ALMOST PERIODIC SOLUTIONS WITH A PRESCRIBED SPECTRUM OF SYSTEMS OF LINEAR AND QUASILINEAR DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS AND CONSTANT TIME LAG (CAUCHY INTEGRAL)

Alexandr Fischer, Praha

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Abstract. This paper generalizes earlier author's results where the linear and quasilinear equations with constant coefficients were treated. Here the method of limit passages and a fixed-point theorem is used for the linear and quasilinear equations with almost periodic coefficients.

Keywords: almost periodic function, Fourier coefficient, Fourier exponent, spectrum of almost periodic function, almost periodic system of differential equations, formal almost periodic solution, almost periodic solution, distance of two spectra, time lag

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1. Introduction

1.1. Preliminaries. One of the methods the author developed in his research works is presented here. This method has been inspired by S. N. Šimanov's paper [9] and is based on the use of Cauchy integrals. Another method, not presented here, is based on the Fourier transform.

1.2. Notation and definitions. We denote: $\mathbb{N}$—the set of all positive integers, $\mathbb{N}_0$—the set of all non-negative integers, $\mathbb{R}$—the set of all real numbers (real axis), $\mathbb{C}$—the set of all complex numbers (complex plane).

If $\mathbb{E}$ is a non-void set and m, n are positive integers then $\mathbb{E}^m$ denotes the Cartesian product $\mathbb{E}\times\mathbb{E}\ldots\times\mathbb{E}$ of m factors and $\mathbb{E}^{m\times n}$ is the set of all matrices of m rows and n columns, the elements of which belong to $\mathbb{E}$; $\mathbb{E}^{1\times 1}=\mathbb{E}^1=\mathbb{E}$. Analogously we could denote more-dimensional matrices.

If $n \in \mathbb{N}$ and $\overline{m} = (m_1, \ldots, m_n) \in \mathbf{N}_0^n$, $\overline{m}' = (m_1', \ldots, m_n') \in \mathbf{N}_0^n$ then the inequality $\overline{m} \leqslant \overline{m}'$ means the system of inequalities $m_j \leqslant m_j'$, $j = 1, 2, \ldots, n$.

If $\mathcal{M}$, $\mathcal{N}$ are non-void subsets of $\mathbb{C}$ or $\mathbb{R}$ and if ω, ξ are complex numbers then $\omega M = \{\omega\lambda\colon \lambda \in \mathcal{M}\}$, $\xi + \mathcal{N} = \{\xi + \mu\colon \mu \in \mathcal{N}\}$, $\mathcal{M} + \mathcal{N} = \{\lambda + \mu\colon \lambda \in \mathcal{M}, \mu \in \mathcal{N}\}$, $\emptyset + \mathcal{N} = \mathcal{M} + \emptyset = \emptyset + \emptyset = \emptyset$ and $S(\mathcal{M})$ stands for the smallest additive semigroup containing $\mathcal{M}$ and $S(\emptyset) = \emptyset$.

The distance of two sets $\mathcal{M}, \mathcal{N}$, of a point z and a set $\mathcal{M}$ and of two points z, w in $\mathbb{C}$ or $\mathbb{R}$, respectively, is denoted by $\operatorname{dist}[\mathcal{M}, \mathcal{N}]$, $\operatorname{dist}[z, \mathcal{N}]$ and $\operatorname{dist}[z, w]$.

The boundary of a set $\mathcal{M}$ is denoted by $\partial\mathcal{M}$.

If α is a positive number then by a strip or an α-strip in the complex plane we mean the set $\pi(\alpha) = \{z \in \mathbb{C}\colon |\operatorname{Re} z| \leqslant \alpha\}$.

If $z_0 \in \mathbb{C}$ and $R \in (0, \infty)$ then $\kappa(z_0, R)$, $\overline{\kappa}(z_0, R)$ and $K(z_0, R)$, respectively, denote an open disc, a closed disc and a circle centred at z_0 with its radius R in the complex plane.

For number vectors or matrices, even more-dimensional, we use the norm $|.|$, which is equal to the sum of absolute values of all coordinates of the vector or all elements of the matrix.

In addition to the usual symbol $\prod\limits_{j=1}^{k} a_j = a_1 a_2 \ldots a_k$ for a product we will use the symbol $\prod\limits_{j=k}^{1} a_j = a_k \ldots a_1$ for the product with the reversed order of factors.

For a vector $\overline{m} = (m_1, \ldots, m_M) \in \mathbb{N}_0^M$, $M \in \mathbb{N}$, we introduce the combinatory number

$$\binom{|\overline{m}|}{\overline{m}} = \frac{|\overline{m}|!}{(m_1!) \ldots (m_M!)}, \quad \text{where } |\overline{m}| = m_1 + \ldots + m_M.$$

1.3. Spaces. We will deal with functions $f\colon \mathbb{R} \to \mathbb{X}$, where $\mathbb{X}$ is one of the spaces $\mathbb{E}$, $\mathbb{E}^m$, $\mathbb{E}^{m \times n}$ and $\mathbb{E} = \mathbb{R}$ or $\mathbb{E} = \mathbb{C}$.

We denote by $C(\mathbb{X})$, $CB(\mathbb{X})$ and $AP(\mathbb{X})$, respectively, the space of all continuous functions $f\colon \mathbb{R} \to \mathbb{X}$, the space of all functions from $C(\mathbb{X})$ bounded on $\mathbb{R}$ and the space of all almost periodic functions from $CB(\mathbb{X})$. The mean value of a function $f \in AP(\mathbb{X})$ is denoted by $M(f)$ or $M_t\{f(t)\}$.

The spaces $CB(\mathbb{X})$ and $AP(\mathbb{X})$ are made Banach spaces (B-spaces) with the norm defined by $|f| = \sup\{|f(t)|\colon t \in \mathbb{R}\}$. For $k = 1$ and $k = 2$ we will denote by $C^k(\mathbb{X})$, $CB^k(\mathbb{X})$ and $AP^k(\mathbb{X})$ the space of all functions from $C(\mathbb{X})$ with continuous derivatives up to the order k on $\mathbb{R}$, the space of all function from $C^k(\mathbb{X})$ which are bounded on $\mathbb{R}$ and have bounded derivatives up to the order k, and the space of all functions from $CB^k(\mathbb{X})$ which are almost periodic and have almost periodic derivatives up to the order k.

The spaces $CB^k(\mathbb{X})$ and $AP^k(\mathbb{X})$ endowed with the norm

$$\begin{aligned} \|f\| &= \max\{|f|, |\dot{f}|\} \quad \text{if } k = 1, \\ \|\|f\|\| &= \max\{|f|, |\dot{f}|, |\ddot{f}|\} \quad \text{if } k = 2, \end{aligned}$$

become B-spaces.

If all elements of a matrix almost periodic function $f \in AP(\mathbb{X})$ are trigonometric polynomials then f is called a trigonometric polynomial.

Remark 1.1. The space $AP(\mathbb{X})$ is the closure of the set of all trigonometric polynomials from $CB(\mathbb{X})$. Analogously $AP^k(\mathbb{X})$ and $CB^k(\mathbb{X}), k = 1, 2$.

1.4. Almost periodic functions. Any almost periodic function from $AP(\mathbb{X})$ has a representation by a Fourier (trigonometric) series which is uniquely determined up to the order of summation. By Λ_f we denote the set of all Fourier exponents of an almost periodic function f and the set $\mathrm{i}\Lambda_f$ will be called the spectrum of f.

If f is an almost periodic function with the Fourier series $\sum\limits_{\lambda} \varphi(\lambda) \exp{(\mathrm{i}\lambda t)}$, $\lambda \in \Lambda_f$, then we denote $\sum(f) = \sum\limits_{\lambda} |\varphi(\lambda)|$, $\lambda \in \Lambda_f$. If the Fourier series of a function f converges absolutely then $\sum(f) < \infty$.

For any function from $AP(\mathbb{X})$ there exists a sequence of the so-called Bochner-Fejér approximation (trigonometric) polynomials B_m, $m = 1, 2, \ldots$ of the function f with their spectra contained in $\mathrm{i}\Lambda_f$ and uniformly convergent to f on $\mathbb{R}$ and moreover $\sum(B_m) \leqslant \sum(f)$, $m = 1, 2, \ldots$, (see [1], [5], [7], [8]).

1.5. Equations with constant coefficients. The basic problem the author dealt with in his paper [6], is to solve the differential equations

$$\dot{x}(t) = a_0 x(t) + b_0 x(t - \tau) + f(t), \tag{1.1}$$

where τ is a positive constant, the so-called time lag, a_0, b_0 belong to $\mathbb{C}^{n\times n}$, where $n \in \mathbb{N}$, $f \in AP^1(\mathbb{C}^{n\times 1})$ and x is an unknown function from $C^1(\mathbb{C}^{n\times 1})$. An important role is played by the properties of the matrix function $\Phi(z) = zE - a_0 - b_0 \exp{(-z\tau)}, z \in \mathbb{C}$, where $E = E_n$ is the unit matrix from $\mathbb{C}^{n\times n}$, and by the properties of its determinant $\Delta(z) = \det\Phi(z)$. This determinant is called the characteristic quasipolynomial and the equation $\Delta(z) = 0$ is called the characteristic equation of the differential equation (1.1).

Under $\sigma(\Delta(z))$ we understand the set of all roots of the characteristic quasipolynomial $\Delta(z)$. The quasipolynomial is a transcendent entire function (in general) of complex variable z and, consequently, the quasipolynomial $\Delta(z)$ has an infinite number of roots without any finite limit point. Each strip $\pi(\alpha), \alpha > 0$, contains only a finite number of roots of the characteristic quasipolynomial $\Delta(z)$ because $\Phi(z)z^{-1}$

is arbitrarily close to the unit matrix E in the strip $\pi(\alpha)$ for z sufficiently large (in absolute value). Hence the matrix $\Phi(z)$ is a regular one for such z. Consequently, the positive number α can be chosen so that the finite set $\pi(2\alpha)\cap\sigma(\Delta(z))$ lies on the imaginary axis of the complex plane. If $\pi(2\alpha)\cap\sigma(\Delta(z))\neq\emptyset$ and this set contains just the points $\mathrm{i}\xi_1,\dots,\mathrm{i}\xi_{j_0}$, $j_0\in\mathbb{N}$, then we set $\theta=\{\xi_j-\xi_k\colon\ j,k=1,\dots,j_0\}$, and if $\pi(2\alpha)\cap\sigma(\Delta(z))=\emptyset$, then we set $\theta=\emptyset$.

1.6. Favard's theorem. In the sequel we will need

Theorem 1.1. (Favard) *If a function $f\in AP(\mathbb{C}^{m\times n}), m,n\in\mathbb{N}$, and if $\Lambda_f\cap(-d,d)=\emptyset$ where d is a positive number, then the primitive function $F(t)=\int_0^t f(s)\,\mathrm{d}s,\ t\in\mathbb{R}$, is an almost periodic function, too, and the estimate*

$$|F-M(F)|\leqslant M_d|f| \tag{1.2}$$

is valid. Here

$$M(F)=\lim_{T\to\infty}\frac{1}{T}\int_0^T F(s)\,\mathrm{d}s$$

is the mean value of the almost periodic function F and M_d is a positive constant depending on d only.

The proof of Favard's theorem was published in [1], [2], [5], [7], [8].

2. Equations with almost periodic coefficients

2.1. Basic equations. In the sequel we study the differential equations

$$\dot{x}(t)=a_0x(t)+b_0x(t-\tau)+a(t)x(t)+b(t)x(t-\tau)+f(t) \tag{2.1}$$

where τ is a positive constant, $a_0,b_0\in\mathbb{C}^{n\times n}$, $a,b\in AP^1(\mathbb{C}^{n\times n})$, for which $\sum(a)<\infty,\sum(b)<\infty$ and $f\in AP^1(\mathbb{C}^{n\times 1})$, $n\in\mathbb{N}$. Our aim is to prove the existence and uniqueness of an almost periodic solution of Equation (2.1) the spectrum of which is contained in a certain apriori given set $\mathrm{i}\Lambda,\Lambda\subset\mathbb{R}$. Such a solution is called an almost periodic Λ-solution.

2.2. Formal solutions. First, we solve the given equation in a formal manner. This means that we are looking for the so-called formal solution x_f represented by a trigonometric series with coefficients from $\mathbb{C}^{n\times 1}$ which formally satisfies Equation (2.1).

For trigonometric series we introduce the so-called formal arithmetic, differential and integral operations, the formal shift and the formal mean value. The formality of these operations consists in the fact that they are performed without any regard to the convergence of the trigonometric series and without any justification (as concerns the convergence) of the operations performed. Hence, not only the resulting trigonometric series but also the series entering into the formal operations need not be convergent. The coefficients of the trigonometric series entering into particular formal operations are supposed to be elements of normed linear spaces such that the operations in question can be accomplished.

Given a trigonometric series

$$x(t) \sim \sum_{\nu} c(\nu) \exp(\mathrm{i}\nu t), \quad \nu \in \Lambda, \tag{2.2}$$

where Λ is an at most countable set of real numbers, then $\mathrm{i}\Lambda$ is called the spectrum of the trigonometric series (2.2). Further, we denote

$$\sum(x) = \sum_{\nu} |c(\nu)|, \quad \nu \in \Lambda,$$

so that the inequality $\sum(x) < \infty$ denotes the absolute convergence of the trigonometric series x.

