# THE OBSTACLE PROBLEM FOR FUNCTIONS OF LEAST GRADIENT

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*Dedicated to Professor Alois Kufner on the occasion of his 65th birthday*

*Abstract.* For a given domain $\Omega \subset \mathbb{R}^n$, we consider the variational problem of minimizing the $L^1$-norm of the gradient on $\Omega$ of a function $u$ with prescribed continuous boundary values and satisfying a continuous lower obstacle condition $u \geqslant \psi$ inside $\Omega$. Under the assumption of strictly positive mean curvature of the boundary $\partial\Omega$, we show existence of a continuous solution, with Hölder exponent half of that of data and obstacle.

This generalizes previous results obtained for the unconstrained and double-obstacle problems. The main new feature in the present analysis is the need to extend various maximum principles from the case of two area-minimizing sets to the case of one sub- and one superminimizing set. This we accomplish subject to a weak regularity assumption on one of the sets, sufficient to carry out the analysis. Interesting open questions include the uniqueness of solutions and a complete analysis of the regularity properties of area superminimizing sets. We provide some preliminary results in the latter direction, namely a new monotonicity principle for superminimizing sets, and the existence of "foamy" superminimizers in two dimensions.

*Keywords*: least gradient, sets of finite perimeter, area-minimizing, obstacle

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## 1. Introduction

A rather complete and extensive literature is now in place concerning existence and regularity of solutions to a wide range of variational problems for which the

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following is prototypical:

$$\inf\left\{\int_\Omega |\nabla u|^p\colon\ u\in W^{1,p}(\Omega), u-g\in W_0^{1,p}(\Omega)\right\}. \tag{1.1}$$

Here, $\Omega\subset\mathbb{R}^n$ is a bounded, open set, $1<p<\infty$ and $g\in W^{1,p}(\Omega)\cap C^0(\overline{\Omega})$. The Euler-Lagrange equation for (1.1) is the $p$-Laplacian $\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0$. The interested reader can consult recent books on this subject and the references therein, [1], [11], and [13]. The theory related to the case corresponding to $p=1$ is far less complete. In spite of the fact that there is a vast literature relating to the least area functional,

$$\inf_u\left\{\int_\Omega\sqrt{1+|\nabla u|^2}\right\},$$

there are many open questions concerning other functionals with linear growth in $|\nabla u|$. Investigations concerning such questions were considered in [20], [19]. In particular, the Dirichlet problem was investigated; that is, for a bounded Lipschitz domain $\Omega\subset\mathbb{R}^n$, and for $g\colon\partial\Omega\to\mathbb{R}^1$ continuous, the questions of existence and regularity of solutions to

$$\inf\{\|\nabla u\|(\Omega)\colon\ u\in\mathrm{BV}(\Omega),\ u=g\text{ on }\partial\Omega\} \tag{1.2}$$

were examined. Here $\|\nabla u\|(\Omega)$ denotes the total variation of the vector-valued measure $\nabla u$ evaluated on $\Omega$. It was shown that a solution $u\in\mathrm{BV}(\Omega)\cap C^0(\Omega)$ exists provided that $\partial\Omega$ satisfies two conditions, namely, that $\partial\Omega$ has non-negative curvature (in a weak sense) and that $\partial\Omega$ is not locally area-minimizing. See Section 2 below for notation and definitions.

In this paper we consider the obstacle problem

$$\inf\left\{\|\nabla u\|(\Omega)\colon\ u\in C^0(\overline{\Omega}),\ u\geqslant\psi\text{ on }\Omega, u=g\geqslant\psi\text{ on }\partial\Omega\right\} \tag{1.3}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz domain, $g\colon\partial\Omega\to\mathbb{R}^1$ is continuous and $\psi$ is a continuous function on $\overline{\Omega}$. The analogous obstacle problem for (1.1) was investigated by several authors and is now well understood, cf. [4], [12], [16], [15]. One of the difficulties encountered in the analysis of both (1.1) and (1.3) is the fact that the compactness in $L^1(\Omega)$ of a sequence whose BV-norms are bounded does not ensure,

*a priori*, continuity of the limiting function or that it will assume the boundary values $g$, thus making the question of existence problematic. In this paper as well as in [20], we rely heavily on the discovery made in [3] that the superlevel sets of a function of least gradient are area-minimizing. This fact, along with the co-area formula (see (2.10) below), suggests that the existence of a function of least gradient subject to an obstacle constraint can be established by actually constructing each of its superlevel sets in such a way that it reflects both the appropriate boundary condition and the obstacle condition. The main thrust of this paper is to show that this is possible. Thus we show that there exists a continuous solution to (1.3) and we also show it inherits essentially the same regularity as the boundary data and obstacle.

As in [20], both existence and regularity are developed by extensive use of BV theory and sets of finite perimeter as well as certain maximum principles. One of the main contributions of this paper is a new maximum principle that involves a super area-minimizing set and an area-minimizing set, Theorem 3.3. The similar result involving two area-minimizing sets, due independently to [14] and [18], played a crucial role in [20].

Our extended maximum principle requires a weak regularity property on one of the sets, that the set be contained in the (topological) closure of its interior. This is clearly satisfied in the contexts that we apply it, for which one of the sets is always area-minimizing. But, an interesting open question is whether or not this technical assumption can be dropped.

This issue leads us to consider a question of interest in its own right: "What is the regularity of a (sub)superminimizing set?" We conclude by presenting some separate, preliminary results on this subject, including a new monotonicity principle for (sub)superminimizing sets, and the existence of unusual, "foamy" (sub)superminimizers in two dimensions. It is our hope that these results will stimulate further investigation into the topic of regularity.

## 2. Preliminaries

The Lebesgue measure of a set $E \subset \mathbb{R}^n$ will be denoted $|E|$ and $H^\alpha(E), \alpha > 0$, will denote $\alpha$-dimensional Hausdorff measure of $E$. Throughout the paper, we almost exclusively employ $H^{n-1}$. The Euclidean distance between two points $x, y \in \mathbb{R}^n$ will be denoted by $|x - y|$. The open ball of radius $r$ centered at $x$ is denoted by $B(x, r)$ and $\overline{B}(x, r)$ denotes its closure.

If $\Omega \subset \mathbb{R}^n$ is an open set, the class of function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distribution are measures with finite total variation in $\Omega$ is denoted by $\mathrm{BV}(\Omega)$ and is called the space of functions of bounded variation on $\Omega$. The space $\mathrm{BV}(\Omega)$ is endowed with the norm

$$\|u\|_{\mathrm{BV}(\Omega)} = \|u\|_{1;\Omega} + \|\nabla u\|(\Omega) \tag{2.1}$$

where $\|u\|_{1;\Omega}$ denotes the $L^1$-norm of $u$ on $\Omega$ and where $\|\nabla u\|$ is the total variation of the vector-valued measure $\nabla u$.

The following compactness result for $\mathrm{BV}(\Omega)$ will be needed later, cf. [10] or [21].

**2.1 Theorem.** *If $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, then*

$$BF(\Omega) \cap \bigl\{u\colon\ \|u\|_{\mathrm{BV}(\Omega)} \leqslant 1\bigr\}$$

*is compact in $L^1(\Omega)$. Moreover, if $u_i \to u$ in $L^1(\Omega)$ and $U \subset \Omega$ is open, then*

$$\liminf_{x\to\infty} \|\nabla u_i\|(U) \geqslant \|\nabla u\|(U).$$

A Borel set $E \subset \mathbb{R}^n$ is said to have *finite perimeter in* $\Omega$ provided the characteristic function of $E$, $\chi_E$, is a function of bounded variation in $\Omega$. Thus, the partial derivatives of $\chi_E$ are Radon measures on $\Omega$ and the perimeter of $E$ in $\Omega$ is defined as

$$P(E,\Omega) = \|\nabla \chi_E\|(\Omega). \tag{2.2}$$

A set $E$ is said to be of *locally finite perimeter* if $P(E,\Omega) < \infty$ for every bounded open set $\Omega \subset \mathbb{R}^n$.

One of the fundamental results in the theory of sets of finite perimeter is that they possess a measure-theoretic exterior normal which is suitably general to ensure the validity of the Gauss-Green theorem. A unit vector $\nu$ is defined as the measure-theoretic exterior normal to $E$ at $x$ provided

$$\lim_{r\to 0} r^{-n} |B(x,r) \cap \{y\colon\ (y-x)\cdot\nu < 0,\ y \notin E\}\,| = 0$$

and

$$\lim_{r\to 0} r^{-n} |B(x,r) \cap \{y\colon\ (y-x)\cdot\nu > 0,\ y \in E\}\,| = 0. \tag{2.3}$$

The measure-theoretic normal of $E$ at $x$ will be denoted by $\nu(x,E)$ and we define

$$\partial_* E = \{x\colon\ \nu(x,E) \text{ exists}\}\,. \tag{2.4}$$

The Gauss-Green theorem in this context states that if $E$ is a set of locally finite perimeter and $V\colon\ \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz vector field, then

$$\int_E \operatorname{div} V(x)\,\mathrm{d}x = \int_{\partial_* E} V(x)\cdot\nu(x,E)\,\mathrm{d}H^{n-1}(x), \tag{2.5}$$

cf. [6, §4.5.6]. Clearly, $\partial_* E \subset \partial E$, where $\partial E$ denotes the topological boundary of $E$. Also, the *topological interior of* $E$ is denoted by $E^i = (\mathbb{R}^n \setminus \partial E) \cap E$, the *topological exterior* by $E^e = (\mathbb{R}^n \setminus \partial E) \cap (\mathbb{R}^n \setminus E)$ and $E^c$ to denote the complement $\mathbb{R}^n \setminus E$. The notation $E \subset\subset F$ means that the closure of $E$, $\overline{E}$, is a compact subset of $F^i$.

