ON WEIGHTED ESTIMATES OF SOLUTIONS OF NONLINEAR ELLIPTIC PROBLEMS

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. The paper is devoted to the estimate

$$|u(x,k)| \leqslant K|k| \left\{ \operatorname{cap}_{p,w}(F) \frac{\varrho^p}{w(B(x,\varrho))} \right\}^{\frac{1}{p-1}},$$

$2 \leqslant p < n$ for a solution of a degenerate nonlinear elliptic equation in a domain $B(x_0,1) \setminus F$, $F \subset B(x_0,d) = \{x \in \mathbb{R}^n \colon |x_0 - x| < d\}$, $d < \frac{1}{2}$, under the boundary-value conditions $u(x,k) = k$ for $x \in \partial F$, $u(x,k) = 0$ for $x \in \partial B(x_0,1)$ and where $0 < \varrho \leqslant \operatorname{dist}(x,F)$, $w(x)$ is a weighted function from some Muckenhoupt class, and $\operatorname{cap}_{p,w}(F)$, $w(B(x,\varrho))$ are weighted capacity and measure of the corresponding sets.

Keywords: degeneracy, Muckenhoupt class, pointwise estimate, nonlinear elliptic equation, capacity, a-priori estimate

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In the study of behaviour of solutions of nonlinear elliptic and parabolic equations an important role is played by special estimates of model problems in domains with small holes (see [1, 2]). In many cases this role is analogous to that of estimates of singular solutions of linear equations. By using these estimates the following problems were studied: asymptotical behaviour and the construction of correctors for nonlinear elliptic and parabolic problems in perforated domains, a necessary condition for the regularity of boundary points, the stability of solutions of nonlinear problems with respect to the variation of domains. The proof and applications of these estimates for elliptic equations are given in [1]. This paper is devoted to the extension of the method of obtaining of pointwise estimates for degenerate nonlinear elliptic equations.

1. Auxiliary lemmas and statement of the result

Let w be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < w(x) < \infty$ almost everywhere. We say that w belongs to the Muckenhoupt class A_t, $1 < t < \infty$, if there exists a constant $c_{t,w}$ such that

$$\frac{1}{|B|}\int\limits_B w\,\mathrm{d}x \leqslant c_{t,w}\Bigl(\frac{1}{|B|}\int\limits_B w^{\frac{1}{1-t}}\,\mathrm{d}x\Bigr)^{1-t} \tag{1}$$

for all balls B in $\mathbb{R}^n$. By $|E|$ we denote the Lebesgue n-measure of a measurable set $E \subset \mathbb{R}^n$.

We shall note only certain properties of functions from Muckenhoupt class.

Lemma 1. *If $w \in A_t$, then*

$$\Bigl(\frac{|E|}{|B|}\Bigr)^t \leqslant c_{t,w}\frac{w(E)}{w(B)}, \tag{2}$$

where B is an arbitrary ball in $\mathbb{R}^n$, E is a measurable subset of B and

$$w(E) = \int\limits_E w(x)\,\mathrm{d}x.$$

Lemma 2. *If $w \in A_t$, $t > 1$, then $w \in A_{t-\varepsilon}$ for some ε, $0 < \varepsilon < t-1$. Moreover, ε and $c_{t-\varepsilon,w}$ depend only on n, t, $c_{t,w}$.*

For the proofs of Lemmas 1, 2 see [3], Chapter 15.

Lemma 3. *Suppose $w \in A_t$ and $s > t$. Then $w \in A_s$.*

This statement immedeatly follows from the Hölder inequality and (1).