In the case $\Lambda = \emptyset$ the associated trigonometric series is equal to zero. If we are given the trigonometric series (2.2) and $\Lambda \subset \widetilde{\Lambda} \subset \mathbf{R}$, where Λ is an at most countable set, then for x we use also the representation

$$x(t) \sim \sum_{c} c(\nu) \exp(\mathrm{i}t\nu), \quad \nu \in \widetilde{\Lambda},$$

in which $c(\nu) = 0$ for $\nu \in \widetilde{\Lambda} \setminus \Lambda$.

Let two trigonometric series

$$a(t) \sim \sum \alpha(\lambda) \exp(\mathrm{i}\lambda t),\ \lambda \in \Lambda_1, \quad b(t) \sim \sum \beta(\mu) \exp(\mathrm{i}\mu t),\ \mu \in \Lambda_2,$$

be given where the sets $\Lambda_j \subset \mathbb{R}$, $j = 1, 2$, are at most countable. If α, β are two complex numbers and $s \in \mathbb{R}$ then we define formal operations

i) the formal linear combination (formal sum, difference and scalar multiple)

$$\alpha a(t) + \beta b(t) \sim \sum [\alpha.\alpha(\nu) + \beta.\beta(\nu)] \exp(\mathrm{i}\nu t), \quad \nu \in \Lambda_1 \cup \Lambda_2,$$

where $\alpha(\nu) = 0$ for $\nu \notin \Lambda_1$, $\beta(\nu) = 0$ for $\nu \notin \Lambda_2$;

ii) the formal product

$$a(t)b(t) \sim \sum_{\nu} \Bigg[\sum_{\lambda+\eta=\nu} \alpha(\lambda)\beta(\eta) \Bigg] \exp(\mathrm{i}\nu t), \quad \nu \in \Lambda_1 + \Lambda_2, \quad \lambda \in \Lambda_1, \quad \eta \in \Lambda_2;$$

iii) the formal derivative (term-by-term differentiation)

$$\dot{a}(t) \sim \sum \mathrm{i}\lambda\alpha(\lambda)\exp(\mathrm{i}\lambda t), \quad \lambda \in \Lambda_1;$$

iv) the formal primitive trigonometric series (term-by-term integration)

$$A(t) = \int a(t)\,\mathrm{d}t \sim A_0 + \sum \frac{1}{\mathrm{i}\lambda}\alpha(\lambda)\exp(\mathrm{i}\lambda t), \quad \lambda \in \Lambda_1, \quad A_0 \in \mathbb{C}^{n\times 1},$$

under the assumption that $0 \notin \Lambda_1$;

v) the formal shift (for a given real number s)

$$a(t+s) \sim \sum [\alpha(\lambda)\exp(\mathrm{i}\lambda s)]\exp(\mathrm{i}\lambda t), \quad \lambda \in \Lambda_1;$$

vi) the formal mean value of a trigonometric series defined to be its absolute term.

R e m a r k 2.1. Let us note that under the assumption of the appropriate convergence of the trigonometric series entering into the formal operations these formal operations coincide with the non-formal ones.

In connection with the formal operations we speak about a formal almost periodic solution (Λ-solution) of the almost periodic differential Equation (2.1). The trigonometric series (2.2) is called a formal almost periodic Λ-solution of Equation (2.1) if this trigonometric series solves Equation (2.1) formally, i.e. after inserting the trigonometric series representing a, b, f, x, $\dot{x}$ into Equation (2.1) and after having formally performed the indicated operations the right and left sides of the equation give rise to trigonometric series the spectra of which are contained in $\mathrm{i}\Lambda$ and for every $\nu \in \Lambda$ the coefficients at $\exp(\mathrm{i}\nu t)$ on both sides are equal. Clearly, every almost periodic solution (Λ-solution) of Equation (2.1) is also its formal almost periodic solution (Λ-solution). The contrary is not true.

2.3. Construction of a formal solution. We begin with the case when a, b and f are trigonometric polynomials.

Theorem 2.1. *If in Equation* (2.1) a, b *and* f *are trigonometric polynomials and if (see at the end* 1.5. *concerning the set* θ*)*

$$\Delta = \inf(\Lambda_a \cup \Lambda_b) > 0, \tag{2.3}$$

$$d_\theta = \begin{cases} \operatorname{dist}[\theta, S(\Lambda_a \cup \Lambda_b)] > 0 & \textit{for } \theta \neq \emptyset, \\ 2 & \textit{for } \theta = \emptyset, \end{cases} \tag{2.4}$$

$$d = \operatorname{dist}[\mathrm{i}\Lambda, \sigma(\Delta(z))] > 0, \tag{2.5}$$

where $\Lambda = \Lambda_f + S(\Lambda_a \cup \Lambda_b \cup \{0\})$, *then there exists a unique formal almost periodic* Λ*-solution* x_f *of Equation* (2.1).

P r o o f. Let $M \in \mathbb{N}$, $N \in \mathbb{N}$ and let

$$a(t) = \sum_{k=1}^{M} \alpha(\mu_k) \exp(\mathrm{i}\mu_k t),$$
$$b(t) = \sum_{k=1}^{N} \beta(\nu_k) \exp(\mathrm{i}\nu_k t),$$
$$f(t) = \sum \varphi(\lambda) \exp(\mathrm{i}\lambda t), \ \lambda \in \Lambda_f.$$

Further, let the sought formal solution x_f have the representation

$$x_f(t) \sim \sum c(\sigma) \exp(\mathrm{i}\sigma t), \quad \sigma \in \Lambda,$$

so that its formal derivative $\dot{x}_f$ has the representation

$$\dot{x}_f(t) \sim \sum \mathrm{i}\sigma c(\sigma) \exp(\mathrm{i}\sigma t), \quad \sigma \in \Lambda.$$

Substituting formally into Equation (2.1) and equating the corresponding coefficients of the exponential functions $\exp(\mathrm{i}\sigma t)$ we get for the coefficients $c(\sigma)$ a system of infinitely many linear algebraic equations

$$\begin{aligned}\Phi(\mathrm{i}\sigma)c(\sigma) = &\sum_{\mu} \alpha(\mu)c(\sigma-\mu) + \sum_{\nu} \beta(\nu)c(\sigma-\nu)\exp(-\mathrm{i}(\sigma-\nu)\tau) \\ &+ \sum_{\lambda} \delta_{\lambda\sigma}\varphi(\lambda),\end{aligned} \tag{2.6}$$

where $\mu \in \Lambda_a$, $\nu \in \Lambda_b$, $\sigma \in \Lambda$, $\sigma - \mu \in \Lambda$, $\sigma - \nu \in \Lambda$, $\lambda \in \Lambda_f$, where $\delta_{\lambda\sigma} = 0$ for $\lambda \neq \sigma$ and $\delta_{\lambda\sigma} = 1$ for $\lambda = \sigma$. By assumptions (2.3), (2.4), (2.5) the matrix $\Phi(z)$ is regular for $z \in \mathrm{i}\Lambda$ so that for every $\sigma \in \Lambda$ we obtain from (2.6) a unique expression

$$\begin{aligned}c(\sigma) = \Phi^{-1}(\mathrm{i}\sigma)\Big[&\sum_{\mu} \alpha(\mu)c(\sigma-\mu) + \sum_{\nu} \beta(\nu)c(\sigma-\nu)\exp(-\mathrm{i}(\sigma-\nu)\tau) \\ &+ \sum_{\lambda} \delta_{\lambda\sigma}\varphi(\lambda)\Big].\end{aligned} \tag{2.7}$$

Thus, the uniqueness of the formal almost periodic Λ-solution x_f is ensured, provided it exists. To prove its existence we complete the solution of the system (2.7). Every $\sigma \in \Lambda$ can be expressed in the form $\sigma = \lambda + \bar{s}\bar{\omega} = \lambda + \bar{m}\bar{\mu} + \bar{n}\bar{\nu}$, where $\lambda \in \Lambda_f$,

$$\bar{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_M \end{pmatrix}, \quad \bar{\nu} = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_N \end{pmatrix}, \quad \bar{\omega} = \begin{pmatrix} \bar{\mu} \\ \bar{\nu} \end{pmatrix},$$

$$\overline{m} = (m_1, \dots, m_M) \in \mathbb{N}_0^{1\times M}, \quad \bar{n} = (n_1, \dots, n_N) \in \mathbb{N}_0^{1\times N}, \quad \bar{s} = (\overline{m}, \bar{n}).$$

Such an expression for $\sigma \in \Lambda$ need not be unique, but owing to the fact that the sets Λ_a, Λ_b, Λ_f are finite the number of such expressions is also only finite. This makes it possible to solve completely the system (2.7). In the system we shall distinguish the coefficients $c(\lambda + \bar{s}\bar{\omega})$ and $c(\lambda' + \bar{s}'\bar{\omega})$, where $\lambda, \lambda' \in \Lambda_f$ and $\bar{s}, \bar{s}' \in \mathbb{N}_0^{1\times(M+N)}$, and also the equations for them if $\lambda \neq \lambda'$ or $\bar{s} \neq \bar{s}'$ even if $\lambda + \bar{s}\bar{\omega} = \lambda' + \bar{s}'\omega$. We say that $\sigma' = \lambda' + \bar{s}'\bar{\omega}$ is lower than $\sigma = \lambda + \bar{s}\bar{\omega}$ if $\lambda' = \lambda$ and $\bar{s}' \leqslant \bar{s}$, $\bar{s}' \neq \bar{s}$.

For every $\lambda \in \Lambda_f$ we formally solve Equation (2.1)—for simplicity and lucidity—for a "harmonic" $\varphi(\lambda)\exp(\mathrm{i}\lambda t)$ separately, i.e. for $f(t) = \varphi(\lambda)\exp(\mathrm{i}\lambda t)$, and by x_λ we denote the corresponding formal almost periodic Λ-solution. Their formal sum for $\lambda \in \Lambda_f$ gives then a formal almost periodic Λ-solution x_f.

Hence, $\lambda \in \Lambda_f$ being fixed we consider the subsystem of the system (2.7) with $\sigma \in \lambda + S(\Lambda_a \cup \Lambda_b \cup \{0\}) \subset \Lambda$. Let $\bar{s} \in \mathbb{N}_0^{1\times(M+N)}$ be fixed; substituting successively from the equations for coefficients $c(\sigma')$ where σ' is lower than $\sigma = \lambda + \bar{s}\bar{\omega}$ into the equation for $c(\lambda + \bar{s}\bar{\omega})$ we obtain such an equation for $c(\lambda + \bar{s}\bar{\omega})$ which contains only $c(\lambda)$ from all coefficients $c(\sigma')$ where σ' is lower than $\sigma = \lambda + \bar{s}\bar{\omega}$. The number of all possible different "descents" from $\lambda + \bar{s}\bar{\omega}$ to λ is

$$\binom{|\bar{s}|}{\bar{s}} = \frac{|\bar{s}|!}{(m_1!)\dots(m_M!)(n_1!)\dots(n_N!)}.$$

Every such "descent" is accomplished by a successive substitution and is uniquely defined by an increasing sequence $P = P(\bar{s})$ of vectors from $\mathbb{N}_0^{1\times(M+N)}$

$$\bar{0} = \overline{P}_0 \leqslant \overline{P}_1 \leqslant \dots \leqslant \overline{P}_{|\bar{s}|} = \bar{s},$$

which satisfies $|\overline{P}_j - \overline{P}_{j-1}| = 1$, $j = 1, \dots, |\bar{s}|$, while $\overline{P}_j = (\overline{Q}_j, \overline{R}_j)$, $\overline{Q}_j \in \mathbb{N}_0^{1\times M}$, $\overline{R}_j \in \mathbb{N}_0^{1\times N}$, $j = 0, 1, \dots, |\bar{s}|$. To every such sequence $P = P(\bar{s})$ for a fixed λ we can associate in a unique manner a sequence $p = p(\bar{s})$ of vectors $\bar{p}_0, \bar{p}_1, \dots, \bar{p}_{|\bar{s}|}$ from $\mathbb{N}_0^{1\times(M+N)}$ satisfying $\bar{p}_0 = \bar{0}$, $|\bar{p}_j| = 1$, $j = 1, \dots, |\bar{s}|$, and $\overline{P}_k = \sum_{j=0}^{k} \bar{p}_j$, $k = 0, 1, \dots, |\bar{s}|$, while $\bar{p}_j = (\bar{q}_j, \bar{r}_j)$, $\bar{q}_j \in \mathbb{N}_0^{1\times M}$, $\bar{r}_j \in \mathbb{N}_0^{1\times N}$, $j = 0, 1, \dots, |\bar{s}|$. This means that $\bar{p}_j = \overline{P}_j - \overline{P}_{j-1}$, $\bar{q}_j = \overline{Q}_j - \overline{Q}_{j-1}$, $\bar{r}_j = \overline{R}_j - \overline{R}_{j-1}$, $j = 1, \dots, |\bar{s}|$. Let us denote by $c_P(\lambda + \bar{s}\bar{\omega})$ the part of the right-hand side of Equation (2.7) for $c(\lambda + \bar{s}\bar{\omega})$ obtained by the successive substitution using the sequence $P = P(\bar{s})$. This procedure yields $c_P(\lambda + \bar{s}\bar{\omega}) = \Phi_P(\mathrm{i}\lambda)\varphi(\lambda)$, where

$$\Phi_P(z) = \prod_{j=|\bar{s}|}^{0} \Phi^{-1}(z + \mathrm{i}\overline{P}_j\bar{\omega})\gamma(\bar{p}_j\bar{\omega}),$$

$\gamma(0)=1$, $\gamma(\bar p_j\bar\omega)=\alpha(\bar q_j\bar\mu)+\beta(\bar r_j\bar\nu)\exp(-\mathrm{i}\overline{P}_{j-1}\bar\omega\tau)$, $j=1,\dots,|\bar s|$, $\alpha(0)=0$, $\beta(0)=0$. We obtain then by a formal sum

$$c(\lambda+\bar s\bar\omega)=\sum_P c_P(\lambda+\bar s\bar\omega)=\sum_P\Phi_P(\mathrm{i}\lambda)\varphi(\lambda)$$

where the summation is over all mutually different sequences $P=P(\bar s)$ with a fixed $\lambda\in\Lambda_f$ and a fixed $\bar s\in\mathbb{N}_0^{1\times(M+N)}$.