For measurable sets $E$, the *measure-theoretic interior*, $E^i_m$, is the set of all points at which the metric density of $E$ is 1 and the *measure-theoretic exterior*, $E^e_m$, is all points at which the metric density is 0. The *measure-theoretic closure*, $\overline{E}_m$, is the complement of $E^e_m$ and the *measure theoretic-boundary* is defined as $\partial_m E := \mathbb{R}^n \setminus (E^i_m \cup E^e_m)$. Clearly, $\partial_* E \subset \partial_m E \subset \partial E$. Moreover, it is well known that

$$E \text{ is of finite perimeter if and only if } H^{n-1}(\partial_m E) < \infty \tag{2.6}$$

and that

$$P(E,\Omega) = H^{n-1}(\Omega \cap \partial_m E) = H^{n-1}(\Omega \cap \partial_* E) \text{ whenever } P(E,\Omega) < \infty \tag{2.7}$$

cf. [6 §4.5]. From this it easily follows that

$$P(E \cup F, \Omega) + P(E \cap F, \Omega) \leqslant P(E,\Omega) + P(F,\Omega), \tag{2.8}$$

thus implying that sets of finite perimeter are closed under finite unions and intersections.

The definition implies that sets of finite perimeter are defined only up to sets of measure 0. In other words, each such set determines an equivalence class of sets of finite perimeter. In order to avoid this ambiguity, we will employ the measure theoretic closure of $E$ as the canonical representative; that is, with this convention

$$x \in E \quad \text{if and only if} \quad \limsup_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} > 0. \tag{2.9}$$

Furthermore, it easy to see that

$$\overline{\partial_* E} = \partial E. \tag{2.10}$$

This convention will apply, in particular, to all competitors of the variational problems (2.22) and (2.23) below as well as to the sets defined by (2.18).

Of particular importance to us are sets of finite perimeter whose boundaries are area-minimizing. If $E$ is a set of locally finite perimeter and $U$ a bounded, open set, then $E$ is said to be *area-minimizing in* $U$ if $P(E,U) \leqslant P(F,U)$ whenever $E \Delta F \subset\subset U$. Also, $E$ is said to be *super area-minimizing in* $U$ (*sub area-minimizing in* $U$) if $P(E,U) \leqslant P(E \cup F, U)$ $(P(E,U) \leqslant P(E \cap F, U))$ whenever $E\Delta F \subset\subset U$.

A tool that will play a significant role in this paper is the co-area formula. It states that if $u \in \mathrm{BV}(\Omega)$, then

$$\|\nabla u\|(\Omega) = \int_{-\infty}^{\infty} P(E_t, \Omega)\, \mathrm{d}t \tag{2.11}$$

where $E_t = \{u \geqslant t\}$. In case $u$ is Lipschitz, we have

$$\int_\Omega |\nabla u|\, \mathrm{d}x = \int_{-\infty}^{\infty} H^{n-1}\left(u^{-1}(t) \cap \Omega\right)\, \mathrm{d}t.$$

Conversely, if $u$ is integrable on $\Omega$ then

$$\int_{-\infty}^{\infty} P(E_t, \Omega)\, \mathrm{d}t < \infty \text{ implies } u \in \mathrm{BV}(\Omega), \tag{2.12}$$

cf. [5], [7].

Another fundamental result is the isoperimetric inequality for sets of finite perimeter. It states that there is a constant $C = C(n)$ such that

$$P(E)^{n/(n-1)} \leqslant C|E| \tag{2.13}$$

whenever $E \subset \mathbb{R}^n$ is a set of finite perimeter. Furthermore, equality holds if and only if $E$ is a ball when $C$ is the best constant.

The regularity of $\partial E$ plays a crucial role in our development. In particular, we will employ the notion of tangent cone. Suppose $E$ is area-minimizing in $U$ and for convenience of notation, suppose $0 \in U \cap \partial E$. For each $r > 0$, let $E_r = \mathbb{R}^n \cap \{x\colon\, rx \in E\}$. It is known (cf. [17, §35]) that for each sequence $\{r_i\} \to 0$, there exists a subsequence (denoted by the full sequence) such that $\chi_{E_i}$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to $\chi_C$, where $C$ is a set of locally finite perimeter. In fact, $C$ is area-minimizing and is called the tangent cone to $E$ at 0. Although it is not immediate, $C$ is a cone and therefore the union of half-lines issuing from 0. It follows from [17, §37.6] that if $\overline{C}$ is contained in $\overline{H}$ where $H$ is any half-space in $\mathbb{R}^n$ with $0 \in \partial H$, then $\partial H$ is regular at 0. That is there exists $r > 0$ such that

$$B(0, r) \cap \partial E \ \text{ is a real analytic hypersurface.} \tag{2.14}$$

Furthermore, $\partial E$ is regular at all points of $\partial_* E$ and

$$H^\alpha\left((\partial E \setminus \partial_* E) \cap U\right) = 0 \textit{ for all } \alpha > n - 8, \tag{2.15}$$

cf. [10, Theorem 11.8]. We let $\operatorname{sing}(\partial E)$ denote the points of $\partial E$ at which $\partial E$ is not regular.

The boundary data $g$ admits a continuous extension $G \in \mathrm{BV}(\mathbb{R}^n \setminus \overline{\Omega}) \cap C^0(\mathbb{R}^n \setminus \Omega)$, [10, Theorem 2.16]. In fact, $G \in C^\infty(\mathbb{R}^n \setminus \overline{\Omega})$, but we only need that $G$ is continuous on the complement of $\Omega$. Clearly, we can require that the support of $G$ is contained in $B(0,R)$ where $R$ is chosen so that $\Omega \subset\subset B(0,R)$. We have

$$G \in \mathrm{BV}(\mathbb{R}^n \setminus \overline{\Omega}) \cap C^0(\mathbb{R}^n \setminus \Omega) \text{ with } G = g \text{ on } \partial\Omega. \tag{2.16}$$

We now introduce sets that will ensure that our constructed solution satisfies the required Dirichlet condition $u = g$ on $\partial\Omega$ and the obstacle condition $u \geqslant \psi$ in $\Omega$. Thus, for each $t \in [a,b]$, let

$$\mathcal{L}_t = (\mathbb{R}^n \setminus \Omega) \cap \{x\colon\ G(x) \geqslant t\}, \tag{2.17}$$

$$L_t = \mathrm{closure}(\{x\colon\ x \in \Omega, \psi(x) > t\}). \tag{2.18}$$

Note that the co-area formula (2.11) and the fact that $G \in \mathrm{BV}(\mathbb{R}^n \setminus \overline{\Omega})$ imply that $P(\mathcal{L}_t, \mathbb{R}^n \setminus \overline{\Omega}) < \infty$ for almost all $t$. For all such $t$, we remind the reader that we employ our convention (2.9) in defining $\mathcal{L}_t$.

We let $[a,b]$ denote the smallest interval containing $g(\partial\Omega) \cup \psi(\overline{\Omega})$ and define

$$T := [a,b] \cap \big\{t\colon\ P(\mathcal{L}_t, \mathbb{R}^n \setminus \overline{\Omega}) < \infty. \tag{2.19}$$

Thus, by (2.7) and the fact that $H^{n-1}(\partial\Omega) < \infty$, we obtain

$$H^{n-1}(\partial_m \mathcal{L}_t) = P(\mathcal{L}_t, \mathbb{R}^n \setminus \overline{\Omega}) + H^{n-1}\left[(\partial_m \mathcal{L}_t) \cap (\partial\Omega)\right] < \infty. \tag{2.20}$$

For each $t \in T$, the variational problems

$$\min \big\{P(E, \mathbb{R}^n)\colon\ E \setminus \overline{\Omega} = \mathcal{L}_t \setminus \overline{\Omega},\ E \supset L_t\big\}, \tag{2.21}$$

$$\max \{|E|\colon\ E \ \text{ is a solution of (2.21)}\} \tag{2.22}$$

will play a central role in our development. In light of Theorem 2.1, a solution to both problems can be obtained from the direct method. (2.20) is also used to obtain existence for (2.21). We will denote by $E_t$ the solution to (2.22). In this regard, note that our convention (2.9) ensures that $E_t \setminus \overline{\Omega} = \mathcal{L}_t \setminus \overline{\Omega}$; furthermore, because of our convention, $\mathcal{L}_t$ need not be a closed set. Also, observe that $E_t$ is super area-minimizing in $\Omega$.

First, we begin with a result which is a direct consequence of a maximum principle for area-minimizing hypersurfaces established independently in [14] and [18].