Lemma 4. (A_t-weighted Poincaré inequality) *Suppose $w \in A_t$ and let for arbitrary $x, s, h, 0 < s \leqslant h$ an inequality*

$$\frac{s}{h}\Bigl(\frac{w(B(x,s))}{w(B(x,h))}\Bigr)^{\frac{1}{q}} \leqslant c\Bigl(\frac{w(B(x,s))}{w(B(x,h))}\Bigr)^{\frac{1}{t}}, \qquad q > t,$$

hold with a constant c independent of x, s, h. Then

$$\left(\frac{1}{w(B)}\int\limits_B \left|v(x)-\frac{1}{|B|}\int\limits_B v(x)\,\mathrm{d}x\right|^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leqslant Cr\left(\frac{1}{w(B)}\int\limits_B \left|\frac{\partial v(x)}{\partial x}\right|^t w(x)\,\mathrm{d}x\right)^{\frac{1}{t}},$$

where $B=B\left(x_0,r\right)$, $v(x)\in C^\infty(B)$ and C is independent of x_0, r, v.

Lemma 5. (A_t-weighted Sobolev inequality) *With the same hypotheses as in Lemma 4 we have*

$$\left(\frac{1}{w(B)}\int\limits_B |v(x)|^q\,w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leqslant Cr\left(\frac{1}{w(B)}\int\limits_B \left|\frac{\partial v(x)}{\partial x}\right|^t w(x)\,\mathrm{d}x\right)^{\frac{1}{t}},$$

where $B=B\left(x_0,r\right)$, $v(x)\in C_0^\infty(B)$ and C is independent of x_0, r, v.

For the proofs of Lemmas 4, 5 see [4].

Definition and basic properties of the Muckenhoupt class A_t were explicitly studied in [3].

Let F be an arbitrary compact set in $\mathbb{R}^n$. Let us denote by d the minimum of the radii of balls containing F, and let x_0 be the center of such a ball with radius d, satisfying $F\subset\overline{B(x_0,d)}$. Here and in the sequel $B(x,r)$ denotes the ball with radius r and center at x.

Let $\psi(x)$ be a function from the class $C_0^\infty(B(x_0,1))$, equal to one in $B\left(x_0,\frac{1}{2}\right)$. If $d<\frac{1}{2}$, then for an arbitrary real k we consider a nonlinear elliptic boundary value problem

$$\sum_{i=1}^n \frac{\mathrm{d}}{\mathrm{d}x_i}a_i\big(x,\frac{\partial u}{\partial x}\big)=0, \qquad x\in D, \tag{3}$$

$$u(x)=k\psi(x), \qquad x\in\partial D. \tag{4}$$

Here $D=B(x_0,1)\setminus F$.

We assume that the functions $a_i(x,g)$, $i=1,\dots,n$, are defined for $x\in\overline{B}$ (here and in the sequel $B=B(x_0,1)$) and $g\in\mathbb{R}^n$, and satisfy the following conditions:

A$_1$) functions $a_i(x,g)$ are continuous in g for almost every $x\in\overline{B}$, measurable in x for all $g\in\mathbb{R}^n$;

A$_2$) there are positive constants ν_1, ν_2 such that for $2 \leqslant p < n$ and $x \in \overline{B}$, g, $q \in \mathbb{R}^n$ the inequalities

$$
\begin{aligned}
&|a_i(x,g)| \leqslant \nu_1 |g|^{p-1} w(x),\\
&\sum_{i=1}^{n} [a_i(x,g) - a_i(x,q)](g_i - q_i) \geqslant 0,\\
&\sum_{i=1}^{n} a_i(x,g) g_i \geqslant \nu_2 |g|^p w(x)
\end{aligned}
$$

hold, where $w(x) \in A_{(p-1)+\frac{p}{n}}(\mathbb{R}^n)$, $[w(x)]^{-\frac{1}{p-1}} \in A_{\frac{p}{p-1}(1-\frac{1}{n})}(\mathbb{R}^n)$.

R e m a r k 1. Let us choose $w(x) = |x|^\alpha, -n + p < \alpha < n(p-1) - (n-p)$ and $p \geqslant 2$. In this case $w(x)$ satisfies A$_2$. This is easily verified by a direct computation.