Thus, for every "harmonic" $\varphi(\lambda)\exp(\mathrm{i}\lambda t)$, $\lambda\in\Lambda_f$, we get a formal almost periodic Λ-solution

$$x_\lambda(t)\sim\sum_{\bar s\geqslant\bar 0}\sum_P\Phi_P(\mathrm{i}\lambda)\varphi(\lambda)\exp(\mathrm{i}(\lambda+\bar s\bar\omega)t), \tag{2.8}$$

and the formal sum of these solutions yields a formal almost periodic Λ-solution of Equation (2.1)

$$x_f(t)=\sum_\lambda x_\lambda\sim\sum_\lambda\sum_{\bar s\geqslant\bar 0}\sum_P\Phi_P(\mathrm{i}\lambda)\varphi(\lambda)\exp(\mathrm{i}(\lambda+\bar s\bar\omega)t),\ \lambda\in\Lambda_f.$$

The proof of Theorem 2.1. is complete. □

3. Almost periodic solutions

3.1. Closed regions G_k, G_P. In the sequel we will take up the case $\theta\neq\emptyset$ but the case $\theta=\emptyset$ when $\Delta(z)$ has no purely imaginary roots would be even easier. Hence, let $\mathrm{i}\xi_1,\dots,\mathrm{i}\xi_{j_0}, j_0\in\mathbb{N}$, be all mutually different purely imaginary roots in $\mathbb{C}$ of the quasipolynomial $\Delta(z)$ and let $\varrho_1,\dots,\varrho_{j_0}$ be their multiplicities. We pick a positive constant $\delta=\frac12\min\{\alpha,\Delta,d_\theta,d,d_\xi,\tau,2\}$, where $d_\xi=\min\{|\xi_j-\xi_k|\colon\ j\neq k;\ j,k=1,\dots,j_0\}$ for $j_0>1$ and $d_\xi=2$ for $j_0=1$ or $\theta=\emptyset$ and where a positive number α is chosen so that $\pi(2\alpha)\cap\sigma(\Delta(z))=\{\mathrm{i}\xi_1,\dots,\mathrm{i}\xi_{j_0}\}$ for $\theta\neq\emptyset$ or $\pi(2\alpha)\cap\sigma(\Delta(z))=\emptyset$ for $\theta=\emptyset$.

Further, unless stated otherwise, we assume that we are given a fixed vector $\bar s$ and a fixed sequence of vectors $P=P(\bar s)$. Recall that $\kappa(z,\delta)$ and $\bar\kappa(z,\delta)$ are the open disc and the closed disc centred at z with their radius δ in the complex plane $\mathbb{C}$. In $\mathbb{C}$ we construct closed regions

$$G_k=\pi(\alpha)\setminus\bigcup_{j=1}^{j_0}\kappa(\mathrm{i}\xi_j-\mathrm{i}\overline{P}_k\bar\omega;\delta),\quad k=0,1,\dots,|\bar s|,$$

and we denote by G_P their intersection, so that

$$G_P = \bigcap_{k=0}^{|\bar{s}|} G_k = \pi(\alpha) \setminus \bigcup_{k=0}^{|\bar{s}|} \bigcup_{j=1}^{j_0} \kappa(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_k \overline{\omega}; \delta).$$

Each of the closed regions G_k is a shift of the region G_0 in the complex plane by $\overline{P}_k\overline{\omega}$ units downward, $k = 0, 1, \ldots, |\bar{s}|$. Since the matrix function $\Phi(z)$ introduced in 1.5. is analytic and regular on G_0, the matrix function $\Phi(z + \mathrm{i}\overline{P}_k\overline{\omega})$ is analytic and regular on G_k and the same property is possessed also by $\Phi^{-1}(z + \mathrm{i}\overline{P}_k\overline{\omega}), k = 0, 1, \ldots, |\bar{s}|$. It follows that the matrix function $\Phi_P(z)$ is analytic on the closed region G_P.

In the case $\theta = \emptyset$ the boundary $L_P = \partial G_P$ of the closed region G_P is formed by two lines $|\operatorname{Re} z| = \alpha$ which form the boundary of the strip $\pi(\alpha)$. For $\theta \neq \emptyset$ the boundary $L_P = \partial G_P$ is formed by two lines $|\operatorname{Re} z| = \alpha$ and by a circle $K_{j,k} = K(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_k\overline{\omega}; \delta)$, $j = 1, \ldots, j_0$; $k = 0, 1, \ldots |\bar{s}|$. In virtue of the assumptions of Theorem 2.1 and of the choice of the positive number δ it is ensured for $\theta \neq \emptyset$ that no point $z \in K_{j,k}$ belongs to any disc $\kappa_{l,m}$. Namely, the distance between the point z and the center $w = \mathrm{i}\xi_l - \mathrm{i}\overline{P}_m\overline{\omega}$ of the open disc $\kappa_{l,m}$ is greater than or equal to the radius δ of this disc. We have

$$\begin{aligned} \operatorname{dist}[z, w] &= |z - \mathrm{i}\xi_l + \mathrm{i}\overline{P}_m\overline{\omega}| \\ &= |z - \mathrm{i}\xi_j + \mathrm{i}\overline{P}_k\overline{\omega} + \mathrm{i}(\xi_j - \xi_l + (\overline{P}_m - \overline{P}_k)\overline{\omega})| \\ &\geqslant |\xi_j - \xi_l + (\overline{P}_m - \overline{P}_k)\overline{\omega}| - |z - \mathrm{i}\xi_j + \mathrm{i}\overline{P}_k\overline{\omega}|. \end{aligned}$$

Since $|z - \mathrm{i}\xi_j + \mathrm{i}\overline{P}_k\overline{\omega}| = \delta$ we have for $(\overline{P}_m - \overline{P}_k)\overline{\omega} \neq 0$ the inequality $\operatorname{dist}[z, w] \geqslant d_\theta - \delta \geqslant \delta$ and for $j \neq l$ and $(\overline{P}_k - \overline{P}_m)\overline{\omega} = 0$ the inequality $\operatorname{dist}[z, w] \geqslant d_\xi - \delta \geqslant \delta$ and for $j = l$ and $(\overline{P}_k - \overline{P}_m)\overline{\omega} = 0$ the equality $\operatorname{dist}[z, w] = |z - \mathrm{i}\xi_j + \mathrm{i}\overline{P}_k\overline{\omega}| = \delta$ because of $w = \mathrm{i}\xi_j - \mathrm{i}\overline{P}_k\overline{\omega}$, $j, l = 1, \ldots, j_0$; $k, m = 0, 1, \ldots |\bar{s}|$.

3.2. Outline of further investigation. We attempt to prove that the obtained formal Λ-solution is a Λ-solution of Equation (2.1). The approach could be the following: first, to prove the absolute and consequently also uniform convergence of the trigonometric series x_λ on $\mathbb{R}$ for every $\lambda \in \Lambda_f$, see (2.8). After inserting into Equation (2.1) to prove the same for the trigonometric series $\dot{x}_\lambda$ which is the formal derivative of x_λ. The formal solution x_λ then becomes an almost periodic Λ-solution of Equation (2.1) for $f(t) = \varphi(\lambda) \exp(\mathrm{i}\lambda t), t \in \mathbb{R}$. It follows that $x_f = \sum_\lambda x_\lambda$, $\lambda \in \Lambda_f$, is an almost periodic Λ-solution of Equation (2.1), (Λ_f is a finite set).

Instead of this, for better economy, we shall prove directly a certain absolute and uniform convergence on $\mathbb{R}$ of the trigonometric series

$$\sum_{\bar{s} \geqslant \bar{0}} \left[\sum_P \sum_\lambda \Phi_P(\mathrm{i}\lambda)\varphi(\lambda) \exp(\mathrm{i}\lambda t) \right] \exp(\mathrm{i}\bar{s}\overline{\omega}t), \tag{3.1}$$

which arises by a rearrangement of the trigonometric series x_f. Namely, the convergence of the series

$$\sum_{\bar{s}\geqslant\bar{0}}\sum_{P}\Big|\sum_{\lambda}\Phi_P(\mathrm{i}\lambda)\varphi(\lambda)\exp(\mathrm{i}\lambda t)\Big| \tag{3.2}$$

will be considered in the sequel. In the case of the one-point spectrum for $f(t)=\varphi(\lambda)\exp(\mathrm{i}\lambda t)$, $t\in\mathbb{R}$ when x_f and x_λ coincide and x_λ coincides with (3.1), the convergence of the series (3.2) ensures the absolute and uniform convergence of x_λ.

Eventually, with the use of passing to limits we proceed to the case when a, b and f are not trigonometric polynomials.

3.3. Integral representation. For a given vector $\bar{s}$ we can choose a sufficiently large positive number R such that all circles $K_{j,l}$, $j=1,\dots,j_0$; $l=0,\dots,|\bar{s}|$ belong to the interior of the closed region $\pi(\alpha)\cap\bar{\kappa}(0;R)$ the boundary of which we denote by L_R.

Now, we use the Cauchy integral for the expression inside the norm in the series (3.2). If we denote by $L_R(P)$ the boundary of the closed region $G_P\cap\bar{\kappa}(0;R)$ then

$$\begin{aligned}\sum_{\lambda}\Phi_p(\mathrm{i}\lambda)\varphi(\lambda)\exp(\mathrm{i}\lambda t)&=\frac{1}{2\pi\mathrm{i}}\oint_{L_R(P)}\Phi_P(z)F(t,z)\,\mathrm{d}z\\&=\frac{1}{2\pi\mathrm{i}}\oint_{L_R}\Phi_P(z)F(t,z)\,\mathrm{d}z\\&\quad-\sum_{k=0}^{|\bar{s}|}\sum_{j=0}^{j_0}\frac{1}{2\pi\mathrm{i}}\oint_{K_{j,k}}\Phi_P(z)F(t,z)\,\mathrm{d}z,\ \lambda\in\Lambda_f,\end{aligned}\tag{3.3}$$

where

$$F(t,z)=\sum_{\lambda}\frac{\exp(\mathrm{i}\lambda t)}{z-\mathrm{i}\lambda}\varphi(\lambda),\qquad\lambda\in\Lambda_f,\qquad t\in\mathbb{R}. \tag{3.4}$$

The function F has the following properties:

i)

$$\lim_{|z|\to\infty}|F(t,z)|=0 \tag{3.5}$$

uniformly with respect to $t\in\mathbb{R}$. This implies the existence of a constant R' such that the inequality

$$|F(t,z)|\leqslant 1 \tag{3.6}$$

holds uniformly with respect to $t \in \mathbb{R}$ for all $z \in \mathbb{C}$, $|z| \geqslant R'$.

ii) Denoting $\|f\| = \max\{|f|, |\dot{f}|\}$ the estimate

$$|F(t,z)| \leqslant \frac{1+\alpha}{\alpha|z|}\|f\| \tag{3.7}$$

holds uniformly with respect to $t \in \mathbb{R}$ for all $z \in \mathbb{C}$ for which $|\operatorname{Re} z| = \alpha$. Indeed, for $|\operatorname{Re} z| \neq 0$ the equalities

$$\begin{aligned}
\frac{\exp(\mathrm{i}\lambda t)}{z - \mathrm{i}\lambda} &= \exp(zt)\int_t^{\infty \operatorname{Re} z} \exp((\mathrm{i}\lambda - z)s)\,\mathrm{d}s \\
&= \exp(\mathrm{i}\lambda t)\int_0^{\infty \operatorname{Re} z} \exp((\mathrm{i}\lambda - z)s)\,\mathrm{d}s, \\
F(t,z) &= \int_0^{\infty \operatorname{Re} z} f(t+s)\exp(-zs)\,\mathrm{d}s \\
&= \frac{1}{z}\left[f(t) + \int_0^{\infty \operatorname{Re} z} \dot{f}(t+s)\exp(-zs)\,\mathrm{d}s\right]
\end{aligned}$$

are valid.

