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# SOME CLASSES OF INFINITELY DIFFERENTIABLE FUNCTIONS

G. S. Balashova, Moskva

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*Dedicated to Professor Alois Kufner on the occasion of his 65th birthday*

*Abstract.* For nonquasianalytical Carleman classes conditions on the sequences $\{\widehat{M}_n\}$ and $\{M_n\}$ are investigated which guarantee the existence of a function in $C_J\{\widehat{M}_n\}$ such that

$$u^{(n)}(a) = b_n, \quad |b_n| \leqslant K^{n+1} M_n, \quad n = 0, 1, \ldots, \quad a \in J.$$

Conditions of coincidence of the sequences $\{\widehat{M}_n\}$ and $\{M_n\}$ are analysed. Some still unknown classes of such sequences are pointed out and a construction of the required function is suggested.

The connection of this classical problem with the problem of the existence of a function with given trace at the boundary of the domain in a Sobolev space of infinite order is shown.

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Nonquasianalytical Carleman classes of one real variable

$$C_J\{\widehat{M}_n\} \equiv \{f(x) \in C^\infty(J) \colon \max_{x\in J} |f^{(n)}(x)| \leqslant K_{(f)}^{n+1} \widehat{M}_n, \quad n = 0, 1, \ldots\} \tag{1}$$

are considered. That means that the sequence $\{\widehat{M}_n\}$ satisfies the following conditions:

$$\lim_{n\to\infty} \widehat{M}_n^{1/n} = \infty, \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{\widehat{M}_n^c}{\widehat{M}_{n+1}^c} < \infty, \tag{3}$$

where $\{\widehat{M}_n^c\}$ is the logarithmically convex regularization of $\{M_n\}$ (cf. [1]).

**Definition 1.** The indices $\{n_i\}$ such that

$$M_{n_i} = M_{n_i}^c$$

are called fundamental indices for the logarithmically convex regularization of $\{M_n\}$.

**Definition 2.** The sequence $\{M_n\}$ is called almost logarithmically convex if for all its fundamental indices the following condition is satisfied:

$$\sup_i (n_{i+1} - n_i) = K < \infty.$$

If $K = 1$ then the sequence $\{M_n\}$ is logarithmically convex.

The family of sequences $\{b_n\}$ such that

$$|b_n| \leqslant K^n M_n, \qquad n = 0, 1 \ldots, \qquad K = K(\{b_n\}),$$

is denoted as $B\{M_n\}$.

Problem. Find conditions on the sequences $\{M_n\}$ and $\{\widehat{M}_n\}$ which guarantee for any sequence $\{b_n\} \in B\{M_n\}$ the existence of a function $f(x) \in C_{\mathbb{R}}\{\widehat{M}_n\}$ satisfying the following conditions:

$$f^{(n)}(0) = b_n, \qquad n = 0, 1 \ldots \tag{4}$$

It is clear that $M_n \leqslant \widehat{M}_n$ for all $n = 0, 1 \ldots$

In particular, the conditions of coincidence of $\{\widehat{M}_n\}$ and $\{M_n\}$ are analysed.

The problem was studied by T. Bang [2], E. Borel [3], T. Carleman [4], L. Carleson [5], G. Wahde [6], B. S. Mitiagin [7], L. Ehrenpreis [8], G. S. Balashova [9] and other authors.

**Theorem 1.** *For any sequence* $\{b_n\} \in B\{M_n\}$ *and any number* $\alpha > 1$ *there exists the function* $f(x) \in C_{\mathbb{R}}\{\widehat{M}_n\}$ *satisfying the condition* (4), *where*

$$\widehat{M}_n = n^{\alpha n} \sum_{k=1}^{n} M'_k \left( \frac{M'_{k+1}}{M'_k} \right)^{n-k}, \quad M'_k = \frac{M_k}{k^{\alpha k}}, \quad k = 1, 2, \ldots$$

Proof. We construct the desired function. It is known that there exists a function $\psi(x) \in C^{\infty}_{(\mathbb{R})}$ satisfying the following conditions:

1) $\psi(x) \geqslant 0$, $\max\limits_{x \in R} \psi(x) = \psi(0) = 1$, $\psi^{(n)}(0) = 0$, $n = 1, 2, \ldots$;

2) $\psi(x) = 0$, if $|x| > 2 \sum\limits_{n=1}^{\infty} \mu_n^{-1} = \delta$;

3) $\max\limits_{|x|<\delta} |\psi^{(n)}(x)| \leqslant \prod\limits_{j=1}^{n} \mu_j$, where $\mu_n > 0$ is an increasing sequence such that $\mu_0 = 1$ and $\sum\limits_{n=1}^{\infty} \mu_n^{-1} < \infty$.