A solution of the boundary value problem (1), (2) is a function $u(x,k) \in W_p^1(D,w)$ such that $u(x,k) - k\psi(x) \in \overset{\circ}{W}{}_p^1(D,w)$ and the integral identity

$$
\sum_{i=1}^{n} \int_D a_i\left(x, \frac{\partial u}{\partial x}\right) \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x = 0 \tag{5}
$$

holds for arbitrary function $\varphi(x) \in \overset{\circ}{W}{}_p^1(D,w)$.

The definitions and properties of weighted Sobolev spaces $W_p^1(\Omega,w), \overset{\circ}{W}{}_p^1(\Omega,w)$ were studied in [3, 4, 5] (here $\Omega \subset \mathbb{R}^n$).

The existence and uniqueness of the function $u(x,k)$ follows from the global theory of monotone operators (for example, see [1]). The function $u(x,k)$ is assumed to be extended to F by the constant k.

For the purpose of formulation of our main result let us introduce the notion of weighted (p,w)-capacity $\mathrm{cap}_{p,w}$ (see [3]).

The number

$$
\mathrm{cap}_{p,w}(E) = \inf \int_B \left| \frac{\partial v(x)}{\partial x} \right|^p w(x) \,\mathrm{d}x \tag{6}
$$

is called the (p,w)-capacity of the closed set $E \subset B(x_0, \frac{1}{2})$. The infimum in (6) is taken over all functions $v(x) \in C_0^\infty(B)$ satisfying the equality $v(x) = 1$ for $x \in E$.

Further, we shall prove the following

Theorem. *Let us assume that conditions* A$_1$, A$_2$, *are satisfied. Then there exists a constant K depending only on n, p, ν_1, ν_2 and the Muckenhoupt constant $c_{p,w}$ of*

w such that, for a solution $u(x,k)$ of the problem (1), (2) *and for an arbitrary point $x \in D$,*

$$|u(x,k)| \leqslant K|k| \Big\{ \operatorname{cap}_{p,w}(F) \frac{\varrho^p}{w(B(x,\varrho))} \Big\}^{\frac{1}{p-1}}, \tag{7}$$

where $0 < \varrho \leqslant \varrho(x,F)$.

Remark 2. In case $w(x) \equiv 1$ the estimate (7) coincides with the pointwise estimate of the solution of nonlinear Dirichlet problem obtained by the first author in [2]. Exactness of (7) follows also from

$$G(x,\xi) \approx \frac{|x-\xi|^2}{w(B(x,|x-\xi|))}$$

for the fundamental solution $G(x,\xi)$ of the operator

$$L = \sum_{i,j=1}^{n} D_{x_i}\left(a_{ij}(x) D_{x_j}\right),$$

where $a_{i,j}(x)$ are real-valued, symmetric and

$$\lambda w(x)|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant \frac{1}{\lambda} w(x)|\xi|^2,$$

whenever $\lambda > 0$, $\xi = (\xi_1,\dots,\xi_n)$, and $w(x)$ is the same as in condition A_2 (case $p=2$). This estimate was obtained in [6].

2. Proof of the main result

Let us assume $k > 0$.

Lemma 6. *Let us assume that conditions A_1, A_2 are satisfied and let $u(x,k)$ be the solution of the problem* (1), (2). *Then for $k \neq 0$*

$$0 \leqslant \tfrac{1}{k} u(x,k) \leqslant 1.$$

Proof. Let us take the test-function $\varphi_1(x) = \min\{u(x,k),0\}$ in the integral identity (5) and use the condition A_2. We obtain

$$\int_{D_1} \left|\frac{\partial u(x,k)}{\partial x}\right|^p w(x)\,\mathrm{d}x \leqslant 0,$$

where $D_1 = \{x \in D\colon\ u(x,k) < 0\}$. From this inequality it follows that $u(x,k) \geqslant 0$. Similarly, replacing $\varphi(x)$ in (5) by $\varphi_2(x) = \max\{u(x,k)-k,0\}$ the inequality $u(x,k) \leqslant k$ is established. $\square$