**3.1 Theorem.** *Let $E_1 \subset E_2$ and suppose both $E_1$ and $E_2$ are area-minimizing in an open set $U \subset \mathbb{R}^n$. Further, suppose $x \in (\partial E_1) \cap (\partial E_2) \cap U$. Then $\partial E_1$ and $\partial E_2$ agree in some neighborhood of $x$.*

**3.2 Lemma.** *For arbitrary measurable sets $A, B \subset \mathbb{R}^n$, it holds that*

$$\begin{aligned} H^{n-1}(\partial_m(A \cup B)) &\leqslant H^{n-1}(\partial_m A \cap (B^i_m)^c) + H^{n-1}(\partial_m B \cap (\overline{A}_m)^c) \\ H^{n-1}(\partial_m(A \cap B)) &\leqslant H^{n-1}(\partial_m A \cap B^i_m) + H^{n-1}(\partial_m B \cap \overline{A}_m). \end{aligned}$$

Proof. It follows immediately from definitions that

$$(\partial_m A \cap (B^i_m)^c) \cup (\partial_m B \cap (A^i_m)^c) = (\partial_m A \cap (B^i_m)^c) \cup (\partial_m B \cap (\overline{A}_m)^c),$$

which yields the first inequality. The result for intersections then follows from $\partial_m A = \partial_m A^c$ and $A \cap B = (A^c \cup B^c)^c$. $\square$

**3.3 Theorem.** *Let $E$ be sub area-minimizing and $F$ super area-minimizing relative to an open set $U$, with $E \subset F$ and $\partial E \cap \partial F \subset\subset U$. Further, suppose that $\overline{E} \cap U = \overline{E^i} \cap U$. Then, relative to $U$, either $\partial E \cap \partial F = \emptyset$ or else $\partial E = \partial F$ in a neighborhood of $\partial E \cap \partial F$.*

Proof. Suppose $\partial E \cap \partial F \neq \emptyset$. The set $\partial E \cap \partial F$ is contained in open neighborhood $V \subset\subset U$ and thus, for sufficiently small $|w|$, $w \in \mathbb{R}^n$, we have

$$(E + w) \setminus F \subset V + w \subset\subset U. \tag{3.1}$$

Choose $x_0 \in \partial E \cap \partial F$. Since $\overline{E} = \overline{E^i}$, there exists $w \in \mathbb{R}^n$ with $|w|$ arbitrarily small such that $x_0 - w \in E^i$, or equivalently

$$x_0 \in (E + w)^i. \tag{3.2}$$

Denote the translated set $E + w$ by $E_w$. By shrinking $U$ if necessary, we can arrange that $E_w$ is sub area-minimizing in $U$.

Now we will show that $F$ is area-minimizing in the open set $U \cap E_w^i$. For, suppose to the contrary that there were a set $G$ with

$$G \Delta F \subset\subset U \cap E_w^i \tag{3.3}$$

and

$$P(G, U \cap E_w^i) < P(F, U \cap E_w^i). \tag{3.4}$$

By (3.3), $G \cap U = F \cap U$ near $\partial E_w$, while, by (3.4),

$$H^{n-1}(\partial_m G \cap E_w^i \cap U) < H^{n-1}(\partial_m F \cap E_w^i \cap U). \tag{3.5}$$

Since $F$ and $G$ agree on $(E_w)_m^i \setminus E_w^i \subset \partial_m E_w$, it follows that

$$H^{n-1}(\partial_m G \cap (E_w)_m^i \cap U) < H^{n-1}(\partial_m F \cap (E_w)_m^i \cap U)). \tag{3.6}$$

On the other hand, super area-minimality of $F$ in $U$ implies that $P(F \cup E_w, U) \geqslant P(F, U)$. With Lemma 3.2, this gives

$$\begin{aligned} H^{n-1}(\partial_m F \cap ((E_w)_m^i)^c \cap U) + H^{n-1}(\partial_m E_w \cap (\overline{F}_m)^c \cap U) & \\ \geqslant H^{n-1}(\partial_m (F \cup E_w) \cap U) & \\ \geqslant H^{n-1}(\partial_m F \cap U) & \\ = H^{n-1}(\partial_m F \cap ((E_w)_m^i)^c \cap U) & \\ + H^{n-1}(\partial_m F \cap (E_w)_m^i \cap U), & \end{aligned}$$

and thus

$$H^{n-1}(\partial_m E_w \cap (\overline{F}_m)^c \cap U) \geqslant H^{n-1}(\partial_m F \cap (E_w)_m^i \cap U). \tag{3.7}$$

Therefore,

$$\begin{aligned} H^{n-1}(\partial_m (G \cap E_w) \cap U) &\leqslant H^{n-1}(\partial_m E_w \cap \overline{G}_m \cap U) + H^{n-1}(\partial_m G \cap (E_w)_m^i \cap U) \\ &= H^{n-1}(\partial_m E_w \cap \overline{F}_m \cap U) + H^{n-1}(\partial_m G \cap (E_w)_m^i \cap U) \\ &< H^{n-1}(\partial_m E_w \cap \overline{F}_m \cap U) + H^{n-1}(\partial_m F \cap (E_w)_m^i \cap U) \\ &\leqslant H^{n-1}(\partial_m E_w \cap \overline{F}_m \cap U) + H^{n-1}(\partial_m E_w \cap (\overline{F}_m)^c \cap U) \\ &= H^{n-1}(\partial_m E_w \cap U), \end{aligned} \tag{3.8}$$

where the first inequality follows by Lemma 3.2, the second by substituting $F$ for $G$ in the vicinity of $\partial_m E_w$, the third by (3.5), the fourth by (3.7), and the last by set decomposition. In other words, $P(G \cap E_w, U) < P(E_w, U)$. But, at the same time,

$$(G \cap E_w) \Delta E_w = E_w \setminus G \subset (E_w \setminus F) \cup (G \Delta F)$$

is compactly supported in $U$, by (3.1) and (3.3), contradicting the sub area-minimality of $E_w$ in $U$. By contradiction, we have that $F$ is area-minimizing in $E_w^i \cap U$, as claimed.

By basic regularity results, we also have that $\overline{F} = \overline{F^i}$ in a neighborhood of $x_0$. By a symmetric argument, it follows that $E$ is area-minimizing near $x_0$ as well, and therefore we can appeal to Theorem 3.1 to obtain our conclusion. □

We do not know whether the hypothesis $\overline{E} \cap U = \overline{E^i} \cap U$ in the previous result is necessary. However, in the case where $E$ is area-minimizing in $U$, the regularity results (2.15) show that the hypothesis is satisfied and this is sufficient for the purposes of this paper. The following result is what we need and it now follows immediately from Theorem 3.3.

**3.4 Corollary.** *Let $E$ be area-minimizing and $F$ super area-minimizing relative to an open set $U$, with $E \subset F$ and $\partial E \cap \partial F \subset\subset U$. Then, relative to $U$, either $\partial E \cap \partial F = \emptyset$ or else $\partial E = \partial F$ in a neighborhood of $\partial E \cap \partial F$.*

## 4. Construction of the solution

In this section we will construct a solution $u$ of (1.3) by using $E_t \cap \overline{\Omega}$ to define the set $\{u \geqslant t\}$ up to a set of measure zero for almost all $t$. This construction will be possible for bounded Lipschitz domains $\Omega$ whose boundaries satisfy the following two conditions.

(i) For every $x \in \partial\Omega$ there exists $\varepsilon_0 > 0$ such that for every set of finite perimeter $A \subset\subset B(x, \varepsilon_0)$

$$P(\Omega, \mathbb{R}^n) \leqslant P(\Omega \cup A, \mathbb{R}^n). \tag{4.1}$$

(ii) For every $x \in \partial\Omega$, and every $\varepsilon \geqslant 0$ there exists a set of finite perimeter $A \subset\subset B(x, \varepsilon)$ such that

$$P\left(\Omega, B(x,\varepsilon)\right) > P\left(\Omega \setminus A, B(x,\varepsilon)\right). \tag{4.2}$$

Clearly, we may assume that $x \in A$.

The first condition states that $\partial\Omega$ has non-negative mean curvature (in the weak sense) while the second states that $\Omega$ is not locally area-minimizing with respect to interior variations. Also, it can be easily verified that if $\partial\Omega$ is smooth, then both conditions together are equivalent to the condition that the mean curvature of $\partial\Omega$ is positive on a dense set of $\partial\Omega$.

An important step in our development is the following lemma.

**4.1 Lemma.** *For almost all $t \in [a,b]$, $\partial E_t \cap \partial\Omega \subset g^{-1}(t)$.*

Proof. First note that if $t > \max\limits_{x\in\partial\Omega} g(x)$, then $\partial E_t \cap \partial\Omega = \emptyset$. So we may assume that $t \in T$ and $t \leqslant \max\limits_{x\in\partial\Omega} g(x)$. The proof will proceed by contradiction and we first show that $\partial E_t$ is locally area minimizing in a neighborhood of each point $x_0 \in (\partial E_t \cap \partial\Omega) \setminus g^{-1}(t)$, i.e., we claim that there exists $\varepsilon > 0$, such that for every set $F$ with the property that $F \Delta E_t \subset\subset B(x_0,\varepsilon)$, we have

$$P\left(E_t, B(x_0,\varepsilon)\right) \leqslant P\left(F, B(x_0,\varepsilon)\right) \tag{4.3}$$

or equivalently, $P(E_t,\mathbb{R}^n) \leqslant P(F,\mathbb{R}^n)$.