3.4. Estimates. Assume that $\theta \neq \emptyset$. Owing to the choice of positive numbers α, δ and to the properties of the matrix $\Phi(z)$ and the quasipolynomial $\Delta(z)$ there exists a positive constant C_1 such that the inequalities

$$\begin{gathered}
|\Phi^{-1}(z)| \leqslant C_1 \text{ for } z \in G_0 \\
|\Phi^{-1}(z)| \leqslant C_1|z|^{-1} \text{ for } z \in G_0 \setminus \{0\}
\end{gathered} \tag{3.8}$$

are valid. If we pass to the limit for $R \to \infty$ on the right-hand side of the equality (3.3) we get the equality

$$\begin{aligned}
\sum_\lambda \Phi_p(\mathrm{i}\lambda)\varphi(\lambda)\exp(\mathrm{i}\lambda t) &= \frac{1}{2\pi\mathrm{i}}\left(-\int_{-\alpha-\mathrm{i}\infty}^{-\alpha+\mathrm{i}\infty} + \int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty}\right)\Phi_P(z)F(t,z)\,\mathrm{d}z \\
&\quad - \sum_{j=1}^{j_0}\sum_{k=0}^{|\bar{s}|}\frac{1}{2\pi\mathrm{i}}\oint_{K_{j,k}} \Phi_P(z)F(t,z)\,\mathrm{d}z, \quad \lambda \in \Lambda_f.
\end{aligned} \tag{3.9}$$

This can be seen by taking into account the estimates (3.6) and (3.7) which imply the absolute convergence of the improper integrals on the right-hand side in (3.9) and the convergence to zero uniformly with respect to $t \in \mathbb{R}$ for $R \to \infty$ of integrals over the arcs of the circle $K(0;R)$ lying in the α-strip.

The quasipolynomial $\Delta(z)$ may be expressed in the form $\Delta(z) = (z - \mathrm{i}\xi_j)^{\varrho_j}\Delta_j(z)$ where $\Delta_j(z) \neq 0$ for $z \in \bar{\kappa}(\mathrm{i}\xi_j;\delta)$, $j = 1,\dots,j_0$. Hence, the inverse matrix Φ^{-1} may be expressed in the form $\Phi^{-1}(z) = (z - \mathrm{i}\xi_j)^{-\varrho_j}\Gamma_j(z)$, where $\Gamma_j(z) = \Delta_j^{-1}(z)\widetilde{\Phi}(z)$, $j = 1, \ldots, j_0$, and where $\widetilde{\Phi}(z)$ is the matrix whose elements with subscripts k, l are equal to the algebraic complements of $\Phi(z)$ with subscripts l, k: $k, l = 1, \ldots, n$. The matrix Γ_j is analytic in the closed disc $\bar{\kappa}(\mathrm{i}\xi_j; \delta)$, $j = 1, \ldots, j_0$. According to this decomposition and in view of

$$\Phi'(z) = \frac{\mathrm{d}}{\mathrm{d}z}\Phi(z) = E - (-\tau)b_0 \exp(-z\tau),$$

$$\Phi^{(h)}(z) = \frac{\mathrm{d}^h}{\mathrm{d}z^h}\Phi(z) = -(-\tau)^h b_0 \exp(-z\tau), \quad h = 2, 3, \ldots$$

it is possible to choose the already defined constant C_1 large enough so that besides the estimates (3.7) also the following ones are true:

$$\begin{aligned} &|(\Phi^{-1}(z))^{(h)}| \leqslant C_1 \quad \text{for } z \in G_0, \\ &|(\Phi^{-1}(z))^{(h)}| \leqslant C_1|z|^{-1} \quad \text{for } z \in G_0 \setminus \{0\}, \\ &|\Gamma_j^{(h)}(\mathrm{i}\xi_j)| \leqslant C_1 \end{aligned} \tag{3.10}$$

for $j = 1, \ldots, j_0$; $h = 0, 1, \ldots, \varrho$, where $\varrho = \max\{\varrho_1, \ldots, \varrho_{j_0}\}$.

Lemma 3.1. *The magnitudes of the integrals*

$$I_{j,l}(p) = -\frac{1}{2\pi\mathrm{i}}\oint_{K_{j,l}} \Phi_P(z)F(t,z)\,\mathrm{d}z$$

are estimated by

$$\begin{aligned} |I_{j,l}(P)| \leqslant \frac{C_1}{|\bar{s}|!}\binom{|\bar{s}|}{\bar{s}} &\Bigg[\prod_{k=1}^{M} \frac{2^\varrho(L+1)C_1|\alpha(\mu_k)|}{2\delta}\Bigg] \\ \times &\Bigg[\prod_{k=1}^{N} \frac{2^\varrho(L+1)C_1|\beta(\nu_k)|}{2\delta}\Bigg] \sum_{k=1}^{\varrho}(2M_d)^k\|f\|, \end{aligned}$$

$j = 1, \ldots j_0$; $l = 0, 1, \ldots, |\bar{s}|$, *where positive constants* L, M_d *do not depend on* $\bar{s}$ *and* $P(\bar{s})$, *either.*

P r o o f. Notice that the development

$$\frac{1}{z - \mathrm{i}\lambda} = \frac{-1}{\mathrm{i}\lambda + \mathrm{i}\overline{P}_l\bar{\omega} - \mathrm{i}\xi_j}\sum_{h=0}^{\infty}\Big(\frac{z + \mathrm{i}\overline{P}_l\bar{\omega} - \mathrm{i}\xi_j}{\mathrm{i}\lambda + \mathrm{i}\overline{P}_l\bar{\omega} - \mathrm{i}\xi_j}\Big)^h$$

is valid for arbitrary $\lambda \in \Lambda_f$ and $z \in K_{j,l}$ because (see (2.5))

$$\operatorname{dist}[\mathrm{i}\lambda + \mathrm{i}\overline{P}_l\varpi;\ \mathrm{i}\xi_j] = |\mathrm{i}\lambda + \mathrm{i}\overline{P}_l\varpi - \mathrm{i}\xi_j| \geqslant d > \delta = |z + \mathrm{i}\overline{P}_l\varpi - \mathrm{i}\xi_j|,$$
$$j = 1, \dots, j_0;\ l = 0, 1, \dots, |\bar{s}|.$$

Next, let us recall the already verified fact that the discs $\kappa_{j,l}$ do not intersect, $j = 1, \dots j_0$; $l = 0, 1, \dots, |\bar{s}|$. For economy in writing we will use the notation

$$\Phi_{P,j,l}(z) = \prod_{k=|\bar{s}|}^{0} \Psi_{k,j,l}(z)\gamma(\bar{p}_k\varpi), \quad \text{where}$$
$$\Psi_{k,j,l}(z) = \Big(\Phi^{-1}(z + \mathrm{i}\overline{P}_k\varpi)\Big)^{1-\delta_{kl}} \Big(\Gamma_j(z + \mathrm{i}\overline{P}_l\varpi)\Big)^{\delta_{kl}}$$

($\delta_{kl} = 0$ for $k \neq l$ and $\delta_{kl} = 1$ for $k = l$) so that

$$\Phi_P(z) = (z + \mathrm{i}\overline{P}_l\varpi - \mathrm{i}\xi_j)^{-\varrho_j}\Phi_{P,j,l}(z), \quad j = 1, \dots, j_0; \quad k, l = 0, 1, \dots, |\bar{s}|.$$

(The function $\Phi_{P,j,l}(z)$ is analytic in the closed ring $\bar{\kappa}_{j,l}$.) We employ this expression when evaluating the integrals

$$\begin{aligned}
I_{j,l}(P) &= -\frac{1}{2\pi\mathrm{i}} \oint_{K_{j,l}} \Phi_P(z)F(t,z)\,\mathrm{d}z \\
&= -\sum_{\lambda} \frac{1}{2\pi\mathrm{i}} \oint_{K_{j,l}} \frac{\Phi_{P,j,l}(z)\,\mathrm{d}z}{(z + \mathrm{i}\overline{P}_l\varpi - \mathrm{i}\xi_j)^{\varrho_j}(z - \mathrm{i}\lambda)}\varphi(\lambda)\exp(\mathrm{i}\lambda t) \\
&= \sum_{\lambda}\sum_{h=0}^{\infty} \frac{1}{2\pi\mathrm{i}} \oint_{K_{j,l}} \frac{(z + \mathrm{i}\overline{P}_l\varpi - \mathrm{i}\xi_j)^h\Phi_{P,j,l}(z)\,\mathrm{d}z}{(\mathrm{i}\lambda + \mathrm{i}\overline{P}_l\varpi - \mathrm{i}\xi_j)^{h+1}(z + \mathrm{i}\overline{P}_l\varpi - \mathrm{i}\xi_j)^{\varrho_j}}\varphi(\lambda)\exp(\mathrm{i}\lambda t) \\
&= \sum_{h=0}^{\varrho_j - 1} \frac{\Phi_{P,j,l}^{(\varrho_j - h - 1)}(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_l\varpi)}{(\varrho_j - h - 1)!} \sum_{\lambda} \frac{\exp(\mathrm{i}\lambda t)}{(\mathrm{i}\lambda + \mathrm{i}\overline{P}_l\varpi - \xi_j)^{h+1}}\varphi(\lambda) \\
&= \exp(\mathrm{i}(\xi_j - \overline{P}_l\varpi)t) \sum_{h=1}^{\varrho_j} \frac{\Phi_{P,j,l}^{(\varrho_j - h)}(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_l\varpi)}{(\varrho_j - h)!} g_{j,l,h}(t), \quad \lambda \in \Lambda_f,
\end{aligned}$$

where

$$g_{j,l,h}(t) = \sum_{\lambda} \frac{\exp(\mathrm{i}(\lambda + \overline{P}_l\varpi - \xi_j)t)}{(\mathrm{i}\lambda + \mathrm{i}\overline{P}_l\varpi - \mathrm{i}\xi_j)^h}\varphi(\lambda), \quad \lambda \in \Lambda_f,$$

and $l = 0, 1, \dots, |\bar{s}|$; $h = 0, 1, \dots, \varrho_j$; $j = 1, \dots, j_0$.

The almost periodic function $g_{j,l,h}$ (being a trigonometric polynomial) is a primitive function to the almost periodic function $g_{j,l,h-1}$ while their spectra have positive distance from the origin in the complex plane since the assumption (2.5) ensures

$|\lambda + \overline{P}_l\varpi - \xi_j| \geqslant d > 0$. Using repeatedly the estimate from the Favard theorem we obtain inequalities

$$|g_{j,l,h}| \leqslant M_d^h |g_{j,l,0}| = M_d^h |f| \leqslant M_d^h \|f\|,$$

$h = 1, \ldots, \varrho_j$; $j = 1, \ldots, j_0$; $l = 0, 1 \ldots, |\bar{s}|$. Denoting by L the smallest non-negative integer satisfying the system of inequalities (for $\theta \neq \emptyset$, otherwise we set $L = 0$)

$$L\Delta - |\xi_j| \geqslant 0, \; j = 1, \ldots, j_0,$$

and using the estimates (3.10) we get the inequalities

$$\begin{aligned}
|\Psi_{k,j,l}^{(h)}(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_l\varpi)| &= |\Gamma_j^{(h)}(\mathrm{i}\xi_j)| \leqslant C_1 \quad \text{ for } \quad k = l,\\
|\Psi_{k,j,l}^{(h)}(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_l\varpi)| &= \Big|\Big(\Phi^{-1}\big(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_l\varpi + \mathrm{i}\overline{P}_k\varpi\big)\Big)^{(h)}\Big| \leqslant C_1\\
&= \frac{|k-l|}{|k-l|}C_1 < \frac{L+1}{|k-l|\delta}C_1 \quad \text{ for } 0 < |k-l| \leqslant L,\\
\big|\Psi_{k,j,l}^{(h)}(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_l\varpi)\big| &= \Big|\Big(\Phi^{-1}\big(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_l\varpi + \mathrm{i}\overline{P}_k\varpi\big)\Big)^{(h)}\Big|\\
&\leqslant \frac{C_1}{|\xi_j + (\overline{P}_k - \overline{P}_l)\varpi|} \leqslant \frac{C_1}{|(\overline{P}_k - \overline{P}_l)\varpi| - L\Delta + (L\Delta - |\xi_j|)}\\
&\leqslant \frac{C_1}{|k-l|\Delta - L\Delta} \leqslant \frac{L+1}{|k-l|\Delta}C_1 \leqslant \frac{L+1}{|k-l|2\delta}C_1
\end{aligned}$$

for $|k-l| > L$, $h = 0, \ldots, \varrho_j$; $j = 1, \ldots, j_0$; $k, l = 0, 1, \ldots, |\bar{s}|$.