Lemma 7. *Assume that conditions* A_1, A_2 *are satisfied. Then there exists a constant* c_1, *depending only on* n, p, ν_1, ν_2, $c_{p,w}$, *such that*

$$\int_D \left|\frac{\partial u(x,k)}{\partial x}\right|^p w(x)\,\mathrm{d}x \leqslant c_1 k^p \operatorname{cap}_{p,w}(F). \tag{8}$$

P r o o f. Let us take the test-function $\varphi = u(x,k) - k\psi(x)$ in the integral identity (5), where $\psi(x)$ is from the class $C_0^\infty(B)$, and ψ is equal to one in F. Using the condition A_2 and Young's inequality we estimate the terms of the obtained equality and get

$$\int_D \left|\frac{\partial u(x,k)}{\partial x}\right|^p w(x)\,\mathrm{d}x \leqslant c_2 k^p \int_B \left|\frac{\partial \psi(x)}{\partial x}\right|^p w(x)\,\mathrm{d}x. \tag{9}$$

Here and in the sequel we denote by c_i constants depending only on the same parameters as the constant K in the formulation of the Theorem.

By virtue of definition (6), inequality (9) proves the estimate (8). □

Let us denote for $0 < \mu < k$

$$E_\mu = \{x \in D\colon\ 0 \leqslant u(x,k) \leqslant \mu\}.$$

Lemma 8. *Let us assume that conditions* A_1, A_2 *are satisfied. Then there exists a constant* c_3 *such that*

$$\int_{E_\mu} \left|\frac{\partial u(x,k)}{\partial x}\right|^p w(x)\,\mathrm{d}x \leqslant c_3 \mu k^{p-1} \operatorname{cap}_{p,w}(F). \tag{10}$$

P r o o f. We substitute $\varphi(x) = u_\mu(x,k) - \frac{\mu}{k}u(x,k)$ in (5), where $u_\mu(x,k) = \min\{u(x,k),\mu\}$. By standard computations and (8) we obtain (10). □

In order to prove Theorem we need some auxiliary results.

Lemma 9. *Let* $2 \leqslant p < n$ *and* $w \in A_{(p-1)+\frac{p}{n}}$, $[w]^{-\frac{1}{p-1}} \in A_{\frac{p}{p-1}(1-\frac{1}{n})}$. *For any function* $v(x) \in \overset{\circ}{W}{}^1_p(B(0,R),w)$ *and any numbers* r, R, *satisfying the conditions* $0 < r \leqslant R$ *the inequality*

$$\int_{B(0,r)} |v(x)|^p w(x)\,\mathrm{d}x \leqslant K_1 r^p \int_{B(0,R)} \left|\frac{\partial v(x)}{\partial x}\right|^p w(x)\,\mathrm{d}x \tag{11}$$

holds with a constant K_1 depending only on n, p, $c_{p,w}$.

P r o o f. Without loss of generality we may assume that $v(x) \in C_0^\infty(B(0,R))$. From A_p-weighted Poincaré inequality we have

(12)
$$\begin{aligned}
\frac{1}{w(B(0,r))} &\int_{B(0,r)} |v(x)|^p w(x)\,\mathrm{d}x \\
&\leqslant \frac{2^{p-1}}{w(B(0,r))} \int_{B(0,r)} \left| v(x) - \frac{1}{|B(0,r)|} \int_{B(0,r)} v(y)\,\mathrm{d}y \right|^p w(x)\,\mathrm{d}x \\
&\quad + 2^{p-1} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |v(y)|\,\mathrm{d}y \right)^p \\
&\leqslant c_4 \frac{r^p}{w(B(0,r))} \int_{B(0,r)} \left| \frac{\partial v(x)}{\partial x} \right|^p w(x)\,\mathrm{d}x + c_4 \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |v(y)|\,\mathrm{d}y \right)^p.
\end{aligned}$$

Now we only need to estimate the last term on the right-hand side of (12).