By our assumption, either $g(x_0) < t$ or $g(x_0) > t$. First consider the case $g(x_0) < t$. Since $G(x_0) = g(x_0) < t$ and $G$ is continuous on $\mathbb{R}^n \setminus \Omega$, there exists $\varepsilon > 0$, such that $B(x_0,\varepsilon) \cap \mathcal{L}_t = \emptyset$. Also, $\psi$ is continuous on $\overline{\Omega}$ and $\psi(x_0) \leqslant g(x_0) < t$, so we may take $\varepsilon$ small enough such that $L_t \cap B(x_0,\varepsilon) = \emptyset$. We will assume that $\varepsilon < \varepsilon_0$, where $\varepsilon_0$ appears in condition (4.1). We proceed by taking a variation $F$ that satisfies $F \Delta E_t \subset\subset B(x_0,\varepsilon)$. Because of (4.1) and (2.8), for every $A \subset\subset B(x_0,\varepsilon)$, note that

$$\begin{aligned} P(A\cup\Omega,\mathbb{R}^n) + P(A\cap\Omega,\mathbb{R}^n) &\leqslant P(A,\mathbb{R}^n) + P(\Omega,\mathbb{R}^n) \\ &\leqslant P(A,\mathbb{R}^n) + P(A\cup\Omega,\mathbb{R}^n) \end{aligned} \tag{4.4}$$

Hence

$$P(A\cap\Omega,\mathbb{R}^n) \leqslant P(A,\mathbb{R}^n). \tag{4.5}$$

Define $F' = (F \setminus B(x_0,\varepsilon)) \cup (F \cap \overline{\Omega})$, clearly

$$\begin{aligned} F' \setminus \overline{\Omega} &= (F \setminus B(x_0,\varepsilon)) \setminus \overline{\Omega} = (F \setminus \overline{\Omega}) \setminus B(x_0,\varepsilon) \\ &= E_t \setminus \overline{\Omega} \setminus B(x_0,\varepsilon) = \mathcal{L}_t \setminus \overline{\Omega} \setminus B(x_0,\varepsilon) = \mathcal{L}_t \setminus \overline{\Omega} \end{aligned}$$

and $F' \supset L_t$. Thus $F'$ is admissible in (2.21) and therefore

$$P(E_t,\mathbb{R}^n) \leqslant P(F',\mathbb{R}^n).$$

Now we will show that $P(F',\mathbb{R}^n) \leqslant P(F,\mathbb{R}^n)$ which, with the previous inequality, will imply (4.3). First observe from $E_t \Delta F \subset\subset B(x_0,\varepsilon)$ and $(E_t \setminus \overline{\Omega}) \cap B(x_0,\varepsilon) = (\mathcal{L}_t \setminus \overline{\Omega}) \cap B(x_0,\varepsilon) = \emptyset$ that $F' \cap B(x_0,\varepsilon) = F \cap B(x,\varepsilon) \cap \overline{\Omega}$ and $F' \Delta F \subset\subset B(x_0,\varepsilon)$. Hence we obtain by (4.5)

$$\begin{aligned} P(F,\mathbb{R}^n) - P(F',\mathbb{R}^n) &= P\left(F, B(x_0,\varepsilon)\right) - P\left(F', B(x_0,\varepsilon)\right) \\ &= P\left(F \cap B(x_0,\varepsilon), B(x_0,\varepsilon)\right) - P\left(F \cap B(x_0,\varepsilon) \cap \Omega, B(x_0,\varepsilon)\right) \\ &= P\left(F \cap B(x_0,\varepsilon), \mathbb{R}^n\right) - P\left(F \cap B(x_0,\varepsilon) \cap \Omega, \mathbb{R}^n\right) \geqslant 0 \end{aligned} \tag{4.6}$$

This establishes (4.3) when $g(x_0) < t$. Now using the facts that $x_0 \in \partial E_t \cap \partial\Omega$ and that near $x_0$, $\partial\Omega$ is super area-minimizing (by (4.1)), $E_t$ is both contained in $\Omega$ and is area-minimizing, we may employ Corollary 3.4 to conclude that $\partial E_t$ and $\partial\Omega$ agree near $x_0$. This implies that $\partial\Omega$ is area-minimizing near $x_0$, which contradicts (4.2).

The argument to establish (4.3) when $g(x_0) > t$ requires a slightly different treatment from the previous case. Since $G(x_0) = g(x_0) > t$, the continuity of $G$ in $\Omega^c$ implies that $\overline{B}(x_0, \varepsilon) \setminus \Omega \subset \mathcal{L}_t$, provided $\varepsilon$ is sufficiently small. Thus, we have $\overline{B}(x_0, \varepsilon) \setminus \Omega \subset E_t$. Clearly, we may assume $\varepsilon$ chosen to be smaller than $\varepsilon_0$ of (4.1). Observe that the assumption that $\partial\Omega$ is locally Lipschitz implies that $P(\Omega, B(x_0, \varepsilon)) = P(\mathbb{R}^n \setminus \Omega, B(x_0, \varepsilon))$. Consequently, we can appeal to (4.1) to conclude that $\mathbb{R}^n \setminus \Omega$ is sub area-minimizing in $B(x_0, \varepsilon)$. On the other hand, $E_t$ is super area-minimizing. Since $E_t \cap B(x_0, \varepsilon) \setminus \Omega \supset B(x_0, \varepsilon) \setminus \Omega$ we may apply Theorem 3.3 to find that $\partial E_t = \partial(\mathbb{R}^n \setminus \Omega) = \partial\Omega$ in some open neighborhood $U$ of $x_0$. This implies that $L_t \cap U = \emptyset$ since $L_t \subset E_t$ and $\partial E_t \cap U \cap \Omega = \emptyset$. Consequently, $E_t$ must be area-minimizing in $U$, which implies that $\partial\Omega$ is also area-minimizing. As in the previous case, we arrive at a contradiction to (4.2). □

In order to ultimately identify $E_t \cap \overline{\Omega}$ as the set $\{u \geqslant t\}$ (up to a set of measure zero) for almost all $t$, we will need the following result.

**4.2 Lemma.** *If $s, t \in T$ with $s < t$, then $E_t \subset\subset E_s$.*

P r o o f. We first show that $E_t \subset E_s$. Note that

$$(E_s \cap E_t) \setminus \overline{\Omega} = (E_s \setminus \overline{\Omega}) \cap (E_t \setminus \overline{\Omega}) = (\mathcal{L}_s \setminus \overline{\Omega}) \cap (\mathcal{L}_t \setminus \overline{\Omega}) = \mathcal{L}_t \setminus \overline{\Omega}$$

and

$$L_t \subset E_t,\ L_t \subset E_s \Longrightarrow L_t \subset E_s \cap E_t.$$

Thus, $E_s \cap E_t$ is a competitor with $E_t$.

Similarly,

$$(E_s \cup E_t) \setminus \overline{\Omega} = (E_s \setminus \overline{\Omega}) \cup (E_t \setminus \overline{\Omega}) = (\mathcal{L}_s \setminus \overline{\Omega}) \cup (\mathcal{L}_t \setminus \overline{\Omega}) = \mathcal{L}_s \setminus \overline{\Omega}$$

and

$$L_t \subset E_t,\ L_s \subset E_s \Rightarrow L_s \subset E_s \cup E_t.$$

So $E_s \cup E_t$ is a competitor with $E_s$. Then employing (2.8), we have

$$P(E_s, \mathbb{R}^n) + P(E_t, \mathbb{R}^n) \leqslant P(E_s \cup E_t, \mathbb{R}^n) + P(E_s \cap E_t, \mathbb{R}^n) \leqslant P(E_s, \mathbb{R}^n) + P(E_t, \mathbb{R}^n),$$

and thus, since $E_t$ and $E_s$ are minimizers,

$$P(E_s \cup E_t, \mathbb{R}^n) = P(E_s, \mathbb{R}^n)$$

and

$$P(E_s \cap E_t, \mathbb{R}^n) = P(E_t, \mathbb{R}^n).$$

Reference to (2.23) yields $|E_s \cup E_t| = |E_s|$, which in turn implies $|E_s \setminus E_t| = 0$. In view of our convention (2.9),

$$x \in E \quad \text{if and only if} \quad \limsup_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} > 0,$$

we conclude that $E_t \subset E_s$.

Now we come to the crucial part of the argument which is to show that this containment is in fact strict. For this purpose, first note that

$$E_t \setminus \overline{\Omega} = \mathcal{L}_t \setminus \overline{\Omega} \subset\subset \mathcal{L}_s \setminus \overline{\Omega} = E_s \setminus \overline{\Omega}. \tag{4.7}$$

Now observe that Lemma 4.1 implies

$$\partial E_t \cap \partial E_s \cap \partial\Omega = \emptyset. \tag{4.8}$$

In review of (4.7) and (4.8), it remains to show that

$$\partial E_t \cap \partial E_s \cap \Omega = \emptyset \tag{4.9}$$

in order to establish the lemma. For this purpose, let $S \equiv \partial E_s \cap \partial E_t \cap \Omega$. Then for $x_0 \in S$, there are three possible cases with case (ii) being the central issue of this paper.

(i) For any $\varepsilon > 0$, $L_s \cap L_t \cap \Omega \cap B(x_0, \varepsilon)$ is non-empty.

(ii) $x_0 \in \overline{L_s}$ and $B(x_0, \varepsilon) \cap L_t = \emptyset$ for some $\varepsilon > 0$.

(iii) $B(x_0, \varepsilon) \cap L_t = \emptyset = B(x_0, \varepsilon) \cap L_s$ for some $\varepsilon > 0$, thus implying that both $\partial E_s$ and $E_t$ are area-minimizing in $B(x_0, \varepsilon)$.

Next, we will prove that the 3 cases above are impossible, i.e. $S = \emptyset$, which implies that $E_t \subset\subset E_s$.

For case (i), we can choose some sequence $\{y_n\} \subset L_s \cap L_t$, such that $\lim_{n\to\infty} y_n = x_0$. Since $\psi$ is continuous, we have $\lim_{n\to\infty} \psi(y_n) = \psi(x_0) \geqslant t$. Since $t > s$, there exists an $\varepsilon > 0$, such that $B(x_0, \varepsilon) \subset E_s$ which contradicts the fact that $x \in \partial E_s$.

For case (ii), first observe that $E_s$ is super area-minimizing and that $E_t$ is area-minimizing near $x_0$. Since $E_t \subset E_s$, if follows from the maximum principle that $\partial E_s$ and $\partial E_t$ agree in a neighborhood of $x_0$.

For case (iii), since $E_s$ and $E_t$ are area minimizing in $B(x_0, \varepsilon)$ and $E_t \subset E_s$, we apply the maximum principle again to conclude that $\partial E_t$ and $\partial E_s$ agree in a neighborhood of $x_0$.

Now combining above (i), (ii) and (iii), we conclude that for each $x \in S$, there exists $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subset \Omega$ and $S$ is area minimizing in $B(x, \varepsilon_x)$. Also, we see that $S$ consists only of components of $\partial E_s$ that do not intersect $\partial\Omega$.