Let us consider vectors $\bar{h} = (h_0, h_1, \ldots, h_{|\bar{s}|}) \in \mathbb{N}_0^{1\times(|\bar{s}|+1)}$ and let h be a non-negative integer. We have

$$\begin{aligned}
|\Phi_{P,j,l}^{(h)}(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_l\varpi)| &\leqslant \sum_{|\bar{h}|=h} \prod_{k=0}^{|\bar{s}|} |\Psi_{k,j,l}^{(h_k)}(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_l\varpi)| \cdot |\gamma(\bar{p}_k\varpi)|\\
&\leqslant \frac{2^{h|\bar{s}|}C_1}{l!(|\bar{s}|-l)!} \prod_{k=1}^{|\bar{s}|} \frac{(L+1)C_1|\gamma(\bar{p}_k\varpi)|}{2\delta}\\
&= \frac{C_1}{|\bar{s}|!}\binom{|\bar{s}|}{l} \prod_{k=1}^{|\bar{s}|} \frac{2^h(L+1)C_1|\gamma(\bar{p}_k\varpi)|}{2\delta},
\end{aligned}$$

$h = 0, 1, \ldots, \varrho_j$; $j = 1, \ldots, j_0$; $k, l = 0, 1, \ldots, |\bar{s}|$, since

$$\sum_{|\bar{h}|=h} \binom{|\bar{h}|}{\bar{h}} = (|\bar{s}|+1)^h \leqslant 2^{h|\bar{s}|}.$$

Setting $\varrho = \max\{\varrho_1, \dots, \varrho_{j_0}\}$ we get the estimates

$$
\begin{aligned}
|I_{j,l}(P)| &\leqslant \sum_{h=1}^{\varrho_j} \frac{1}{(\varrho_j - h)!} |\Phi_{P,j,l}^{(h)}(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_l\overline{\omega})| \cdot |g_{j,l,h}| \\
&\leqslant \frac{C_1}{|\bar{s}|!} \binom{|\bar{s}|}{l} \sum_{h=1}^{\varrho_j} (2M_d)^h \|f\| \prod_{k=1}^{|\bar{s}|} \frac{1}{2\delta} 2^h (L+1) C_1 |\gamma(\bar{p}_k\overline{\omega})| \\
&\leqslant \frac{C_1}{|\bar{s}|! 2^{|\bar{s}|}} \binom{|\bar{s}|}{l} \sum_{h=1}^{\varrho} (2M_d)^h \|f\| \prod_{k=1}^{|\bar{s}|} \frac{1}{\delta} 2^{\varrho} (L+1) C_1 |\gamma(\bar{p}_k\overline{\omega})|.
\end{aligned}
$$

Since

$$
\prod_{k=1}^{|\bar{s}|} |\gamma(\bar{p}_k\overline{\omega})| = \Big[\prod_{k=1}^{M} (|\alpha(\mu_k)|)^{m_k}\Big] \prod_{k=1}^{N} (|\beta(\nu_k)|)^{n_k}, \tag{3.11}
$$

where $\bar{s} = (\overline{m}, \bar{n}), \overline{m} = (m_1, \dots, m_M)$, $\bar{n} = (n_1, \dots, n_N)$, Lemma 3.1 is proved. □

Lemma 3.2. *The improper integrals*

$$
I_-(P) + I_+(P) = \frac{1}{2\pi\mathrm{i}} \Big(-\int_{-\alpha-\mathrm{i}\infty}^{-\alpha+\mathrm{i}\infty} + \int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} \Big) \Phi_P(z) F(t,z) \,\mathrm{d}z
$$

converge absolutely and the following estimate is valid:

$$
|I_-(P) + I_+(P)| \leqslant \frac{1+\alpha}{\alpha^2} C_1 \Big[\prod_{k=1}^{M} \Big(\frac{\sqrt{2}C_1|\alpha_k|}{\delta}\Big)^{m_k}\Big]\Big[\prod_{k=1}^{N} \Big(\frac{\sqrt{2}C_1|\beta_k|}{\delta}\Big)^{n_k}\Big] \frac{\|f\|}{|\bar{s}|!},
$$

$$
|\bar{s}| = 0, 1, \dots; \ \alpha_k = \alpha(\mu_k), \ k = 1, \dots, M; \ \beta_k = \beta(\nu_k), \ k = 1, \dots, N.
$$

P r o o f. We distinguish two cases: $|\bar{s}| = 0$ and $|\bar{s}| > 0$. First, we consider $|\bar{s}| = 0$ when $(I_- = I_-(\bar{0}), I_+ = I_+(\bar{0}))$

$$
\begin{aligned}
|I_- + I_+| \leqslant |I_-| + |I_+| &\leqslant \frac{1+\alpha}{2\pi\alpha} C_1 \int_{-\infty}^{\infty} \Big(\frac{1}{|\alpha - \mathrm{i}v|^2} + \frac{1}{|\alpha + \mathrm{i}v|^2}\Big) \,\mathrm{d}v \cdot \|f\| \\
&= \frac{1+\alpha}{\alpha^2} C_1 \|f\|.
\end{aligned}
$$

Now, let us consider $|\bar{s}| > 0$. It suffices to estimate the magnitude of the improper integral $I_+(P)$ since the estimate for the magnitude of $I_-(P)$ can be obtained analogously with the same result:

$$
|I_+(P)| = \frac{1}{2\pi} \Big| \int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} \Phi_P(z) F(t,z) \,\mathrm{d}z \Big| \leqslant \frac{1+\alpha}{2\pi\alpha^2} \int_{-\infty}^{\infty} |\Phi_P(\alpha + \mathrm{i}v)| \,\mathrm{d}v \cdot \|f\|.
$$

The inequality $|\alpha+\mathrm{i}v| \geqslant (\alpha+|v|)/\sqrt{2}$ holds true clearly for every $v \in \mathbb{R}$ so that according to the estimate (3.8) we get the inequality

$$
\begin{aligned}
|\Phi_P(\alpha+\mathrm{i}v)| &\leqslant \prod_{j=0}^{|\bar{s}|} \left|\Phi^{-1}\left(\alpha+\mathrm{i}\left(v+\overline{P}_j\varpi\right)\right)\right| \left|\gamma\left(\bar{p}_j\varpi\right)\right| \\
&\leqslant \sqrt{2}C_1\Bigg[\prod_{j=1}^{|\bar{s}|}\sqrt{2}C_1|\gamma(\bar{p}_j\varpi)|\Bigg]\prod_{j=0}^{|\bar{s}|}\frac{1}{\alpha+|v+\overline{P}_j\varpi|}.
\end{aligned}
$$

We split the improper integral into the sum of three integrals

$$
\int_{-\infty}^{\infty} = \int_{-\infty}^{-\bar{s}\varpi} + \int_{-\bar{s}\varpi}^{0} + \int_{0}^{\infty}
$$

and estimate the magnitude of each term separately. In virtue of the choice of the positive constant δ (see at the beginning of 3.1.) one easily finds that

$$
\begin{aligned}
\int_0^\infty \left(\prod_{j=0}^{|\bar{s}|}\left(\alpha+\overline{P}_j\varpi+v\right)\right)^{-1}\mathrm{d}v &\leqslant \frac{1}{(2\delta)^{|\bar{s}|}}\int_0^\infty\left(\prod_{j=0}^{|\bar{s}|}(j+1+v)\right)^{-1}\mathrm{d}v \\
&\leqslant \frac{1}{|\bar{s}|!(2\delta)^{|\bar{s}|}}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\int_{-\infty}^{-\bar{s}\varpi}\left(\prod_{j=0}^{|\bar{s}|}\left(\alpha+|\overline{P}_j\varpi+v|\right)\right)^{-1}\mathrm{d}v &= \int_0^\infty\left(\prod_{j=0}^{|\bar{s}|}\left(\alpha+\left(\bar{s}-\overline{P}_j\right)\varpi+v\right)\right)^{-1}\mathrm{d}v \\
&\leqslant \frac{1}{|\bar{s}|!(2\delta)^{|\bar{s}|}}.
\end{aligned}
$$

Let us note that the expression $w_{j,l} = \alpha+|(\overline{P}_j-\overline{P}_l)\varpi+v|$ satisfies the relations

$$
\begin{aligned}
&w_{j,l} = \alpha+(\overline{P}_l-\overline{P}_j)\varpi-v \geqslant (l-j)2\delta \quad \text{for } 0\leqslant j<l-1,\\
&w_{j,l} = \alpha+\bar{p}_l\varpi-v \quad \text{for } j=l-1,\\
&w_{j,l} = \alpha+v \quad \text{for } j=l,\\
&w_{j,l} = \alpha+(\overline{P}_j-\overline{P}_l)\varpi+v \geqslant (j-l+1)2\delta \quad \text{for } j>l,
\end{aligned}
\tag{3.12}
$$

for $0\leqslant v\leqslant \bar{p}_l\varpi$; $j,l=0,1,\ldots,|\bar{s}|$. Further, let us note that

$$
\int_0^{\bar{p}_l\varpi}\frac{2\delta\,\mathrm{d}v}{(\alpha+v)(\alpha+\bar{p}_l\varpi-v)} \leqslant \frac{2\delta\bar{p}_l\varpi}{\alpha(\alpha+\bar{p}_l\varpi)} < \frac{2\delta}{\alpha} \leqslant 1
\tag{3.13}
$$

since $(\alpha+v)(\alpha+\bar{p}_l\bar{\omega}-v)\geqslant\alpha(\alpha+\bar{p}_l\bar{\omega})$ for $0\leqslant v\leqslant\bar{p}_l\bar{\omega}$ for $l=0,1,\dots,|\bar{s}|$. Using (3.12) and (3.13) we obtain the desired estimate

$$\begin{aligned}\int_{-\bar{s}\bar{\omega}}^{0}\Bigl(\prod_{j=0}^{|\bar{s}|}w_{j,0}\Bigr)^{-1}\mathrm{d}v&=\sum_{l=1}^{|\bar{s}|}\int_{-\bar{P}_l\bar{\omega}}^{-\bar{P}_{l-1}\bar{\omega}}\Bigl(\prod_{j=0}^{|\bar{s}|}w_{j,0}\Bigr)^{-1}\mathrm{d}v\\&=\sum_{l=1}^{|\bar{s}|}\int_{0}^{\bar{p}_l\bar{\omega}}\Bigl(\prod_{j=0}^{|\bar{s}|}w_{j,l}\Bigr)^{-1}\mathrm{d}v\\&\leqslant\sum_{l=1}^{|\bar{s}|}\frac{1}{\bar{l}!(|\bar{s}|-l+1)!(2\delta)^{|\bar{s}|}}\int_{0}^{\bar{p}_l\bar{\omega}}\frac{2\delta\,\mathrm{d}v}{(\alpha+v)(\alpha+\bar{p}_l\bar{\omega}-v)}\\&\leqslant\frac{1}{(|\bar{s}|+1)!(2\delta)^{|\bar{s}|}}\sum_{l=1}^{|\bar{s}|}\binom{|\bar{s}|+1}{l}=\frac{2^{|\bar{s}|+1}-2}{(|\bar{s}|+1)!(2\delta)^{|\bar{s}|}}.\end{aligned}$$

Summing up the above estimates we get the inequality

$$\int_{-\infty}^{\infty}\Bigl(\prod_{j=0}^{|\bar{s}|}(\alpha+|\bar{P}_j\bar{\omega}+v|)\Bigr)^{-1}\mathrm{d}v\leqslant\frac{2(|\bar{s}|+1)+2^{|\bar{s}|+1}}{(|\bar{s}|+1)!(2\delta)^{|\bar{s}|}}\leqslant\frac{2}{|\bar{s}|!\delta^{|\bar{s}|}},$$

so that

$$|I_+(P)|\leqslant 2C_1\frac{1+\alpha}{\pi\alpha^2}\frac{\|f\|}{|\bar{s}|!}\prod_{j=1}^{|\bar{s}|}\frac{\sqrt{2}C_1|\gamma(\bar{p}_j\bar{\omega})|}{\delta},\quad |\bar{s}|=1,2,\dots.$$

The same estimate is valid for $|I_-(P)|$. Because of the inequality $2\sqrt{2}/\pi<1$ and owing to (3.11) the validity of the inequality in Lemma 3.2 is established. □

3.5. Almost periodic solutions. Now we show that the obtained formal solution is an almost periodic solution of Equation (2.1).

Theorem 3.3. *The formal almost periodic Λ-solution x_f from Theorem* 2.1, *is an almost periodic Λ-solution of Equation* (2.1). *Moreover, it is unique and satisfies the estimate*

$$\|x_f\|\leqslant A\|f\| \tag{3.14}$$

where the positive constant A depends only on a_0, b_0, d, d_θ, Δ, τ, S, T where $S=\sum|\alpha(\mu)|=\sum(a)$, $\mu\in\Lambda_a$; $T=\sum|\beta(\nu)|=\sum(b)$, $\nu\in\Lambda_b$.