Let $\omega = \frac{x}{|x|}$. A straightforward calculation yields

$$|v(x)| = \left| \int_{|x|}^{R} \frac{\mathrm{d}}{\mathrm{d}t} v(\omega t)\,\mathrm{d}t \right| \leqslant \left| \int_{|x|}^{R} \left| \frac{\partial v}{\partial x}(\omega t) \right| \mathrm{d}t \right|. \tag{13}$$

Transforming the last integral on the right-hand side of (12) into spherical coordinates with respect to the variables $|x| \in [0,r]$, $\omega = \frac{x}{|x|} \in S_1(0)$, using (13) and Hölder inequality, we obtain

(14)
$$\begin{aligned}
I_1 &= \left\{ \frac{1}{|B(0,r)|} \int_{B(0,r)} |v(y)|\,\mathrm{d}y \right\}^p = \left\{ \frac{1}{|B(0,r)|} \int_0^r \int_{S_1(0)} |v(|x|\omega)|\,|x|^{n-1}\,\mathrm{d}\omega\,\mathrm{d}|x| \right\}^p \\
&\leqslant \left\{ \frac{1}{|B(0,r)|} \int_0^r \int_{S_1(0)} \int_{|x|}^{R} \left| \frac{\partial v}{\partial x}(\omega t) \right| \mathrm{d}t |x|^{n-1}\,\mathrm{d}\omega\,\mathrm{d}|x| \right\}^p \\
&\leqslant c_5 \left\{ \frac{1}{|B(0,r)|^p} \int_0^r \int_{|x|}^{R} \int_{S_1(0)} \left| \frac{\partial v}{\partial x}(\omega t) \right|^p w(\omega t) t^{n-1}\,\mathrm{d}\omega\,\mathrm{d}t |x|^{n-1}\,\mathrm{d}|x| \right\} \\
&\quad \times \left\{ \int_0^r \int_{|x|}^{R} \int_{S_1(0)} [w(\omega t)]^{-\frac{1}{p-1}} t^{-\frac{n-1}{p-1}}\,\mathrm{d}\omega\,\mathrm{d}t |x|^{n-1}\,\mathrm{d}|x| \right\}^{p-1}
\end{aligned}$$

$$\leqslant c_6\Big\{\frac{1}{r^{n(p-1)}}\int\limits_{B(0,R)}\Big|\frac{\partial v}{\partial x}(x)\Big|^p w(x)\,\mathrm{d}x\Big\}$$

$$\times\Big\{\int\limits_0^r |x|^{n-1}\int\limits_{|x|}^{R}\int\limits_{S_1(0)}[w(\omega t)]^{-\frac{1}{p-1}}t^{-\frac{n-1}{p-1}}\,\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}|x|\Big\}^{p-1}.$$

Now we estimate separately the integral

$$I_2=\int\limits_{|x|}^{R}\int\limits_{S_1(0)}[w(\omega t)]^{-\frac{1}{p-1}}t^{-\frac{n-1}{p-1}}\,\mathrm{d}\omega\,\mathrm{d}t=\int\limits_{|x|\leqslant|z|\leqslant R}|z|^{\frac{-np+p}{p-1}}[w(z)]^{-\frac{1}{p-1}}\,\mathrm{d}z$$

$$\leqslant\sum_{j=1}^{\infty}\int\limits_{2^{j-1}|x|\leqslant|z|\leqslant 2^j|x|}|z|^{\frac{-np+p}{p-1}}[w(z)]^{-\frac{1}{p-1}}\,\mathrm{d}z \tag{15}$$

$$\leqslant c_7\sum_{j=1}^{\infty}\big(2^j|x|\big)^{\frac{-np+p}{p-1}}\int\limits_{|z|\leqslant 2^j|x|}[w(z)]^{-\frac{1}{p-1}}\,\mathrm{d}z.$$