We proceed to conclude the proof by showing that $S = \emptyset$. Intuitively, the reason for this is that any component of $S$ must be a cycle (and therefore a bounding cycle). But locally area-minimizing bounding cycles do not exist. The rigorous justification of this is essentially contained in the proof of [17, Corollary 37.8], which we include for the reader's convenience.

Let $S'$ be a component of the set of regular points of $S$. Our first step is to show that $S'$ is a cycle in the sense of currents; that if, we wish to show that

$$\int_{S'} \mathrm{d}\varphi = 0 \tag{4.10}$$

whenever $\varphi$ is a smooth $(n-2)$-form on $\mathbb{R}^n$ with compact support. For each $x \in S$, we use the area-minimizing property of $S$ in $B(x, \varepsilon_x)$ and the monotonicity formula, cf. [10, Remark 5.13] to conclude that $H^{n-1}(B(x,r) \cap S')r^{-(n-1)}$ is a nondecreasing function of $r$ on $(0, \varepsilon_x)$ where $S'$ is the component of $S$ containing $x$. Thus, it follows that only a finite number of components of $S$ can intersect any given compact subset of $B(x, \varepsilon_x)$, in particular, $\operatorname{spt}\varphi \cap \overline{B}(x, \varepsilon_x/2)$. Thus, there exists a concentric ball $B \subset \overline{B}(x, \varepsilon_x/2)$ such that for any smooth function $\zeta$ with $\operatorname{spt}\zeta \subset B$, we have

$$\int_{S' \cap \overline{B}} \mathrm{d}(\zeta\varphi) = \int_{\partial E_s} \mathrm{d}(\zeta\varphi) = 0.$$

As this holds for each $x \in S'$, using a partition of unity, we conclude that

$$\int_{S'} \mathrm{d}\varphi = 0$$

as required.

This shows that $S'$ is an $(n-1)$-rectifiable cycle in the sense of currents; that is, $\partial S' = 0$. From the theory of integral currents, it follows that $S'$ is a bounding cycle. That is, we can appeal to [6, Thm. 4.4.2] or [17, 27.6] to conclude that there is a set $F \subset \mathbb{R}^n$ of finite perimeter such that $\partial F = S'$. It follows from elementary considerations that there is a vector $\nu \in \mathbb{R}^n$ and a corresponding hyperplane, $P$, with normal $\nu$ such that $P \cap \overline{S'} \neq \emptyset$ and

$$F \subset \{x\colon\ (x - x_0) \cdot \nu \leqslant 0\}$$

where $x_0 \in P \cap \overline{S'}$. Corollary 3.4 implies $P \cap \overline{S'}$ is open as well as closed in $P$, thus leading to a contradiction since $S \cap \partial\Omega = \emptyset$. $\square$

Now we are in a position to construct the solution $u$ to problem (1.3). For this purpose, we first define for $t \in T$,

$$A_t = \overline{E_t \cap \Omega}.$$

With the help of Lemma 4.2, observe that for $t \in T$,

$$\{g > t\} \subset (E_t)^i \cap \partial\Omega \subset A_t \cap \partial\Omega \tag{4.11}$$

$$\overline{\{g > t\}} \subset A_t \cap \partial\Omega \subset \overline{E_t} \cap \partial\Omega = [(E_t)^i \cup \partial E_t] \cap \partial\Omega \subset \{g \geqslant t\}. \tag{4.12}$$

Finally, note that (4.12) and Lemma 4.1 imply

$$A_t \subset\subset A_s \tag{4.13}$$

relative to the topology on $\overline{\Omega}$ whenever $s, t \in T$ with $s < t$. We now define our solution $u$ by

$$u(x) = \sup\{t\colon\ x \in A_t\}. \tag{4.14}$$

**4.3 Theorem.** *The function $u$ defined by* (4.14) *satisfies the following:*
(i) *$u = g$ on $\partial\Omega$*
(ii) *$u$ is continuous on $\overline{\Omega}$,*
(iii) *$A_t \subset \{u \geqslant t\}$ for all $t \in T$ and $|\{u \geqslant t\} - A_t| = 0$ for almost all $t \in T$.*
(iv) *$u \geqslant \psi$ on $\overline{\Omega}$.*

Proof. To show that $u = g$ on $\partial\Omega$, let $x_0 \in \partial\Omega$ and suppose $g(x_0) = t$. If $s < t$, then $G(x) > s$ for all $x \in \Omega^c$ near $x_0$. Hence, $x_0 \in (E_s)^i \cap \partial\Omega$ by (4.11) and consequently, $x_0 \in A_s$ for all $s \in T$ such that $s < t$. By (4.14), this implies $u(x) \geqslant t$. To show that $u(x) = t$ suppose by contradiction that $u(x) = \tau > t$. Select $r \in (t, \tau) \cap T$. Then $x \in A_r$. But $A_r \cap \partial\Omega \subset \{g \geqslant r\}$ by (4.12), a contradiction since $g(x) = t < r$.

For the proof of (ii), it is easy to verify that

$$\{u \geqslant t\} = \left\{\bigcap A_s\colon\ s \in T, s < t\right\} \text{ and } \{u > t\} = \left\{\bigcup A_s\colon\ s \in T, s > t\right\}.$$

The first set is obviously closed while the second is open relative to $\overline{\Omega}$ by (4.13). Hence, $u$ is continuous on $\overline{\Omega}$.

For (iii), it is clear that $\{u \geqslant t\} \supset A_t$. Now, $\{u \geqslant t\} - A_t \subset u^{-1}(t)$. But $|u^{-1}(t)| = 0$ for almost all $t$ because $|\Omega| < \infty$.

In (iv), it is sufficient to show $u(x_0) \geqslant \psi(x_0)$ for $x_0 \in \Omega$. Let $t = u(x_0)$ and $r = \psi(x_0)$ and suppose $t < r$. Then $x_0 \in L_{r'} \subset E_{r'}$ for $t < r' < r$. But then, $x_0 \notin A_{r'}$ by the definition of $u$, a contradiction. □

**4.4 Theorem.** *If $\Omega$ is a bounded Lipschitz domain that satisfies* (4.1) *and* (4.2), *then the function $u$ defined by* (4.14) *is a solution to* (1.3).

Proof. Let $v \in \mathrm{BV}(\Omega)$, $v = g$ on $\partial\Omega$ be a competitor in problem (1.3). We recall the extension $G \in \mathrm{BV}(\mathbb{R}^n - \overline{\Omega})$ of $g$, (2.17). Now define an extension $\overline{v} \in \mathrm{BV}(\mathbb{R}^n)$ of $v$ by $\overline{v} = G$ in $\mathbb{R}^n - \overline{\Omega}$. Let $F_t = \{\overline{v} \geqslant t\}$. It is sufficient to show that

$$P(E_t, \Omega) \leqslant P(F_t, \Omega) \tag{4.15}$$

for almost every $t \in T$ (see (2.19)), because then $v \in \mathrm{BV}(\Omega)$ and (2.11) would imply

$$\int_a^b P(E_t, \Omega)\,\mathrm{d}t \leqslant \int_{-\infty}^{\infty} P(F_t, \Omega)\,\mathrm{d}t = \|\nabla v\|(\Omega) < \infty.$$

Hence, by (2.12), $u \in \mathrm{BV}(\Omega)$; furthermore, $\|\nabla u\|(\Omega) \leqslant \|\nabla v\|(\Omega)$ by (2.11) .

We know that $E_t$ is a solution of

$$\min\left\{P(E, \mathbb{R}^n)\colon\ E \setminus \overline{\Omega} = \mathcal{L}_t \setminus \overline{\Omega},\ E \supset L_t\right\}, \tag{4.16}$$

while $F_t - \overline{\Omega} = \mathcal{L}_t - \overline{\Omega}$ and $F_t \supset L_t$. Hence,

$$P(E_t, \mathbb{R}^n) \leqslant P(F_t, \mathbb{R}^n). \tag{4.17}$$

Next, note that

$$\begin{aligned} P(E_t, \mathbb{R}^n) &= H^{n-1}(\partial_* E_t - \overline{\Omega}) + H^{n-1}(\partial_* E_t \cap \partial\Omega) + H^{n-1}(\partial_* E_t \cap \Omega) \\ &\geqslant H^{n-1}(\partial_* \mathcal{L}_t - \overline{\Omega}) + P(E_t, \Omega). \end{aligned} \tag{4.18}$$

We will now show that

$$\begin{aligned} P(F_t, \mathbb{R}^n) &= H^{n-1}(\partial_* \mathcal{L}_t - \overline{\Omega}) + H^{n-1}(\partial_* F_t \cap \Omega) \\ &= H^{n-1}(\partial_* \mathcal{L}_t - \overline{\Omega}) + P(F_t, \Omega), \end{aligned} \tag{4.19}$$

which will establish (4.15) in light of (4.17) and (4.18).