The required function is

$$f(x) = \sum_{k=0}^{\infty} \frac{b_k}{k!} \psi_k(d_k x),$$

where $\psi_k(x)$ satisfies the conditions 1)–3) with

$$\mu_n^{(k)} = (n+k)^{\alpha}, \quad d_k = K(\alpha) \frac{M'_{k+1}}{M'_k}.$$

□

Corollary. If the sequence $\{M_n\}$ has the property that for some $\alpha > 1$ the sequence $\{M_k k^{-\alpha k}\}$ is almost logarithmically convex, then $\widehat{M}_n = M_n$.

Examples. 1°. If $M_n = n^{\alpha n} \ln^{\beta n} n$, $\alpha > 1$, $\beta \geqslant 0$, then $\widehat{M}_n = M_n$. When $\beta = 0$, we obtain the known result of L. Carleson, L. Ehrenpreis and B. Mitiagin.

2°. If $M_n = a^{n\alpha}(n^{\beta} \ln^{\gamma} n)^n$, $a > 1$, $\alpha > 1$, $\beta \geqslant 0$, $\gamma \geqslant 0$, then $\widehat{M}_n = M_n$.

If the sequence $\{M_n\}$ grows slowlier than $n^{\alpha n}$, $\alpha > 1$, then the following is true:

**Theorem 2.** *If $M_n = (n \ln_r^{\gamma} n \ln_{r+s}^{\beta} n)^n$, $\gamma > 0$, $\beta \geqslant 0$, $r \geqslant 1$, $s \geqslant 1$, then there exists a function $f(x) \in C_{\mathbb{R}}(\widehat{M}_n)$ satisfying the condition (4), where $\widehat{M}_n = (n \ln_r^{\gamma+1} n \ln_{r+s}^{\beta} n)^n (\ln n \ln\ln n \ldots \ln_{r-1} n)^n$, $\ln_r n$ means $r$-times iterated logarithm.*

Proof is of a constructive character. The required function looks like $f(x) = \sum\limits_{k=0}^{\infty} \frac{b_k}{k!} x^k \psi_k(dx)$, where the constant $d$ is chosen, and the sequence $\mu_n^{(k)}$ is built as follows $\mu_n^{(k)} = (n+k)\ln(n+k)\ln\ln(n+k)\ldots\ln_{r-1}(n+k)\ln_r^{\gamma+1}(n+k)\ln_{r+s}^{\beta}(n+k)$.

Remark. When $r = 1$, $\gamma = 1$, $\beta = 0$, $M_n = (n \ln n)^n$, we obtain $\widehat{M}_n = (n \ln^2 n)^n$, which is the known result of L. Carleson.

While studying estimates of the norm of the $n$-th order derivative of a function $f(x)$ on the Lebesgue space of $p$-integrable functions ($1 \leqslant p < \infty$) there was obtained

**Theorem 3.** *If the sequence $\{\widehat{M}_n\}$ is logarithmically convex and for some $\alpha > 1$ the sequence $\{\widehat{M}_n n^{-\alpha n}\}$ is almost logarithmically convex, then for any sequence*

*$\{b_n\} \in B\{M_n\}$, where $M_n = \widehat{M}_{n+1}^{1/p}\widehat{M}_n^{1-\frac{1}{p}}$, there exists an infinitely differentiable function on $\mathbb{R}$ such that*

$$f^{(n)}(0) = b_n \quad \textit{and} \quad ||f^{(n)}_{(x)}||_{L_p(\mathbb{R})} \leqslant K^{n+1}\widehat{M}_n, \quad n = 0, 1, \dots$$

Remark. Theorem 3 makes sense only for such sequences $\{M_n\}$, for which the ratio $\frac{M_{n+1}}{M_n}$ grows in $n$ faster than the geometrical progression (for example, $M_n = 2^{n^s}$, $s > 2$, $n = 1, 2 \dots$).