Since $w^{-\frac{1}{p-1}}\in A_{\frac{p}{p-1}(1-\frac{1}{n})}$, by Lemma 2, $w^{-\frac{1}{p-1}}\in A_{\frac{p}{p-1}(1-\frac{1}{n})-\varepsilon_1}$, where $\varepsilon_1>0$. Now, using (2), from (15) we obtain

$$I_2\leqslant c_8\sum_{j=1}^{\infty}\big(2^j|x|\big)^{\frac{-np+p}{p-1}}\big(2^j\big)^{\frac{np-p}{p-1}}\big(2^j\big)^{-n\varepsilon_1}\int\limits_{|z|\leqslant|x|}[w(z)]^{-\frac{1}{p-1}}\,\mathrm{d}z$$

$$\leqslant c_9\,|x|^{\frac{-np+p}{p-1}}\int\limits_{|z|\leqslant|x|}[w(z)]^{-\frac{1}{p-1}}\,\mathrm{d}z. \tag{16}$$

By virtue of Lemma 3, $w\in A_p$, and estimating the integral on the right-hand side of (16) with the help of (1) we have

$$I_2\leqslant c_{10}\,|x|^{\frac{-np+p}{p-1}}\,|x|^{\frac{np}{p-1}}\,[w(B(0,|x|))]^{-\frac{1}{p-1}}=c_{10}\,|x|^{\frac{p}{p-1}}\,[w(B(0,|x|))]^{-\frac{1}{p-1}}\,. \tag{17}$$

Since $w\in A_{p-1+\frac{p}{n}}$, by Lemma 2, $w\in A_{p-1+\frac{p}{n}-\varepsilon_2}$, where $\varepsilon_2>0$. Using this, we obtain from Lemma 1

$$[w(B(0,|x|))]^{-\frac{1}{p-1}}\leqslant c_{11}\Big[\frac{r}{|x|}\Big]^{n+\frac{p-\varepsilon_2 n}{p-1}}[w(B(0,r))]^{-\frac{1}{p-1}}\,. \tag{18}$$

Thus from (14), (17), (18) we have

$$
\begin{aligned}
I_1 &\leqslant c_{12}\,[w(B(0,r))]^{-1}\, r^{p-\varepsilon_2 n}\\
&\quad\times\Big\{\int\limits_{B(0,R)} \Big|\frac{\partial v}{\partial x}(x)\Big|^p w(x)\,\mathrm{d}x\Big\}\Big\{\int\limits_0^r |x|^{n-1}|x|^{\frac{p}{p-1}}|x|^{-n-\frac{p}{p-1}+\frac{\varepsilon_2 n}{p-1}}\,\mathrm{d}|x|\Big\}^{p-1}\\
&\leqslant c_{13}\frac{r^p}{w(B(0,r))}\int\limits_{B(0,R)} \Big|\frac{\partial v}{\partial x}(x)\Big|^p w(x)\,\mathrm{d}x.
\end{aligned}
$$

Now the desired estimate follows from (12) and the last inequality. □

R e m a r k 3. In case $p=2$ the statement of Lemma 9 coincides with the statement of Lemma 2.2 in [7].

P r o o f o f T h e o r e m. Let ξ be an arbitrary point of D and for $0<\varrho\leqslant\varrho(\xi,F)$ we define the numerical sequence

$$\varrho_j=\tfrac{\varrho}{4}[3-2^{-j}],\quad j=1,2,\dots$$

Let functions $\psi_j(x)$ be equal to one on $B_j=B\left(\xi,\varrho_j\right)$ and to zero outside B_{j+1}, and such that $0\leqslant\psi_j(x)\leqslant 1$, $\Big|\frac{\partial\psi_j(x)}{\partial x}\Big|\leqslant\frac{2^{j+4}}{\varrho}$.

Substitute $\varphi(x)=[u(x,k)]^{\sigma+1}[\psi_j(x)]^{\tau+p}$ into (5), where σ,τ are arbitrary positive numbers. Using A_2 and Young's inequality, we obtain

$$\int\limits_D\Big|\frac{\partial u}{\partial x}\Big|^p u^\sigma\psi_j^{\tau+p}w\,\mathrm{d}x\leqslant c_{14}(\tau+p)^p\frac{2^{jp}}{\varrho^p}[m_{j+1}]^{p-1}\int\limits_D u^{\sigma+1}\psi_j^\tau w\,\mathrm{d}x, \tag{19}$$

where $m_j=\max\{u(x,k)\colon\ x\in\overline{B_j}\}$.