Observe

$$P(F_t, \mathbb{R}^n) = H^{n-1}(\partial_* \mathcal{L}_t - \overline{\Omega}) + H^{n-1}(\partial_* F_t \cap \partial\Omega) + H^{n-1}(\partial_* F_t \cap \Omega).$$

We claim that $H^{n-1}(\partial_* F_t \cap \partial\Omega) = 0$ for almost all $t$ because $\partial_* F_t \subset \partial F_t \subset \overline{v}^{-1}(t)$ since $\overline{v} \in C^0(\mathbb{R}^n)$. But $H^{n-1}(\overline{v}^{-1}(t) \cap \partial\Omega) = 0$ for all but countably many $t$ since $H^{n-1}(\partial\Omega) < \infty$. $\square$

**5.1 Lemma.** *Suppose $\Omega$ is a bounded, open subset of $\mathbb{R}^n$ whose boundary is $C^2$ with mean curvature bounded below by $a > 0$. Assume $g \in C^{0,\alpha}(\partial\Omega)$, and $\psi \in C^{0,\alpha/2}(\Omega)$ for some $0 < \alpha \leqslant 1$. Let $u \in C^0(\overline{\Omega}) \cap \mathrm{BV}(\Omega)$ be a solution to (1.3). Then, there exist positive numbers $\delta$ and $C$ depending only on $a$, $\|g\|_{C^{0,\alpha}(\partial\Omega)}$, $\|g\|_{C^0(\partial\Omega)}$, $\|\psi\|_{C^{0,\alpha/2}(\Omega)}$ and $\|u\|_{C^0(\Omega)}$ such that*

$$|u(x) - u(x_0)| \leqslant C|x - x_0|^{\alpha/2}.$$

*whenever $x_0 \in \partial\Omega$ and $x \in \overline{\Omega}$ with $|x - x_0| < \delta$.*

Proof. For each $x_0 \in \partial\Omega$ we will construct functions $\omega^+, \omega^- \in C^0(\overline{U})$ where $U = U(x_0, \delta) := B(x_0, \delta) \cap \Omega$ and $\delta > 0$ is sufficiently small, such that

(i) $\omega^+(x_0) = \omega^-(x_0) = g(x_0)$,

(ii) for $x \in U(x_0, \delta)$

$$|\omega^+(x) - g(x_0)| \leqslant C|x - x_0|^{\alpha/2}$$
$$|\omega^-(x) - g(x_0)| \leqslant C|x - x_0|^{\alpha/2},$$

(iii) $\omega^- \leqslant u \leqslant \omega^+$ in $U(x_0, \delta)$.

We begin with the construction of $\omega^-$. To this end, let $d(x) = \operatorname{dist}(x, \partial\Omega)$. Since $\partial\Omega \in C^2$ recall that $d \in C^2(\{x\colon\ 0 \leqslant d(x) < \delta_0\})$ for some $\delta_0 > 0$, cf. [9, Lemma 14.16]. Furthermore, since $\partial\Omega$ has positive mean curvature and $|\nabla d| = 1$, it follows that

$$\operatorname{div}\left(\frac{\nabla d}{|\nabla d|}\right) = \Delta d \leqslant -a, \tag{5.1}$$

in $\{x\colon\ 0 \leqslant d(x) < \delta_0\}$ for some $a > 0$. For each $\varepsilon > 0$, set

$$v(x) = |x - x_0|^2 + \lambda d(x)$$
$$\omega^-(x) = \max\{\psi, -Kv^{\alpha/2}(x) + g(x_0)\},$$

where $\lambda > 0$ and $K$ are to be determined later. Clearly (i) is satisfied.

Next, in the open set $\{\omega^- > \psi\}$, observe that

$$\begin{aligned} |\nabla\omega^-| &= K\tfrac{\alpha}{2}v^{\frac{\alpha}{2}-1}|\nabla v|, \\ |\nabla v| &= |2(x - x_0) + \lambda\nabla d| \geqslant \lambda|\nabla d| - 2|x - x_0| \\ &= \lambda - 2|x - x_0| > 0, \end{aligned}$$

provided we choose $\delta$ and $\lambda$ such that $\lambda > 2\delta$. Further, we note that

$$\operatorname{div}\left(\frac{\nabla\omega^-}{|\nabla\omega^-|}\right) = -\operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) = \frac{-1}{|\nabla v|^3}Av,$$

where $Av = |\nabla v|^2\Delta v - D_ivD_jvD_{ij}v$. Finally, observe that $Av < 0$ for $\lambda$ sufficiently large and $\delta$ sufficiently small. Indeed, using $D_idD_{ij}d = 0$ for any $j$, one readily obtains

$$Av = |\nabla v|^2(\lambda\Delta d + 2(n-1)) - 4\lambda(x-x_0)_i(x-x_0)_jD_{ij}d$$

and

$$|\nabla v|^2 = \lambda^2 + 4|x-x_0|^2 + 4\lambda(x-x_0)\cdot\nabla d$$

so that

$$Av \leqslant -a\lambda^3 + O(\lambda^2), \text{ as } \lambda\to\infty$$

uniformly for $\delta < \delta_0$.

Clearly, we can choose $K$ sufficiently large so that $\omega^- = \psi$ on $\partial U(x_0,\delta)$ and that (ii) is satisfied, where $K$ depends only on $\|g\|_{C^{0,\alpha}(\partial\Omega)}$, $\|g\|_{C^0(\partial\Omega)}$, $\|\psi\|_{C^{0,\alpha/2}(\Omega)}$ and $\|u\|_{C^0(\Omega)}$. Also, on $\Delta := \{\omega^- > u\}\cap U(x_0,\delta)$, we have $\omega^- = -Kv^{\alpha/2} + g(x_0)$ and therefore

$$|\nabla\omega^-| > 0 \quad\text{and}\quad \operatorname{div}\left(\frac{\nabla\omega^-}{|\nabla\omega^-|}\right) > 0 \text{ on } \Delta. \tag{5.2}$$

We now proceed to show that $\Delta = \emptyset$, which will establish the first of the inequalities in (iii). For this purpose, note that $\omega^- \in \mathrm{BV}(\Delta)$. Next, for $t > 0$, let $\Delta_t := \{\omega^- - t > u\}$ and note that

$$\Delta = \cup_{t>0}\Delta_t \qquad \Delta_t \subset\subset \Delta \subset \Omega. \tag{5.3}$$

Let $\omega^* := \max(u, \omega^- - t)$ and note that $\omega^* \in \mathrm{BV}(\Omega)\cap C^0(\overline{\Omega})$ since $\omega^- - t = \psi - t < u$ on $\partial\Delta_t$. For all but countably many $t > 0$, it follows from basic measure theory that

$$\|\nabla\omega^*\|(\partial\Delta_t) = 0 = \|\nabla u\|(\partial\Delta_t). \tag{5.4}$$

For the remainder of this argument, we will consider only such $t$. Since $\omega^* \geqslant u \geqslant \psi$, it follows that

$$\|\nabla u\|(\Omega) \leqslant \|\nabla\omega^*\|(\Omega). \tag{5.5}$$

Now let $\eta \in C_0^\infty(\Delta)$ satisfy $\eta = 1$ on $\Delta_t$ and $0 \leqslant \eta \leqslant 1$ in $\Delta$. Set

$$h = \eta\frac{\nabla\omega^-}{|\nabla\omega^-|}$$

so that $h \in [C_0^1(\Delta)]^n$. Since $\omega^* = u$ on $\Delta - \Delta_t$, it follows from

$$\int_\Delta u \operatorname{div} h \, dx = -\nabla u(h),$$

and

$$\int_\Delta \omega^* \operatorname{div} h \, dx = -\nabla \omega^*(h),$$

that

$$\int_\Delta u - \omega^* \, dx = \int_{\Delta_t} (u - \omega^- + t) \operatorname{div} h \, dx = [\nabla(\omega^* - u)](h).$$

It follows from (5.4) and the definition of the BV norm that

$$\|\nabla \omega^*\| (\partial \Delta_t) \leqslant \|\nabla u\| (\partial \Delta_t) + \int_{\partial \Delta_t} |\nabla \omega^*| \, dx = 0,$$

so that

$$\begin{aligned} \int_{\Delta_t} (u - \omega^* + t) \operatorname{div} h \, dx &= \nabla \omega^*(h\chi_{\Delta_t}) - \nabla u(h\chi_{\Delta_t}) \\ &\geqslant \int_{\Delta_t} |\nabla \omega^*| \, dx - \|\nabla u\| (\Delta_t). \end{aligned}$$

Since $u - \omega^- + t < 0$ and $\operatorname{div} h > 0$ on $\Delta_t$, we have

$$\int_{\Delta_t} |\nabla \omega^*| \, dx < \|\nabla u\| (\Delta_t).$$

That is,

$$\|\nabla \omega^*\| (\Delta_t) < \|\nabla u\| (\Delta_t).$$

Since $\omega^* = u$ on $\mathbb{R}^n \setminus \Delta_t$, we obtain from (5.4) that $\|\nabla \omega^*\| (\Omega) < \|\nabla u\| (\Omega)$, which contradicts (5.5). Thus we conclude that $\omega^- \leqslant u$ on $U(x_0, \delta)$.

The proof of the second inequality in (iii) is obtained by a similar argument using $\omega^+(x) := K v^{\alpha/2}(x) + g(x_0)$. □

**5.2 Theorem.** *Suppose $\Omega$ is a bounded, open subset of $\mathbb{R}^n$ with $C^2$ boundary having mean curvature bounded below by $a > 0$. Suppose $g \in C^{0,\alpha}(\partial\Omega)$, and $\psi \in C^{0,\alpha/2}$ for some $0 < \alpha \leqslant 1$. If $u \in C^0(\overline{\Omega}) \cap \mathrm{BV}(\Omega)$ is a solution to* (1.3), *then $u \in C^{0,\alpha/2}(\overline{\Omega})$.*

Proof. For $s < t$, consider the superlevel sets $E_s, E_t$ of $u$ and assume that $\operatorname{dist}(\partial E_s, \partial E_t) = |y - x|$ where $x \in E_t$ and $y \in E_s$. Assume $t - s$ is small enough to ensure that $|y-x| < \delta$, where $\delta$ is given by Lemma 5.1. Observe that $L_t \subset E_t \subset\subset E_s$. Theorems 4.3 and 4.4 imply that $u$ is continuous on $\overline{\Omega}$ and therefore bounded. Hence it is sufficient to show that $|u(y)-u(x)| = |t-s| \leqslant C|x-y|^{\alpha/2}$ whenever $|y-x| < \delta$. This will be accomplished by examining the following five cases.

