *Assume* (f0), (f1), (f2), (g0), (g1), (h0), (h1) *and* (h2)$'$ *hold. For any* $\bar\lambda \in (0, \lambda_1^+)$, *there exist* $\varrho > 0$ *and* α^* *with* $\alpha^* \leqslant \alpha_1$, *such that for any* $\lambda \in [\bar\lambda, \lambda^*)$, $\alpha \in (0, \alpha^*)$ *and* $u \in \Lambda_\lambda^- \cap U_{2R}$, $-\langle \Psi_\lambda'(u), u\rangle \geqslant \varrho$.

Proof. We first show that for some $\eta > 0$, depending only on α_1 and λ^*, $\|u\| \geqslant \eta$ for $u \in \Lambda_\lambda^-$. Indeed, if for some $u_n \in \Lambda_\lambda^-$, $u_n \to 0$, then we have, by

Lemma 4.3,

$$0<\sigma \leqslant J_\lambda(v_n)=\alpha\int h|v_n|^q\cdot\|u_n\|^{q-p}+\int f|v_n|^{p^*}\cdot\|u_n\|^{p^*-p}\to 0,$$

a contradiction, where $v_n=u_n/\|u_n\|$.

Using Lemma 4.3 we get

$$\begin{aligned}-\langle\Psi'_\lambda(u),u\rangle&=(p^*-p)J_\lambda(u)-\alpha(p^*-q)\int h|u|^q\\&\geqslant(p^*-p)J_\lambda(u)-\alpha(p^*-q)\|h\|_Q\cdot\|u\|^q\\&\geqslant(p^*-p)\sigma\eta^p-\alpha(p^*-q)\|h\|_Q(2R)^q>c>0,\end{aligned}$$

for α small enough. The lemma is proved. □

Lemma 4.6 implies that $\Lambda_\lambda^-\cap U_{2R}$ is a closed set (in fact one can prove that Λ_λ^- is a closed set). Replacing Λ_λ^- by $\Lambda_\lambda^-\cap U_{2R}$, and noting that any minimizing sequence in $\Lambda_\lambda^-\cap U_{2R}$ will be a positive distance from the boundary $\|u\|=2R$, we can check straightforwardly that the proofs of Lemmas 3.2, 3.4, 3.5, and 3.6 remain valid. So we can state our result.

Theorem 4.7. *Assume that* (f0), (f1), (f2), (f3) *or* (f3)$'$, (g0), (g1), (h0), (h1), *and* (h2)$'$ *hold. Then there exist* $\lambda^*>\lambda_1^+$ *and* $\alpha^*>0$, *such that for any* $\lambda\in[\lambda_1^+,\lambda^*)$ *and* $\alpha\in(0,\alpha^*)$, *the problem*

$$-\Delta_p u=\lambda g(x)|u|^{p-2}u+\alpha h(x)|u|^{q-2}u+f(x)|u|^{p^*-2}u$$

has at least one positive solution in V.

Remark 4.8. As we remarked earlier, for $\lambda\in(0,\lambda_1^+)$, Theorems 3.8 and 4.7 hold without the integral conditions (f2), (h2) and (h2)$'$, and can be proved via Mountain Pass argument. Cf. [1], [2], [7] and [12].

Remark 4.9. We note that the proofs are applicable to Dirichlet problems on bounded domains and similar results hold. We can also deal with

$$-\Delta_p u+a(x)|u|^{p-2}u=\lambda g(x)|u|^{p-2}u+\alpha h(x)|u|^{q-2}u+f(x)|u|^{p^*-2}u$$

in $\mathbb{R}^N$, where $a(x)\in L^\infty_{\rm loc}(\mathbb{R}^N)$, $a(x)\geqslant 0$.

Remark 4.10. Similarly, one can consider the negative principal eigenvalue $\lambda_1^-<0$ given by

$$-\Delta_p u=\lambda g(x)|u|^{p-2}u\quad\text{in }\mathbb{R}^N,\qquad\int g(x)|u|^p<0.$$

Existence of positive solutions of $(1.1)_\lambda$ for $\lambda<0$ can be obtained provided conditions similar to (f2), (h2) and (h2)$'$ hold.

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