Remark. When $p = 1$ we have $M_n = \widehat{M}_{n+1}$. That result gives the best estimation for $\widehat{M}_n$ as it is evident that $\widehat{M}_{n+1} \geqslant M_n$. In fact, $K^{n+2}\widehat{M}_{n+1} \geqslant \|f^{(n+1)}(x)\|_{L_1(\mathbb{R})} \geqslant \int\limits_0^\infty |f^{(n+1)}(x)| \,\mathrm{d}x \geqslant |\int\limits_0^\infty f^{(n+1)}(x) \,\mathrm{d}x| = |f^{(n)}(0)| = |b_n|$.

The problem of the existence of a function with the given trace at the boundary of the domain $G \in \mathbb{R}$ in the space

$$(5) \qquad W^\infty\{a_n, p\}_{(G)} \equiv \left\{u(x) \in C^\infty_{(G)}\colon\ \varrho(u) = \sum_{n=0}^\infty a_n \|D^n u(x)\|^p_{L_p(G)} < \infty\right\}$$

is very closely related to the one mentioned above (see [10], [11]). Here $a_n \geqslant 0$, $1 \leqslant p < \infty$. These spaces are the energy spaces for the differential equations of infinite order the model example of which is the following

$$(6) \qquad \sum_{n=0}^\infty (-1)^n D^n(a_n|D^n u|^{p-2} D^n u) = h(x), \qquad x \in G = (0, a)$$

$$(7) \qquad D^n u(0) = b_n, \quad D^n u(a) = c_n, \quad n = 0, 1, \dots$$

For the solvability of the problem (6), (7) we should first of all investigate the conditions of existence of a function in the space (5), satisfying the conditions (7).

We will suppose that the space (5) is nontrivial which means that the space

$$\overset{\circ}{W}{}^\infty\{a_n, p\}_{(0,a)} \equiv \{u(x) \in C^\infty_0(0, a), \varrho(u) < \infty\}$$

contains at least one function other than that which is identical to zero. Yu. Dubinskij [11] showed that this is the case if and only if the sequence $\{M_n\}$ defined by $M_n = a_n^{-1/p}$ for $a_n \neq 0$ and $M_n = \infty$ for $a_n = 0$, specifies a nonquasianalytic Carleman class (1), i.e., the conditions (2), (3) hold for $\{M_n\}$.

**Theorem 4.** *A necessary and sufficient condition for the sequence* $\{b_n\}$ *to be extendable in any space* $W^\infty\{a_n, p\}_{(0,a)}$ *is*

$$\overline{\lim_{n\to\infty}} \tfrac{1}{n}|b_n|^{1/n} = K < \infty. \tag{8}$$

We shall call a trace satisfying the condition (8) analytical.

R e m a r k. For any space $W^\infty\{a_n, p\}_{(0,a)}$ there exists a nonanalytic trace extendable in this space.

**Theorem 5.** *For the sequence* $\{b_n\}$ *to be extendable in the space* $W^\infty\{a_n, p\}_{(0,a)}$, *the following condition is necessary:*

$$\sum_{n=0}^{\infty} a_{n+1}^{1/p} a_n^{1-\frac{1}{p}} |b_n|^p < \infty. \tag{9}$$

**Theorem 6.** *Let the sequence* $\{a_n\}$ *be such that*

$$1 > a_n^q \geqslant a_{n+1}, \quad n = 0, 1, \ldots, \quad a_0 > 0, \tag{10}$$

*for some* $q > 1$. *Then for the existence of a function* $u(x) \in W^\infty\{a_n, p\}_{(0,a)}$ *with the given trace* $\{b_n\}$, *the condition*

$$\sum_{n=0}^{\infty} |b_n|^p (M_n^c)^{-(1-\frac{1}{p})} (M_{n+1}^c)^{\frac{-1}{p}} < \infty \tag{11}$$

*is necessary and sufficient.*

R e m a r k. If the sequence $\{a_n\}$ satisfies the condition (10) and the sequence $\{a_n^{-1}\}$ is almost logarithmically convex, then $M_n^c = a_n^{-1}$ and the condition (11) coincides with the condition (9).

R e m a r k. Proofs of Theorems 4–6 can be found in the paper [10].

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*Author's address*: *G. S. Balashova*, Department of Mathematics, Power Engineering Institute, Krasnokazarmennaja 14, 111 250 Moscow, Russia, e-mail: `balashov@cs.isa.ac.ru`.