Let t, s be arbitrary positive numbers satisfying the inequalities

$$t+p>\frac{pnp_0}{np_0-p},\quad s+p>\frac{pnp_0}{np_0-p},$$

where $1<p_0<p-1+\frac{p}{n}$ and p_0 depends only on n, p, $c_{p,w}$. Then

$$\left\{u^{t+p}(x,k)\psi_j^{s+p}(x)\right\}^{\frac{np_0-p}{pnp_0}}\in\overset{\circ}{W}{}^1_p\left(B\left(\xi,\tfrac34\varrho\right),w\right).$$

By virtue of A_p-weighted Sobolev inequality and (19) we have

$$\begin{aligned}
&\left[\frac{1}{w\left(B\left(\xi,\frac{3}{4}\varrho\right)\right)}\int\limits_{B\left(\xi,\frac{3}{4}\varrho\right)}\left(\{u^{t+p}\psi_j^{s+p}\}^{\frac{np_0-p}{pnp_0}}\right)^{\frac{pnp_0}{np_0-p}}w\,\mathrm{d}x\right]^{\frac{np_0-p}{pnp_0}}\\
&\qquad\leqslant c_{15}\varrho\left[w\left(B\left(\xi,\tfrac{3}{4}\varrho\right)\right)\right]^{-\frac{1}{p}}\\
&\qquad\times\left\{(t+p)^p\int\limits_{B\left(\xi,\frac{3}{4}\varrho\right)}\left|\frac{\partial u}{\partial x}\right|^p u^{(t+p)\frac{np_0-p}{np_0}-p}\psi_j^{(s+p)\frac{np_0-p}{np_0}}w\,\mathrm{d}x\right.\\
&\qquad\quad\left.+(s+p)^p\int\limits_{B\left(\xi,\frac{3}{4}\varrho\right)}u^{(t+p)\frac{np_0-p}{np_0}}\left|\frac{2^j}{\varrho}\right|^p\psi_j^{(s+p)\frac{np_0-p}{np_0}-p}w\,\mathrm{d}x\right\}^{\frac{1}{p}}\\
&\qquad\leqslant c_{16}(t+s+p)^2\left[w\left(B\left(\xi,\tfrac{3}{4}\varrho\right)\right)\right]^{-\frac{1}{p}}2^j\left[m_{j+1}\right]^{\frac{p-1}{p}}\\
&\qquad\times\left\{\int\limits_{B\left(\xi,\frac{3}{4}\varrho\right)}u^{(t+p)\frac{np_0-p}{np_0}-p+1}\psi_j^{(s+p)\frac{np_0-p}{np_0}-p}w\,\mathrm{d}x\right\}^{\frac{1}{p}}.
\end{aligned}$$

Using Lemma 1, we obtain from the last inequality

$$\begin{aligned}
\int\limits_{B_{j+1}}u^{t+p}\psi_j^{s+p}w\,\mathrm{d}x &\leqslant c_{17}(t+s+p)^{\frac{2pnp_0}{np_0-p}}[w(B(\xi,\varrho))]^{-\frac{p}{np_0-p}}\\
&\times\left[2^{jp}m_{j+1}^{p-1}\right]^{\frac{np_0}{np_0-p}}\left[\int\limits_{B_{j+1}}u^{(t+p)\frac{np_0-p}{np_0}-p+1}\psi_j^{(s+p)\frac{np_0-p}{np_0}-p}w\,\mathrm{d}x\right]^{\frac{np_0}{np_0-p}}.
\end{aligned}$$