Proof. With the aid of the estimates from Lemmas 3.1 and 3.2 for the magnitudes of the integrals $I_{j,l}$, $I_-(P)$, $I_+(P)$ we shall prove the convergence of the series

(3.2), which yields the absolute and uniform convergence with respect to $t \in \mathbb{R}$ of the trigonometric series x_f:

$$
\begin{aligned}
\Big|\sum_{\lambda} c_P(\lambda + \bar{s}\bar{\omega}) \exp(\mathrm{i}\lambda t)\Big| &\leqslant |I_-(P) + I_+(P)| + \sum_{j=1}^{j_0}\sum_{l=0}^{|\bar{s}|} |I_{j,l}(P)| \\
&\leqslant \frac{C_1}{|\bar{s}|}\|f\|\Big[\frac{1+\alpha}{\alpha^2}\prod_{j=1}^{|\bar{s}|}\frac{2^{\varrho}(L+1)C_1|\gamma_j|}{\delta} \\
&\quad + j_0\sum_{h=1}^{\varrho}(2M_d)^h\prod_{j=1}^{|\bar{s}|}\frac{2^{\varrho}(L+1)C_1|\gamma_j|}{\delta}\Big] \\
&\leqslant \frac{C_1C_2}{|\bar{s}|!}\|f\|\prod_{j=1}^{|\bar{s}|}\frac{2^{\varrho}(L+1)C_1|\gamma_j|}{\delta},
\end{aligned}
$$

where $\gamma_j = \gamma(\bar{p}_j\bar{\omega})$, $j = 1, \ldots, |\bar{s}|$; $C_2 = \frac{1+\alpha}{\alpha^2} + j_0\sum_{h=1}^{\varrho}(2M_d)^h$ and the positive constant M_d depends only on d and is defined by Theorem 1.1. This implies

$$
\begin{aligned}
\sum_P\Big|\sum_{\lambda} c_P(\lambda + \bar{s}\bar{\omega}) \exp(\mathrm{i}\lambda t)\Big| &\leqslant \binom{|\bar{s}|}{\bar{s}}\frac{C_1C_2}{|\bar{s}|!}\|f\|\prod_{j=1}^{|\bar{s}|}\frac{2^{\varrho}(L+1)C_1|\gamma_j|}{\delta} \\
&\leqslant C_1C_2\|f\|\Big[\prod_{k=1}^{M}\frac{1}{m_k!}\Big(\frac{2^{\varrho}(L+1)C_1|\alpha_k|}{\delta}\Big)^{m_k}\Big] \\
&\quad \times\prod_{k=1}^{N}\frac{1}{n_k!}\Big(\frac{2^{\varrho}(L+1)C_1|\beta_k|}{\delta}\Big)^{n_k}
\end{aligned}
$$

and the convergence of the series (3.2) follows since

$$
\begin{aligned}
&\sum_{\bar{s}\geqslant\bar{0}}\sum_P\Big|\sum_{\lambda} c_P(\lambda + \bar{s}\bar{\omega}) \exp(\mathrm{i}\,(\lambda + \bar{s}\bar{\omega})\,t)\Big| \\
&\quad = \sum_{\bar{s}\geqslant\bar{0}}\sum_P\Big|\sum_{\lambda} c_P(\lambda + \bar{s}\bar{\omega}) \exp(\mathrm{i}\lambda t)\Big| \\
&\quad \leqslant C_1C_2\|f\|\sum_{\bar{s}\geqslant\bar{0}}\Big[\prod_{k=1}^{M}\frac{1}{m_k!}\Big(\frac{2^{\varrho}(L+1)C_1|\alpha_k|}{\delta}\Big)^{m_k}\Big] \\
&\quad \quad \times\prod_{k=1}^{N}\frac{1}{n_k!}\Big(\frac{2^{\varrho}(L+1)C_1|\beta_k|}{\delta}\Big)^{n_k} \\
&\quad = C_1C_2\|f\|\exp(2^{\varrho}(L+1)C_1(S+T)/\delta).
\end{aligned}
$$

If we denote $\widetilde{A} = C_1 C_2 \exp\left(2^{\varrho}(L+1)C_1(S+T)/\delta\right)$ then $|x_f| \leqslant \widetilde{A}\|f\|$. Inserting into Equation (2.1) we get

$$|\dot{x}_f| \leqslant (|a_0| + |b_0| + |a| + |b|)|x_f| + |f| \leqslant [(|a_0| + |b_0| + S + T)\widetilde{A} + 1]\|f\|.$$

Setting $A = (|a_0| + |b_0| + S + T)\widetilde{A} + 1$ we conclude that the estimate (3.14) holds. □

Corollary 3.4. *Let Λ_1, Λ_2 be two non-void sets of real numbers and let S, T be two positive constants. If a, b, f from Equation* (2.1) *are trigonometric polynomials with $\Lambda_f \subset \Lambda_1$, $\Lambda_a \subset \Lambda_2$, $\Lambda_b \subset \Lambda_2$ and $\sum(a) \leqslant S$, $\sum(b) \leqslant T$ and if*

$$\Delta' = \inf \Lambda_2 > 0, \tag{3.15}$$

$$d'_\theta = \begin{cases} \operatorname{dist}[\theta, S(\Lambda_2)] > 0 & \text{for } \theta \neq \emptyset, \\ 2 & \text{for } \theta = \emptyset, \end{cases} \tag{3.16}$$

$$d' = \operatorname{dist}\left[\mathrm{i}\Lambda'; \sigma(\Delta(z))\right] > 0, \tag{3.17}$$

where $\Lambda' = \Lambda_1 + S(\Lambda_2 \cup \{0\})$, then there exists exactly one almost periodic Λ'-solution x_f of Equation (2.1). *This solution satisfies the estimate* (3.14) *where the positive constant A depends only on a_0, b_0, Δ', d'_θ, d', τ, S, T.*

P r o o f. The existence of an almost periodic Λ'-solution x_f follows from Theorem 3.3 which ensures the existence of an almost periodic Λ-solution where $\Lambda = \Lambda_f + S(\Lambda_a \cup \Lambda_b \cup \{0\})$, so that $\Lambda \subset \Lambda'$ and an almost periodic Λ-solution is also an almost periodic Λ'-solution.

The uniqueness of an almost periodic Λ'-solution follows from the fact that the system (2.7) for coefficients $c(\sigma)$ for $\sigma \in \Lambda'$ coincides with the system (2.7) for $\sigma \in \Lambda$ since $\alpha(\mu) = 0$ for $\mu \in \Lambda_2 \setminus \Lambda_a$ and $\beta(\nu) = 0$ for $\nu \in \Lambda_2 \setminus \Lambda_b$ and $\varphi(\lambda) = 0$ for $\lambda \in \Lambda_1 \setminus \Lambda_f$.

The construction of the constant A is the same as before with the only exception that the constants Δ, d_θ, d are replaced by the constants Δ', d'_θ, d', respectively, for which it is apparent that $\Delta' \leqslant \Delta$, $d'_\theta \leqslant d_\theta$, $d' \leqslant d$ so that the constant A could at worst increase. □

R e m a r k 3.5. Corollary 3.4 ensures the validity of the estimate (3.14) with a constant A common for all almost periodic Λ'-solutions x_f of Equation (2.1) of the whole class of trigonometric polynomials a, b, f from Corollary 3.4.

3.6. Limit passages. The conclusions obtained under the assumption that a, b, f are trigonometric polynomials remain valid even under more general assumptions.

Theorem 3.6. *If in Equation* (2.1) *a, b are trigonometric polynomials and f is an almost periodic function with an almost periodic derivative $\dot{f}$ and if the conditions*

(2.3), (2.4), (2.5) *from Theorem* 2.1 *are fulfilled then Equation* (2.1) *has exactly one almost periodic* Λ*-solution* x_f *and this solution satisfies the estimate* (3.14).

Remark 3.7. Equation (2.1) may admit infinitely many almost periodic solutions but only one of them has its spectrum contained in iΛ (hence is an almost periodic Λ-solution).

Proof of Theorem 3.6. There exists a sequence of Bochner-Fejér approximation polynomials B_m, $m = 1, 2, \ldots$ of the function f (with spectra contained in iΛ_f) uniformly convergent to f on $\mathbb{R}$ such that the sequence of derivatives $\dot{B}_m$, $m = 1, 2, \ldots$ forms a sequence of Bochner-Fejér approximation polynomials of the almost periodic function $\dot{f}$ which converges uniformly on $\mathbb{R}$ to $\dot{f}$.

If we choose $\Lambda_1 = \Lambda_f$, $\Lambda_2 = \Lambda_a \cup \Lambda_b$ then $\Lambda' = \Lambda$ and for Equation (2.1) with $f = B_m$ we have satisfied the assumptions from Corollary 3.4 which coincide in this case with the assumptions (2.3), (2.4), (2.5), $m = 1, 2, \ldots$. The equation $\dot{x}(t) = a_0x(t) + b_0x(t-\tau) + a(t)x(t) + b(t)x(t-\tau) + B_m(t)$ has exactly one almost periodic Λ-solution x_m and this solution satisfies the estimate $\|x_m\| \leqslant A\|B_m\|$, $m = 1, 2, \ldots$. Since the spectrum of the trigonometric polynomial $B_{m+k} - B_m$ is contained in iΛ_f, the equation $\dot{x}(t) = a_0x(t) + b_0x(t-\tau) + a(t)x(t) + b(t)x(t-\tau) + B_{m+k}(t) - B_m(t)$ has also exactly one almost periodic Λ-solution, namely $x_{m+k} - x_m$, and the estimate $\|x_{m+k} - x_m\| \leqslant A\|B_{m+k} - B_m\|$ holds ; $m, k = 1, 2, \ldots$. In virtue of the uniform convergence of the sequences of trigonometric polynomials B_m and $\dot{B}_m$ to the almost periodic functions f and $\dot{f}$, respectively, it is readily seen that the sequences of almost periodic functions $\{x_m\}$, $\{\dot{x}_m\}$ converge uniformly on $\mathbb{R}$ and the limit functions $x_f = \lim x_m$, $\dot{x}_f = \lim \dot{x}_m$ satisfy Equation (2.1). Thus, x_f is an almost periodic Λ-solution of Equation (2.1) and the validity of the estimate (3.14) can be verified by using the limit passage for $m \to \infty$ in the estimates for the magnitude of x_m, $m = 1, 2, \ldots$.

It remains to check the uniqueness which could be damaged by the limit passage. So, let us suppose the existence of an almost periodic Λ-solution y of Equation (2.1). Taking into account that a, b, f have almost periodic derivatives of the first order, the almost periodic function y has besides the first also the second almost periodic derivative $\ddot{y}$. In such a case there exists a sequence y_m, $m = 1, 2, \ldots$ of Bochner-Fejér approximation polynomials of the almost periodic function y to which they converge uniformly on $\mathbb{R}$ and their derivatives $\dot{y}_m$ and $\ddot{y}_m$, $m = 1, 2, \ldots$, form sequences of Bochner-Fejér approximation polynomials of the almost periodic functions $\dot{y}$ and $\ddot{y}$, respectively, to which they converge uniformly on $\mathbb{R}$. It is easy to verify that the sequences of trigonometric polynomials $f_m(t) = \dot{y}_m(t) - a_0y_m(t) - b_0y_m(t-\tau) - a(t)y_m(t) - b(t)y_m(t-\tau)$ and $\dot{f}_m(t) = \ddot{y}_m(t) - a_0\dot{y}_m(t) - b_0\dot{y}_m(t-\tau) - a(t)\dot{y}_m(t) - b(t)\dot{y}_m(t-\tau) - \dot{a}(t)y_m(t) - \dot{b}(t)y_m(t-\tau)$, $m = 1, 2, \ldots$, converge uniformly on $\mathbb{R}$ to the almost periodic functions f and $\dot{f}$, respectively. Denoting $\Lambda_1 = \Lambda = \Lambda_f + S(\Lambda_a \cup \Lambda_b \cup \{0\})$, $\Lambda_2 = \Lambda_a \cup \Lambda_b$ then $\Lambda' = \Lambda_1 + S(\Lambda_2 \cup \{0\})$ and the assumptions (3.15), (3.16), (3.17) are satisfied which coincide here with the assumptions (2.3), (2.4), (2.5). The spectra of the trigonometric polynomials f_m and consequently also the spectra of the trigonometric polynomials $B_m - f_m$ are contained in $\mathrm{i}\Lambda$, $m = 1, 2, \ldots$, so that by Corollary 3.4 the equation $\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a(t)x(t) + b(t)x(t-\tau) + B_m(t) - f_m(t)$ has exactly one almost periodic Λ-solution, namely $w_m = x_m - y_m$, which satisfies the estimate $\|w_m\| = \|x_m - y_m\| \leqslant A\|B_m - f_m\|$, $m = 1, 2, \ldots$. However, $\|x_f - y\| = \lim \|x_m - y_m\| = 0$ and hence $x_f = y$. $\square$

Corollary 3.8. *Let Λ_1, Λ_2 be two non-void sets of real numbers and let S, T be two positive constants. If the assumptions* (3.15), (3.16), (3.17) *are satisfied and if f is an almost periodic function with its spectrum contained in* $\mathrm{i}\Lambda_1$ *having the almost periodic derivative $\dot{f}$ and if a, b are trigonometric polynomials with their spectra contained in* $\mathrm{i}\Lambda_2$ *for which $\sum(a) \leqslant S$, $\sum(b) \leqslant T$, then Equation* (2.1) *has exactly one almost periodic Λ'-solution x_f where $\Lambda' = \Lambda_1 + S(\Lambda_2 \cup \{0\})$ and this solution satisfies the estimate* (3.14) *where the positive constant A depends only on a_0, b_0, d'_θ, d', Δ', τ, S, T.*

P r o o f. The validity of Corollary 3.8 can be verified by passing to the limit analogously as in the proof of Theorem 3.3. $\square$

R e m a r k 3.9. Corollary 3.8 ensures the validity of the estimate (3.14) with a constant A common for all almost periodic Λ'-solutions x_f of Equation (2.1) of the whole class of trigonometric polynomials a, b and an almost periodic function f from Corollary 3.8.