(i) If either $x$ or $y$ belongs to $\partial\Omega$, then our result follows from Lemma 5.1.

(ii) $y \in \partial E_s \setminus L_s$, $x \in \partial E_t \cap L_t$: Since $L_s \supset L_t$, there exists $y' \in \partial L_s$ such that $y' - x = c(y-x)$, $0 < c < 1$, and therefore

$$|u(y) - u(x)| = |t-s| = |\psi(y') - \psi(x)| \leqslant C|y'-x|^{\alpha/2} \leqslant C|y-x|^{\alpha/2}. \tag{5.6}$$

(iii) $y \in \partial E_s \cap L_s$, $x \in \partial E_t \cap L_t$: This is treated as in the previous case.

(iv) $y \in \partial E_s \cap L_s$, $x \in \partial E_t \setminus L_t$: Let $[\partial E_t]_v$ denote the translation of $\partial E_t$ by the vector $v := y - x$. Since $E_s$ is super area-minimizing and $E_t$ is area-minimizing in some neighborhoods of $y$ and $x$ respectively, we can apply Corollary 3.4 to conclude that $\partial E_s$ and $[\partial E_t]_v$ agree on some open set containing their intersection. Let $S$ be the connected component of the intersection that contains $y$. If $z$ is a limit point of $S$, then $z \in S$ since $S$ is closed and $z = y' - x'$ where $y' \in \partial E_s$ and $x' \in \partial E_t$. There are several possibilities to consider. First, if $y'$ and $x'$ can be treated by the first three cases, then nothing more is required as our desired conclusion is reached. If not, then either $y' \in \partial E_s \cap L_s$, $x' \in \partial E_t \setminus L_t$, or $y' \in \partial E_s \setminus L_s$, $x' \in \partial E_t \setminus L_t$. In the first of these last two possibilities, observe that $E_s$ is super area-minimizing and $[\partial E_t]_v$ is area-minimizing while in the second of the last two possibilities, both $\partial E_s$ and $[\partial E_t]_v$ are area-minimizing. Hence, in each of these two possibilities, we may apply Corollary 3.4 to conclude that $\partial E_s$ and $[\partial E_t]_v$ agree on some open set containing their intersection. Thus, if $y'$ and $x'$ cannot be treated by the first three cases, it follows that $S$ is area-minimizing in some open set containing $S$. With $S'$ denoting a component of the regular set of $S$, we are precisely in the situation encountered in the proof of (4.10) in Lemma 4.2, which leads to a contradiction.

(v) $y \in \partial E_s \setminus L_s$, $x \in \partial E_t \setminus L_t$: This is treated as in the previous case. □

## 6. A monotonicity principle for superminimizing sets

An issue left open in our development is whether the regularity requirement $\overline{E} \cap U = \overline{E^i} \cap U$ is necessary in Theorem 3.3, the extended maximum principle for sub and superminimizing sets.

This suggests the question, of interest in its own right, of what regularity, if any, is enjoyed by (sub)superminimizing sets. For example, do (sub)superminimizers have tangent cones? Are they $C^1$ or analytic $H^{n-1}$ almost-everywhere? And, the question begged by Theorem 3.3, is a subminimizer necessarily the closure of its interior? In the next section, we will give an explicit example showing that the last conjecture is false. In this section, we present some preliminary results in the direction of regularity, consisting of a new *monotonicity principle* and consequent *one-sided mass bound* for (sub)superminimizing sets.

Let $B_r = B(0,r)$ denote the ball of radius $r$ about the origin in $\mathbb{R}^n$. Let $F$ be a superminimizing set in $U$, and without loss of generality, assume $B_1 \subset U$.

**6.1 Lemma.** *Let $\widetilde{A} = \{x \in A^c\colon$ the metric density of $A$ is one at $x\}$. Then, $H^{n-1}(\partial B_r \cap \widetilde{A}) = 0$ for almost all $r$.*

P r o o f. The Lebesgue measure of $\widetilde{A} \cap B_1$ is zero. But, by the co-area formula, (2.11), it is also equal to $\int_0^1 H^{n-1}(\partial B_r \cap \widetilde{A})\,\mathrm{d}r$, whence the result follows. □

**6.2 Lemma.** *Let $E$ area subminimizing in $U$, $B_1 \subset U$, and $r$ such that $H^{n-1}(\partial B_r \cap \widetilde{E}) = 0$. Then, $P(E, B_r) \leqslant H^{n-1}(E \cap \partial B_r)$.*

P r o o f. The set $G = E \backslash B_r$ is a competitor to $E$. Exterior to $B_r$, $G$ has the same reduced boundary as does $E$, but interior to $B_r$, it has no reduced boundary. On $\partial B_r$, $G$ has reduced boundary contained in the set of points at which $E$ has density one, which by assumption is contained in $E$ except for a set of $H^{n-1}$-measure zero.

Therefore, by the subminimality of $E$, we have

$$0 \leqslant P(E \setminus B_r, U) - P(E, U) \leqslant H^{n-1}(E \cap \partial B_r) - P(E, B_r),$$

giving the result. □

Define the dimension-dependent constant $0 < \delta(n) < 1/2$ by

$$\delta(n) = |D_1|/|B_1|,$$

where $D_1 \subset B_1$ is a set bounded by a hemispherical cap of radius one, orthogonal to $\partial B_1$.

**6.3 Lemma.** *If $|A \cap B_r|/|B_r| \leqslant \delta(n)$, then $P(A, B_r)/H^{n-1}(\partial B_r) \geqslant |A \cap B_r|/|B_r|$.*

R e m a r k. Another way of stating this result is that $P(A, B_r) \geqslant \frac{n}{r}|A \cap B_r|$. It could also be rephrased as an isoperimetric inequality.

P r o o f. By rearrangement, we find that the set $D$ of minimum perimeter $P(D, B_r)$ subject to $|D \cap B_r| = |A \cap B_r|$ is the set bounded by a hemispherical cap meeting $\partial B$ orthogonally. Trivially, we have

$$P(D, B) \leqslant P(A, B). \tag{6.1}$$

Let $D_r$ be the set bounded by a spherical cap of radius $r$, intersecting $\partial B_r$ orthogonally, so that $|D \cap B_r|/|B_r| = \delta(n)$. Since $|D \cap B_r|/|B_r| = |A \cap B_r|/|B_r| \leqslant \delta(n)$, we thus have that $|D| \leqslant |D_r|$ and so the radius of the hemispherical cap bounding $D$ is less than or equal to $r$. It follows by elementary geometry that

$$H^{n-1}(\partial D \cap \partial B) \leqslant P(D, B). \tag{6.2}$$

(To see this, e.g., one can reflect the hemispherical cap $D$ about the plane of its intersection with $B_r$, to obtain a surface oriented in the same direction as the patch $\overline{D} \cap \partial B_r$ and containing the patch in its interior. Since the patch has positive mean curvature, it follows that this outer surface has greater area than does $\overline{D} \cap \partial B_r$.)

But, $D$ is entirely contained in the cone $C$ from $\partial D \cap \partial B_r$ to the center of $B_r$ and tangent to $D$ at $\partial B_r$. That is, $|A \cap B_r| \leqslant |C|$. On the other hand, the volume ratio $|C|/|B_r|$ for a cone is exactly its surface ratio, $H^{n-1}(\partial D \cap \partial B_r)/H^{n-1}(\partial B_r)$. Combining these facts with (6.2) and (6.1), we have

$$\begin{aligned} |A|/|B_r| &\leqslant |C|/|B_r| = H^{n-1}(\partial D \cap \partial B_r)/H^{n-1}(\partial B_r) \\ &\leqslant P(D, B_r)/H^{n-1}(\partial B_r) \leqslant P(A, B_r)/H^{n-1}(\partial B_r), \end{aligned}$$

which leads to our desired conclusion. □

We now prove our main result, a *volume* monotonicity principle for superminimizing sets.

**6.4 Proposition.** *Let $E$ be subminimizing in $U$, $B_1 \subset U$. If $|E \cap B_1|/|B_1| < \delta(n)$ ($0 < \delta(n) < 1/2$ as defined above Lemma 6.3), then the ratio $|E \cap B_r|/|B_r|$ is increasing in $r$ for $0 \leqslant r \leqslant 1$.*

Proof. From Lemmas 6.2 and 6.3, we have

$$H^{n-1}(E \cap \partial B_r)/H^{n-1}(\partial B_r) \geqslant P(E, B_r)/H^{n-1}(\partial B_r) \geqslant |E \cap B_r|/|B_r)|$$

for almost all $r$, so long as $|E \cap B_r|/|B_r| < \delta(n)$.

By the co-area formula, (2.11),

$$(\mathrm{d}/\mathrm{d}r)|B_r| = H^{n-1}(\partial B) \text{ and } (\mathrm{d}/\mathrm{d}r)|E \cap B_r| = H^{n-1}(E \cap \partial B_r).$$

Thus,

$$\mathrm{d}|E \cap B_r|/\mathrm{d}|B_r| = H^{n-1}(E \cap \partial B_r)/H^{n-1}(\partial B) \geqslant |E \cap B_r|/|B_r|,$$

giving monotonicity so long as $|E \cap B_r|/|B_r| < \delta(n)$. But, because of monotonicity, this property persists for all $0 \leqslant r \leqslant 1$. □

This property has many implications. Among them is the following important one, a one-sided bound on the average density.