Choosing

$$\begin{aligned}
t=t_i&=\left[p+\frac{np_0}{p}(p-1)\right]\left(\frac{np_0}{np_0-p}\right)^i-\frac{np_0}{p}(p-1)-p,\\
s=s_i&=[p+np_0]\left(\frac{np_0}{np_0-p}\right)^i-np_0-p,
\end{aligned}$$

we rewrite the last inequality in the form

$$J_i\leqslant c_{18}\left(\frac{np_0}{np_0-p}\right)^{2i\frac{pnp_0}{np_0-p}}[w(B(\xi,\varrho))]^{-\frac{p}{np_0-p}}\left[2^{jp}m_{j+1}^{p-1}\right]^{\frac{np_0}{np_0-p}}[J_{i-1}]^{\frac{np_0}{np_0-p}},\tag{20}$$

where $J_i=\int\limits_{B_{j+1}}u^{t_i+p}\psi^{s_i+p}w\,\mathrm{d}x$.

By iterating we arrive at

(21)

$$[J_i]^{\left(\frac{np_0-p}{np_0}\right)^i} \leqslant \left\{ c_{19} \left[2^{jp} m_{j+1}^{p-1}\right]^{\frac{np_0}{np_0-p}} [w(B(\xi,\varrho))]^{\frac{-p}{np_0-p}} \right\}^{\frac{np_0-p}{np_0}+\left(\frac{np_0-p}{np_0}\right)^2+\ldots+\left(\frac{np_0-p}{np_0}\right)^i}$$
$$\times \left\{ \left(\frac{np_0}{np_0-p}\right)^{2p} \right\}^{1+2\frac{np_0-p}{np_0}+\ldots+i\left(\frac{np_0-p}{np_0}\right)^{i-1}} \times J_0.$$

When i tends to infinity, then (21) yields

$$[m_j]^{p+\frac{np_0}{p}(p-1)} \leqslant c_{20} 2^{jnp_0} [w(B(\xi,\varrho))]^{-1} [m_{j+1}]^{\frac{np_0}{p}(p-1)} \int\limits_{B_{j+1}} u^p \psi_j^p w \, \mathrm{d}x. \tag{22}$$

Now we estimate the integral on the right-hand side of (22) by Lemma 8 and Lemma 9:

$$\begin{aligned} \int\limits_{B_{j+1}} u^p \psi_j^p w \, \mathrm{d}x &\leqslant \int\limits_{B_{j+1}} \left[u_{m_{j+1}}\right]^p w \, \mathrm{d}x \\ &\leqslant c_{21} \varrho^p \int\limits_{E_{m_{j+1}}} \left|\frac{\partial u}{\partial x}\right|^p w \, \mathrm{d}x \\ &\leqslant c_{22} \varrho^p m_{j+1} k^{p-1} \operatorname{cap}_{p,w}(F). \end{aligned} \tag{23}$$

By virtue of (22), (23) implies

$$[m_j]^{p+\frac{np_0}{p}(p-1)} \leqslant c_{23} 2^{jnp_0} \frac{\varrho^p}{w(B(\xi,\varrho))} [m_{j+1}]^{1+\frac{np_0}{p}(p-1)} k^{p-1} \operatorname{cap}_{p,w}(F), \tag{24}$$

Further we shall use the following:

Lemma 10. *Let $\{\alpha_i\}$ be a bounded number sequence satisfying*

$$\alpha_i \leqslant A \alpha_{i+1}^{\sigma} a^i, \quad i = 1, 2, \ldots$$

with positive constants $A, a, \sigma \in (0,1)$. Then we have

$$\alpha_1 \leqslant c A^{\frac{1}{1-\sigma}}$$

with a constant c depending only on σ and a.

For the proof of Lemma 10 see [1], Chapter 5.

Finally, from (24) and Lemma 10 we have

$$m_1 \leqslant c_{24} k \left\{ \operatorname{cap}_{p,w}(F) \frac{\varrho^p}{w(B(\xi,\varrho))} \right\}^{\frac{1}{p-1}},$$

and so the inequality (7) is established.

This completes the proof of Theorem.

□

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