Now, we abandon the assumptions that a, b are trigonometric polynomials.

Theorem 3.10. *If a and b are almost periodic functions with absolutely convergent Fourier series having almost periodic first derivatives and f is the function from Theorem* 3.6. *and if the assumptions* (2.3), (2.4), (2.5) *are satisfied, then Equation* (2.1) *has exactly one almost periodic Λ-solution x_f, where $\Lambda = \Lambda_f + S(\Lambda_a \cup \Lambda_b \cup \{0\})$, and this solution satisfies the estimate* (3.14) *in which the positive constant A depends only on a_0, b_0, Δ, d_θ, d, τ, $S = \sum(a)$, $T = \sum(b)$.*

P r o o f. As a consequence of the fact that the almost periodic functions a and b have almost periodic derivatives $\dot{a}$ and $\dot{b}$, respectively, there exist sequences a_m and b_m, $m = 1, 2, \ldots$, of Bochner-Fejér approximation polynomials of the almost periodic functions a and b, respectively, to which they converge uniformly on $\mathbb{R}$, the derivatives $\dot{a}_m$ and $\dot{b}_m$ of which form sequences of Bochner-Fejér approximation polynomials of the almost periodic functions $\dot{a}$ and $\dot{b}$, respectively, to which they converge uniformly on $\mathbb{R}$.

If we denote $\Lambda_2 = \Lambda_a \cup \Lambda_b$, $\Lambda_1 = \Lambda_f + S(\Lambda_2 \cup \{0\})$ then $\Lambda' = \Lambda_1 + S(\Lambda_2 \cup \{0\}) = \Lambda$, $\Lambda_{a_m} \subset \Lambda_2$, $\Lambda_{b_m} \subset \Lambda_2$, $m = 1, 2, \ldots$; $\Lambda_f \subset \Lambda_1$. Moreover, $\sum(a_m) \leqslant S$, $\sum(b_m) \leqslant T$, $m = 1, 2, \ldots$. According to the choice of Λ_1, Λ_2 the assumptions of Corollary 3.8 are satisfied for the equation $\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a_m(t)x(t) + b_m(t)x(t-\tau) + f(t)$. Therefore, this equation has exactly one almost periodic Λ-solution x_m and for this solution we have the estimate $\|x_m\| \leqslant A\|f\|$, $m = 1, 2, \ldots$. Corollary 3.8 implies that the equation $\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a_m(t)x(t) + b_m(t)x(t-\tau) + f_{m,k}(t)$, where $f_{m,k}(t) = (a_{m+k}(t) - a_m(t))x_{m+k}(t) + (b_{m+k}(t) - b_m(t))x_{m+k}(t-\tau)$, has exactly one almost periodic Λ-solution. It is evident that this solution is $x_{m+k} - x_m$ and for this solution the estimate $\|x_{m+k} - x_m\| \leqslant A\|f_{m,k}\|$ holds true, $m = 1, 2, \ldots$.

Since any two almost periodic functions u, v with almost periodic derivatives $\dot{u}$, $\dot{v}$ satisfy $\|uv\| \leqslant 2\|u\|\,\|v\|$ we get the inequality

$$\begin{aligned} \|x_{m+k} - x_m\| &\leqslant A\|f_{m,k}\| \leqslant 2A(\|a_{m+k} - a_m\| + \|b_{m+k} - b_m\|)\|x_{m+k}\| \\ &\leqslant 2A^2(\|a_{m+k} - a_m\| + \|b_{m+k} - b_m\|)\|f\|; \qquad m, k = 1, 2, \ldots. \end{aligned}$$

But this means that $\lim \|x_{m+k} - x_m\| = 0$ for $m \to \infty$ uniformly with respect to $k = 1, 2, \ldots$, so that the almost periodic function $x_f = \lim x_m$ is an almost periodic Λ-solution of Equation (2.1) and satisfies the estimate (3.14).

Again, it is necessary to verify the uniqueness of this solution which could be lost by the passage to the limit. Let y be also an almost periodic Λ-solution of Equation (2.1). Then the almost periodic function $w = x_f - y$ is a unique almost periodic Λ-solution of the equation $\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a_m(t)x(t) + b_m(t)x(t-\tau) + F(t)$ where $F(t) = (a(t) - a_m(t))w(t) + (b(t) - b_m(t))w(t-\tau)$ and this solution satisfies the estimate $\|w\| = \|x_f - y\| \leqslant A\|F\| \leqslant 2A(\|a - a_m\| + \|b - b_m\|)\|w\|$, $m = 1, 2, \ldots$. The right-hand side converges to zero for $m \to \infty$, so that $y = x_f$. □

Corollary 3.11. *Let Λ_1, Λ_2 be two non-void sets of real numbers and let S, T be two positive constants. If the assumptions* (3.15), (3.16), (3.17) *are satisfied and if f is an almost periodic function with its spectrum contained in* $\mathrm{i}\Lambda_1$ *having the almost periodic derivative $\dot{f}$ and if a, b are almost periodic functions with their spectra contained in* $\mathrm{i}\Lambda_2$ *satisfying $\sum(a) \leqslant S$, $\sum(b) \leqslant T$, then Equation* (2.1) *has exactly one almost periodic Λ'-solution x_f where $\Lambda' = \Lambda_1 + S(\Lambda_2 \cup \{0\})$ and this solution satisfies the estimate* (3.14) *where the positive constant A depends only on a_0, b_0, Δ', d'_θ, d', τ, S, T.*

P r o o f. Analogous reasoning as in the proof of Theorem 3.10. □

R e m a r k 3.12. Corollary 3.11 ensures the validity of the estimate (3.14) with a constant A common for all almost periodic Λ'-solutions x_f of Equation (2.1) of the whole class of almost periodic functions a, b, f from Corollary 3.11.

4. Quasilinear equations

4.1. Functions of several variables. Let $g = g(t,x)$ be a continuous function $g\colon \mathbb{R}\times D \to \mathbb{C}^{p\times q}$, where $D \subset \mathbb{C}^{m\times n}$ is a non-void set. The function g is said to be

a) almost periodic in the variable t on $\mathbb{R}\times D$ if $g(t,x)$ is almost periodic as a function of t for any fixed $x \in D$;

b) uniformly almost periodic in the variable t on $\mathbb{R}\times D$ if $g(t,x)$ is almost periodic in t on $\mathbb{R}\times D$ and for any $\varepsilon > 0$ there exists a set $\{\tau\} \subset \mathbb{R}$ relatively dense in $\mathbb{R}$ such that $|g(t+\tau,x) - g(t,x)| < \varepsilon$ for every $\tau \in \{\tau\}$, $t \in \mathbb{R}$, $x \in D$;

c) locally uniformly almost periodic in the variable t on $\mathbb{R}\times D$ if for any compact set $K \subset D$ the restriction g_K of the function g on $\mathbb{R}\times K$ is uniformly almost periodic in the variable t on $\mathbb{R}\times K$.

Lemma. *Let $g\colon \mathbb{R}\times D \to \mathbb{C}^{p\times q}$ be a function almost periodic in t on $\mathbb{R}\times D$. A necessary and sufficient condition for g to be locally uniformly almost periodic in t is that g be continuous in x uniformly with respect to $t \in \mathbb{R}$ on $\mathbb{R}\times D$.*

In the proof it is sufficient to take $p = q = 1$. To prove the sufficiency, let $K \subset D$ be a compact set and $\varepsilon > 0$. The restriction g_K is uniformly continuous in x uniformly with respect to $t \in \mathbb{R}$ on $\mathbb{R}\times K$. Hence, there exists $\delta = \delta(\varepsilon/3)$ such that for any $x, y \in K$ and $t \in \mathbb{R}$ it holds $|g_K(t,x) - g_K(t,y)| < \varepsilon/3$ in the case $|x-y| < \delta$. Further, there exists a finite δ-net for K, namely, $x_1,\dots,x_N \in K$ such that $\min\{|x-x_j|\colon j = 1,\dots,N\} < \delta$ for any $x \in K$.

Since the functions $h_j(t) = g(t,x_j)$, $j = 1,\dots,N$, are almost periodic, there exists a set $\{\tau\} \subset \mathbb{R}$ of $\varepsilon/3$-almost periods common for the functions $h_1,\dots,h_N$ which is relatively dense in $\mathbb{R}$, i.e. $|h_j(t+\tau) - h_j(t)| < \varepsilon/3$ for any $t \in \mathbb{R}, \tau \in \{\tau\}$ and $j = 1,\dots,N$. Now, let $\tau \in \{\tau\}$, $t \in \mathbb{R}$, $x \in K$. Choose j so that $|x-x_j| < \delta$. Then

$$\begin{aligned}|g_K(t+\tau,x) - g_K(t,x)| &\leqslant |g_K(t+\tau,x) - g_K(t+\tau,x_j)| \\ &\quad + |g_K(t+\tau,x_j) - g_K(t,x_j)| + |g_K(t,x_j) - g_K(t,x)| < \varepsilon.\end{aligned}$$

Thus, g is locally uniformly almost periodic in t on $\mathbb{R}\times D$ and the sufficiency is proved. Let us remark that, on the same vein, the function g_K may be shown to be uniformly continuous on $\mathbb{R}\times K$.

On the other hand, to prove the necessity we take an arbitrary compact set $K \subset D$ and $\varepsilon > 0$ and use the uniform continuity of g_K on $[0, l] \times K$ where $l = l(\varepsilon/3)$ is an inclusion length of the relative density of the set $\{\tau\}$ of $\varepsilon/3$-almost periods. For any $x, y \in K$ and $t, s \in [0, l]$ we have $|g_K(t, x) - g_K(s, y)| < \varepsilon/3$ in the case $|t - s| + |x - y| < \delta$. For any $t \in \mathbb{R}$ there exist $\tau = \tau(t) \in \{\tau\}$ such that $t + \tau \in [0, l]$. Consequently, for any $x, y \in K$, $|x - y| < \delta$ and $t \in \mathbb{R}$ we get

$$\begin{aligned} |g_K(t, x) - g_K(t, y)| &\leqslant |g_K(t, x) - g_K(t + \tau, x)| \\ &\quad + |g_K(t + \tau, x) - g_K(t + \tau, y)| + |g_K(t + \tau, y) - g_K(t, y)| < \varepsilon \end{aligned}$$

and the assertion follows.

In the sequel we deal with the cases in which the conditions for the locally uniform almost periodicity of introduced function are fulfilled.

4.2. Harmonic analysis. Let $g\colon \mathbb{R} \times D \to \mathbb{C}^{p\times q}$ be a function almost periodic in t on $\mathbb{R} \times D$. For any $x \in D$ there exists the Bohr transformation

$$a(\lambda, x) = a(\lambda, x, g) = \lim_{T\to\infty} \frac{1}{T} \int_s^{s+T} g(t, x) \exp(-\mathrm{i}\lambda t)\,\mathrm{d}t$$

for each $\lambda \in \mathbb{R}$ uniformly with respect to $s \in \mathbb{R}$. If $a(\lambda, x)$ is non-zero for a given $\lambda \in \mathbb{R}$ for at least one point $x \in D$, i.e. $a(\lambda, x) \not\equiv 0$, $x \in D$, then λ is called the Fourier exponent and $a(\lambda, x)$, $x \in D$, is called the Fourier coefficient of the function g. The set of all Fourier exponents of the function g is denoted by Λ_g. If D is a compact set, then the set Λ_g is at most countable. Due to the compactness of D there exists a countable set $\{x_j\} \subset D$, which is dense in D, i.e. the equality $\inf_j |x - x_j| = 0$ holds for each $x \in D$. If $a(\lambda, x_j) = 0$, $j = 1, 2, \ldots$, for some $\lambda \in \mathbb{R}$, then $|a(\lambda, x)| = |a(\lambda, x - x_j)| \leqslant \inf\limits_j \sup\limits_t |g(t, x) - g(t, x_j)| = 0$. Thus $a(\lambda, x) \not\equiv 0$ only for $\lambda \in \bigcup\limits_j \Lambda_j = \Lambda_g$, where Λ_j is the set of all Fourier exponents of the almost periodic function $g(t, x_j)$, $t \in \mathbb{R}$, so that Λ_j is an at most countable set, $j = 1, 2, \ldots$, and thus also Λ_g is an at most countable set.

If the set D is a region (open connected non-void set), then there exists a monotonous sequence of compact sets $K_1 \subset K_2 \subset \ldots \subset K_m \subset \ldots \subset D$ for which $\lim K_m = D$. In such a case the equality $\Lambda_g = \bigcup\limits_m \Lambda_m$ holds, where Λ_m is the set of all Fourier exponents of the restriction of the function g on $\mathbb{R} \times K_m$, $m = 1, 2, \ldots$, and thus Λ_g is an at most countable set.