**6.5 Proposition.** *Let $E$ be subminimizing in $U$, $B_1 \subset U$. If $0 \in \partial E$, then $|E \cap B_1|/|B_1| \geqslant \delta(n)$.*

Proof. Suppose to the contrary that $|E \cap B_1|/|B_1| < \delta(n)$. Then, for some $R < 1$, $|E \cap B(x,R)| < \delta(n)$ for every $x \in B_{1-R}$. By the monotonicity property of Proposition 6.4, we thus have $|E \cap B(x,r)|/|B(x,r)| < \delta(n)$ for $r \leqslant R$. Thus,

$$|E \cap \widetilde{B}|/|\widetilde{B}| < \delta(n) < \tfrac{1}{2}$$

for any ball contained in $B_{1-R}$; hence the density of $E$ is strictly less than $1/2$ at each point of $B_{1-R}$.

But, since the density of $E$ must be zero or one at almost every point of $B_{1-R}$, the density of $E$ must be zero at almost every point in $B_{1-R}$, and therefore $|E \cap B_{1-R}| = 0$. But, by our convention in choosing set representatives, this would imply that $B_{1-R} \subset E^e$, in particular $0 \in E^e$, a contradiction. □

**6.6 Corollary.** *If $E$ is subminimizing, then $\overline{E^i_m} = (E^e_m)^c = \overline{E}$.*

Proof. By Proposition 6.5, the density of $E$ at any $x \in \partial E$ is strictly greater than 0, hence $\partial E \cap E^e_m = \emptyset$. It follows that $\partial E$, and therefore $\overline{E}$ as well, is contained in $(E^e_m)^c \subset \overline{E^i_m}$. Since $\overline{E^i_m}$ is always contained in $\overline{E}$, we thus obtain

$$\overline{E^i_m} = (E^e_m)^c = \overline{E},$$

as claimed. □

**6.7 Corollary.** *Let $E$ be minimizing in $U$ and $x \in \partial E$. Then, in any ball $B(x,r) \subset U$, the relative volume fractions of $E$ and $E^c$ are bounded below by $\delta(n) > 0$.*

Proof. By the previous Proposition applied to $E$ and $E^c$, we find that violation of this bound would imply that $x$ were in the interior of $E$ or of $E^c$. But, $x \in \partial E$ by assumption, a contradiction. □

**6.8 Corollary.** *Let $E$ be minimizing in $U$ and $x \in \partial E$. Then, in any ball $B(x,r) \subset U$, $P(E, B_r) \geqslant \delta r^{n-1}$, where $\delta > 0$ is an independent constant.*

Proof. This follows from Corollary 6.7 plus the explicit form of the minimizer of $P(A, B_r)$ among sets with $|A| = |E|$. □

R e m a r k . Propositions 6.4 and 6.5 give an alternative, and more elementary route to regularity of minimizing sets than the usual path via the Isoperimetric Theorem for minimal surfaces, cf., [10, Chapter 8]. Using Corollary 6.8, one can go on to show existence of tangent cones, etc. This standard result is usually proved by reference to the Isoperimetric Theorem for minimal surfaces, cf. [10, Chapter 5].

## 7. "Foamy" sets

We conclude by demonstrating existence of sparse, "foamy" superminimizing sets having topological boundary with positive Lebesgue measure, thus indicating possible limitations of a regularity theory for (sub)superminimizing sets.

For $\overline{B}(x_1,r), \overline{B}(x_0,R) \subset U \subset \mathbb{R}^2$, $\overline{B}(x_1,r) \cap \overline{B}(x_0,R) = \emptyset$, consider the obstacle problem

$$\inf\{P(F,U)\colon\ B(x_1,r) \cup B(x_0,R) \subset F \subset\subset U\}. \tag{7.1}$$

**7.1 Lemma.** *For $r$ sufficiently small, the solution of* (7.1) *is*

$$E = B(x_1,r) \cup B(x_0,R).$$

*Moreover, for any connected set $\widetilde{F}$ containing $B(x_1,r) \cup B(x_0,R)$, there holds*

$$P(\widetilde{F},U) > P(E,U) + \delta, \tag{7.2}$$

*for some $\delta > 0$.*

P r o o f . Without loss of generality, take $U$ to be all of $\mathbb{R}^2$. Since we are in two dimensions, minimal surfaces for (7.1) are easily characterized as arcs of $\partial B(x_0,R)$, $\partial B(x_1,r)$ joined by straight lines. By explicit comparison, it is then found that the connected competitor $\widetilde{F}$ with least perimeter is the convex hull of $\partial B(x_0,R)$, $\partial B(x_1,r)$, which for $r$ sufficiently small satisfies (7.2). Among disconnected competitors, the best is $E = B(x_1,r) \cup B(x_0,R)$, by (2.13). □

**7.2 Proposition.** *For any open $V \subset\subset U \subset \mathbb{R}^2$, and any $\varepsilon > 0$, there exists a superminimizing set $F$ in $U$ such that $\overline{F} = \overline{V}$ and $|F| \leqslant \pi\varepsilon^2$.*

P r o o f . Enumerate the rationals as $\{x_j\}$. □

**Claim.** *For suitably chosen $r_j$,*

$$F_J := \bigcup_{j \leqslant J} B(x_j, r_j)$$

*has the properties:*

(i) *Any set $F_J \subset G \subset\subset U$ with a connected component containing two $B(x_j, r_j)$ with $j \leqslant J$, satisfies*

$$P(G,U) > P(F_J, U) + \delta_J, \quad \delta_J > 0,$$

(ii)

$$\sum_{j=J+1}^{\infty} P(B(x_j, r_j) < \delta_J. \tag{7.3}$$

P r o o f  o f  c l a i m. The radii $r_j$ may be chosen inductively, as follows:

Choose $r_1 < \varepsilon/2$ sufficiently small that $B(x_1, r_1) \subset V$. If $x_{j+1} \in \overline{F_j}$, then take $r_{j+1} = 0$. Otherwise, choose $r_{j+1}$ so small that $B(x_{j+1}, r_{j+1}) \subset V \setminus F_j$,

$$P((B(x_{j+1}, r_{j+1}), U) < \tfrac{\delta_j}{2}, \tag{7.4}$$

and, by Lemma 7.1, any connected set $G$ containing $B(x, r_{j+1})$ and any $B(x_k, r_k)$, $k \leqslant j$ satisfies

$$P(G,U) > P(B(x_{j+1}, r_{j+1}), U) + P(B(x_k, r_k), U) + \delta_{j+1} \tag{7.5}$$

for some $\delta_{j+1} > 0$. By (7.4), (ii) is clearly satisfied. Further, (7.4) and (7.5) together give (i). For, if $G$ has a component containing any $B(x_k, r_k)$, $B(x_l, r_l)$, $k \neq l \leqslant j$, then (7.3) holds by the induction hypothesis. Likewise, if no component of $G$ contains $B(x_{j+1}, r_{j+1})$ and any $B(x_k, r_k)$, $k \leqslant j$. The remaining case is that precisely one $B(x_k, r_k)$, $k \leqslant j$, lies in a component with $B(x_{j+1}, r_{j+1})$, and the rest lie each in distinct components. In this case, (7.3) follows by (7.5) and (2.13).

Defining $F := \bigcup_j B(x_j, r_j)$, we find that $F$ is superminimizing in $U$. For, let $G$ be any competitor. If $G$ has any component containing $B(x_j, r_j)$ and $B(x_k, r_k)$, $j < k$, then (i)–(ii) together give

$$P(G,U) > P(F_k, U) + \delta_k > P(F_k, U) + \sum_{k+1}^{\infty} P(B(x_j, r_j), U) \geqslant P(F,U).$$

On the other hand, if each $B(x_j, r_j)$ lies in a distinct component $G_j$ of $G$, then either $G_j \equiv B(x_j, r_j)$, or, by the Isoperimetric Theorem, $P(G_j, U) \geqslant P(B(x_j, r_j), U)$, with strict inequality for some $J$. Noting that $P(G,U) \geqslant \sum_{j=1}^{k} P(G_j,U)$ for any finite sum, and recalling (ii), we thus obtain $P(G,U) > P(F,U)$ as claimed.

By (ii), and the choice $r_1 < \varepsilon$, we have $|F| \leqslant \pi\varepsilon^2 \sum_{j=1}^{\infty} (1/2)^{2j} < \pi\varepsilon^2$. But, clearly, also, $F$ is dense in $V$, giving $\overline{F} = \overline{V}$ as claimed. □

Remark. It is not clear whether such a construction can be carried out in higher dimensions, since Lemma 7.1 no longer holds with positive $\delta$.

Consequences. 1. The construction of Proposition 7.2 shows that in general $\overline{E} = \overline{E^i}$ is false for subminimizing sets $E$, in contrast to the result of Corollary 6.6. It would seem that some form of connectivity must be assumed on $E$, if this property is to hold.

2. A similar construction with $U = B(0,1)$ yields a superminimizing set of the form $B^+(0,1) \cup G$, where $G$ is the union of a disjoint family of discs dense in $U \setminus B^+(0,1)$. Here $B^+(0,1)$ denotes the upper half-ball $\{x \in B(0,1)\colon\ x_n \geqslant 0\}$. Taking $E = B^+(0,1)$ and $F = E \cup G$, we find that the strong maximum principle as stated in Theorem 3.3 is violated, although the regularity assumption $\overline{E} = \overline{E_i}$ is satisfied. This shows that the assumption $\partial E \cap \partial F \subset\subset U$ is important. However, we remark that in the original form as stated in [18], the conclusion of the theorem was that $\partial E$ and $\partial F$ should agree on their components of $x_0$. This version of the theorem remains valid also for the above example, though the two statements are equivalent for minimizing sets.

Evidently, the issue of a maximum principle for sub- and superminimizing sets is a delicate one, requiring ideas beyond those in this paper. This would appear to be an interesting area for further study.

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