If g is locally uniformly almost periodic in the variable t on $\mathbb{R} \times D$ and D is a region, then the Fourier series $g(t, x) \sim \sum\limits_\lambda a(\lambda, x) \exp(\mathrm{i}\lambda t)$, $\lambda \in \Lambda_g$, can be uniquely determined except for its order of summation. If the function g is also analytic in the variable x on a closed ball lying in D and containing the set $\mathbb{R}_f$ of all values of the almost periodic function f, then $\Lambda_F \subset \Lambda_g + S(\Lambda_f \cup \{0\})$ is valid for the function $F(t) = g(t, f(t))$, $t \in \mathbb{R}$.

4.3 Derivatives. Now we will deal with a function $g = g(t, u, v, \varepsilon)\colon \mathbb{R} \times D = \mathbb{R} \times \mathbb{C}^{n\times 1} \times \mathbb{C}^{n\times 1} \times \bar{\kappa}_0 \to \mathbf{C}^{n\times 1}$, where $\bar{\kappa}_0 \subset \mathbb{C}$. In order to avoid complicated expressions, we will use the symbolic records of Jacobi matrices, as for example

$$\begin{aligned}
g_t &= \frac{\partial g}{\partial t} = \frac{\partial(g_1, \dots, g_n)}{\partial t} = \begin{pmatrix} \frac{\partial g_1}{\partial t} \\ \vdots \\ \frac{\partial g_n}{\partial t} \end{pmatrix}, \\
g_u &= \frac{\partial g}{\partial u} = \frac{\partial(g_1, \dots, g_n)}{\partial(u_1, \dots, u_n)} = \begin{pmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial u_1} & \cdots & \frac{\partial g_n}{\partial u_n} \end{pmatrix} = \Big(\frac{\partial g_j}{\partial u_k}\Big)_{j,k=1,\dots,n}, \\
g_{tu} &= \frac{\partial^2 g}{\partial t \partial u} = \frac{\partial(g_{1t}, \dots, g_{nt})}{\partial(u_1, \dots, u_n)} = \Big(\frac{\partial^2 g_j}{\partial t \partial u_k}\Big)_{j,k=1,\dots,n}, \\
g_{uv} &= \frac{\partial^2 g}{\partial u \partial v} = \Big(\frac{\partial^2 g_j}{\partial u_k \partial v_l}\Big)_{j,k,l=1,\dots,n},
\end{aligned}$$

where the last matrix is three dimensional. Analogously, $g_v, g_{tv}, g_{uu}, g_{vv}$ can be expressed. These Jacobi matrices will be called the derivatives of the function g. The norm of a matrix is the sum of absolute values of all its elements, for example $|g_{uv}| = \sum\limits_j \sum\limits_k \sum\limits_l |\frac{\partial^2 g_j}{\partial u_k \partial v_l}|$.

4.4. Quasilinear equations. Using the Banach contraction principle we shall deal with following quasilinear (weakly nonlinear) system

$$\begin{aligned}
\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a(t)x(t) + b(t)x(t-\tau) + f(t) \\
+ \varepsilon g(t, x(t), x(t-\tau), \varepsilon),
\end{aligned} \tag{4.1}$$

where ε is a small complex parameter. For $\varepsilon = 0$ we get the generating equation (2.1) with its conditions for a_0, b_0, a, b, f. Assume that the function $g = g(t, u, v, \varepsilon)$ together with its derivative g_t are locally uniformly almost periodic in the variable t on $\mathbb{R} \times D$, where $D = \mathbb{C}^{n\times 1} \times \mathbb{C}^{n\times 1} \times \bar{\kappa}_0$ and $\bar{\kappa}_0 = \bar{\kappa}(0, \delta_0)$, $\delta_0 > 0$, and g is analytic in the variables u, v, ε.

Put $\Lambda = S(\Lambda_f \cup \Lambda_g + S(\Lambda_a \cup \Lambda_b \cup \{0\}))$. If $\Lambda_\xi \subset \Lambda$ for a function $\xi \in AP(\mathbb{C}^{n\times 1})$, then the composite function $F(t) = F(t, \xi) = g(t, \xi(t), \xi(t-\tau), \varepsilon)$, $t \in \mathbb{R}$, is an almost periodic function whose spectrum is contained in iΛ for each $\varepsilon \in \bar{\kappa}_0$, as $\Lambda_F \subset \Lambda_g + S(\Lambda_f \cup \{0\}) \subset \Lambda_f \cup \Lambda_g + S(\Lambda \cup \{0\}) \subset \Lambda$ is valid due to the analyticity of the function g in the variables u, v. Thus the "spectrum" iΛ is wide enough in order to allow the existence of an almost periodic Λ-solution of Equation (4.1).

If a positive number R is given then the norm $\|g\|_R$ is the maximum value among the least upper bounds of magnitudes of function g and its derivatives $g_t, g_u, g_v, g_{tu}, g_{tv}, g_{uu}, g_{uv}, g_{vv}$ on the (metric) space $\Omega_R = \mathbb{R} \times \mathbb{C}_R^{n\times 1} \times \mathbb{C}_R^{n\times 1} \times \bar{\kappa}_0$, where $\mathbb{C}_R^{n\times 1} = \{w \in \mathbb{C}^{n\times 1}\colon |w| \leqslant R\}$. For any given two points $U = [t, u, v, \varepsilon]$, $\widetilde{U} = [t, \widetilde{u}, \widetilde{v}, \varepsilon]$ from the space Ω_R the inequality

$$\begin{aligned}\max\{|g(U) - g(\widetilde{U})|, |g_t(U) - g_t(\widetilde{U})|,\ |g_u(U) - g_u(\widetilde{U})|,\ |g_v(U) - g_v(\widetilde{U})|\}\\ \leqslant \|g\|_R |U - \widetilde{U}| = \|g\|_R(|u - \widetilde{u}| + |v - \widetilde{v}|)\end{aligned}$$

holds.

Theorem 4.1. *If the conditions* (3.15), (3.16), (3.17) *are fulfilled for*

$$\Lambda = S(\Lambda_f \cup \Lambda_g + S(\Lambda_a \cup \Lambda_b \cup \{0\})),$$

then for each positive number $R > A\|f\|$, where A is from the estimate (3.14), *there exists such a positive number $\varepsilon(R)$ that the equation* (4.1) *has a unique almost periodic Λ-solution x_ε with the norm $\|x_\varepsilon\| \leqslant R$ for each $\varepsilon \in \bar{\kappa}_0$ for which $|\varepsilon| < \varepsilon(R)$ holds.*

Proof. Let us consider the Banach space $H(\Lambda) = \{\xi \in AP^1(\mathbb{C}^{n\times 1})\colon\ \Lambda_\xi \subset \Lambda\}$ with the norm $\|.\|$. If a non-negative number R is given, then we define the metric closed subspace $H_R(\Lambda) = \{\xi \in H(\Lambda)\colon\ \|\xi\| \leqslant R\}$ of the space $H(\Lambda)$.

If $\xi \in H(\Lambda)$, $R \geqslant \|\xi\|$ and $\varepsilon \in \bar{\kappa}_0$, then the function

$$\gamma(t) = \gamma(t, \varepsilon) = g(t, \xi(t), \xi(t-\tau), \varepsilon),\ t \in \mathbb{R},$$

is almost periodic and belongs again to $H(\Lambda)$ and

$$|\gamma| \leqslant \|g\|_R,\ |\dot{\gamma}| = |g_t + g_u\dot{\xi}(t) + g_v\dot{\xi}(t-\tau)| \leqslant (1+2R)\|g\|_R.$$

Thus $\|\gamma\| \leqslant (1+2R)\|g\|_R$.

Define an operator $\mathcal{A} = \mathcal{A}(\varepsilon)$ on the Banach space $H(\Lambda)$ for each $\varepsilon \in \bar{\kappa}_0$ such that the operator $\mathcal{A}$ maps any function $\xi \in H(\Lambda)$ onto the function $\mathcal{A}\xi \in H(\Lambda)$, which is the unique almost periodic Λ-solution of the equation

$$\begin{aligned}\dot{x}(t) = a_0x(t) + b_0x(t-\tau) + a(t)x(t) + b(t)x(t-\tau) + f(t)\\ + \varepsilon g(t, \xi(t), \xi(t-\tau), \varepsilon)\end{aligned}$$

(uniqueness is guaranteed by Theorem 3.10) and which satisfies the estimate (3.14), i.e. $\|\mathcal{A}\xi\| \leqslant A\|f + \varepsilon\gamma\|$. Due to Corollary 3.11 the constant A is common for all functions from $H(\Lambda)$ for $\Lambda_1 = \Lambda$, $\Lambda_2 = \Lambda_a \cup \Lambda_b$ as $\Lambda' = \Lambda$. Thus the final estimate is $\|\mathcal{A}\xi\| \leqslant A[\|f\| + \varepsilon(1+2R)\|g\|_R]$.

If a positive number R is chosen such that $R > A\|f\|$, then the operator $\mathcal{A} = \mathcal{A}(\varepsilon)$ maps the space $H_R(\Lambda)$ into itself for any $\varepsilon \in \bar{\kappa}_0$ for which $|\varepsilon| \leqslant (R - A\|f\|)/((1+2R)A\|g\|_R)$.

Further, it is necessary to find out for which $\varepsilon \in \bar{\kappa}_0$ the operator $\mathcal{A} = \mathcal{A}(\varepsilon)$ is contractive. If two functions ξ, η belong to $H_R(\Lambda)$ and $\varepsilon \in \bar{\kappa}_0$ is given, then we put $\gamma_\xi(t) = g(t, \xi(t), \xi(t-\tau), \varepsilon)$ and $\gamma_\eta(t) = g(t, \eta(t), \eta(t-\tau), \varepsilon), t \in \mathbb{R}$.

The function $w = \mathcal{A}\xi - \mathcal{A}\eta$ is the unique almost periodic Λ-solution of the equation

$$\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a(t)x(t) + b(t)x(t-\tau) + \varepsilon(\gamma_\xi(t) - \gamma_\eta(t))$$

and satisfies the inequality

$$\|w\| = \|\mathcal{A}\xi - \mathcal{A}\eta\| \leqslant |\varepsilon| A \|\gamma_\xi - \gamma_\eta\| \leqslant |\varepsilon| 4(1+R)A\|g\|_R \|\xi - \eta\|,$$

as

$$|\gamma_\xi - \gamma_\eta| \leqslant 2\|g\|_R \|\xi - \eta\|, \ |\dot{\gamma}_\xi - \dot{\gamma}_\eta| \leqslant 4(1+R)\|g\|_R \|\xi - \eta\|.$$

In order to get a contractive operator $\mathcal{A}$ on $H_R(\Lambda)$ it is sufficient to put $|\varepsilon| < 1/(4(1+R)A\|g\|_R)$.

The operator $\mathcal{A}$ maps the space $H_R(\Lambda)$ into itself and turns out to be a contraction on $H_R(\Lambda)$ for $|\varepsilon| < \varepsilon(R)$, where

$$\varepsilon(R) = \min\Big\{\delta_0, \frac{R - A\|f\|}{(1+2R)A\|g\|_R}, \frac{1}{4(1+R)A\|g\|_R}\Big\}.$$

Consequently, there exists a unique function x_ε from $H_R(\Lambda)$ for $|\varepsilon| < \varepsilon(R)$, $R > A\|f\|$, such that $\mathcal{A}x_\varepsilon = x_\varepsilon$, i.e. there exists a unique almost periodic Λ-solution x_ε of Equation (4.1) for each $\varepsilon \in \bar{\kappa}_0$ if $|\varepsilon| < \varepsilon(R)$. This completes the proof of Theorem 4.2. □

Conclusion. The method developed in this paper for the construction of almost periodic solutions of almost periodic systems of differential equations can be used also for finding an approximative solution of this problem.

References

[1] *Amerio, L.; Prouse, G.*: Almost Periodic Functions and Functional Equations. N.Y. Van Nostrand Reinhold Company, 1971.

[2] *Bohr, H.*: Zur Theorie der fastperiodischen Funktionen. Acta Math. *45* (1925), 29–127; *46* (1925), 101–214; *47* (1926), 237–281.

[3] *Bochner, S.*: Beiträge zur Theorie der fastperiodeschen Funktionen. Math. Ann. *96* (1927), 119–147.

[4] *Bochner S.*: Abstrakte fastperiodische Funktionen. Acta Math. *61* (1933), 149–184.

[5] *Fink, A. M.*: Almost Periodic Differential Equations. Lecture Notes in Mathematics, Springer, New York, 1978.

[6] *Fischer, A.*: Existence of almost periodic solution of systems of linear and quasilinear differential equations with time lag. Czechoslovak Math J. *106* (1981).

[7] *Levitan, B. M.*: Almost Periodic Functions. GIZTL, Moskva, 1953. (In Russian.)

[8] *Levitan, B. M., Žikov, V. V.*: Almost Periodic Functions and Differential Equations. IMU, Moskva, 1978. (In Russian.)

[9] *Šimanov, S. N*: Almost periodic functions and differential equations with time lag. VUZ SSSR, Matematika *4(5)* (1959). (In Russian.)

Author's address: *Alexandr Fischer*, Czech Technical University, Faculty of Mechanical Engineering, Dept. of Technical Mathematics, Karlovo nám. 13, 121 35 Praha 2, Czech Republic.