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KILLING'S EQUATIONS IN DIMENSION TWO AND SYSTEMS OF FINITE TYPE

G. Thompson, Toledo[1]

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Abstract. A PDE system is said to be of finite type if all possible derivatives at some order can be solved for in terms lower order derivatives. An algorithm for determining whether a system of finite type has solutions is outlined. The results are then applied to the problem of characterizing symmetric linear connections in two dimensions that possess homogeneous linear and quadratic integrals of motions, that is, solving Killing's equations of degree one and two.

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1. Introduction

A system of partial differential equations is said to be of *finite type* if every possible derivative of some order, say r, can be solved for in terms of lower order derivatives and independent and dependent variables [1]. In more abstract terms if $\pi\colon E \to M$ is a bundle with M the space of "independent variables" and the fibres of π corresponding to the "dependent variables" a system of finite type defines a section $s\colon J^{r-1}E \to J^rE$ over $J^{r-1}E$. In case r is 1 a system of finite type is nothing but a "total differential system". Another invariant characterization of a finite type system is that its symbol should vanish.

Many of the PDE systems that occur in the classical problems of differential geometry are of finite type or become so after a number of prolongations. For example the problem of determining a metric that is compatible with a given symmetric

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connection, deciding whether a connection admits a parallel field of vectors and Killing's equations for the existence of a homogeneous polynomial integral of the geodesic flow all fit into the finite type scheme.

The main purpose of the present article is to investigate two of the problems just mentioned for symmetric linear connections in dimension two. Section 3 investigates the existence of parallel vector and line element fields. The equations determining the existence of a parallel vector field do fit into the finite system scheme but it is not necessary to use the theory developed in Section 2. Section 4 is concerned with Killing's equations for degree one integrals and Section 5 and 6 for integrals of degree two. In Section 2 we outline an algorithm for determining whether a system of finite type has a solution. The study of Killing's equations in Section 4, 5 and 6 is of interest in its own right and serves as an illustration of the scope and limitations of the theory developed in Section 2.

In Section 3 we use mainly coordinate free language whereas Sections 4 and 5 are entirely local in nature, it being understood that all calculations are carried out in a coordinate chart (x^i) on the two dimensional smooth manifold M. We use R and K to denote the Riemann curvature and Ricci tensors associated to a symmetric linear connection ∇ on M.

2. Solutions to systems of finite type

Suppose that a^A denote the dependent variables of some PDE system and that the independent variables are (x^i). Suppose also that $1 \leqslant i \leqslant n$ and $1 \leqslant A \leqslant m$. We assume that the PDE system is of finite type and that $r = 3$ in the definition given in Section 1. This latter assumption is purely for the purpose of simplifying the exposition and is in no way an essential restriction.

We write the PDE system in the form

$$a^A_{ijk} = f^A_{ijk}, \tag{2.1}$$

where the left hand side of (2.1) represents a third order derivative of a^A and the function f^A_{ijk} contains at most x^i, a^A, a^a_j and a^a_{jk}. We assume that there are precisely $m\binom{n+3}{4}$ equations of the form (2.1). We could also have some equations of order less than three but none of order greater than three. We assume also that the f^A_{ijk} are smooth i.e. infinitely differentiable functions again for the sake of simplifying the exposition.

Now differentiate (2.1) with respect to x^l. We obtain

$$a^A_{ijkl} = f^A_{ijkl} + \frac{\partial f^A_{ijk}}{\partial a^\beta} a^\beta_l + \frac{\partial f^A}{\partial a^\beta_m} a^\beta_{lm} + \frac{\partial f^A_{ijk}}{\partial a^\beta_{mn}} a^\beta_{lmn} \tag{2.2}$$

[The summation convention over repeated indices applies in (2.2).] Next form the analogous expression for a^{a}_{ijlk} and equate the corresponding right hand sides. Finally replace a^{β}_{mnl} and a^{β}_{mnk} by their values as given by (2.1). The result is an equation which contains at most x^i, a^i, a^i_j, and a^i_{jk}.

The next stage of the process consists of choosing a collection of *independent* second order conditions generated by the method described above using all equations of the form (2.1). By "independent" we mean functional independence with respect to second order derivatives, taking into account that the original PDE system may contain some equations of purely second order. It is also possible that the new second order conditions could be manipulated so as to obtain first order or even zeroth order conditions i.e. relations among the a^A's. Again we add to the PDE system only relations which are independent of equations already contained in the system.

At this stage various possibilities may occur. First of all, it is conceivable that the "new" equations are algebraically inconsistent with the original system. In this case there are no solutions and the algorithm is finished. Secondly it is possible that the augmented system is not inconsistent and becomes of finite type of order two. In this case we iterate the procedure to try to produce new first or zeroth order conditions. Thirdly the new conditions may be satisfied identically or by virtue of equations occurring in the original system which we have tacitly assumed is not algebraically inconsistent.

In this latter case the general solution depends on a certain number of arbitrary constants. The easiest way to understand the number of such constants is to imagine developing a power series solution to the PDE system, which is not to say that the theory is restricted only to real analytic systems. The number of arbitrary constants is the number of "free" parameters in a Taylor series where some of the first and second order derivatives are solved for in terms of derivatives of order no higher than themselves using the implicit function theorem if necessary.

A fourth possibility is that by differentiating some of the zeroth or first order conditions, either "new" equations or ones originally there, one can produce a system of finite type of order two, one or conceivably zero. In the first case one returns to the original procedure to produce new zeroth or first order conditions. In the second case one differentiates the first order conditions and uses these second order equations in conjunction with second order equations already present to produce yet new first or zeroth order conditions. Again in the third case where all the dependent variables are determined one still has to look at all the derivatives of first and second order and check for consistency with conditions already contained in the system. The general principle to bear in mind is that if at some stage we arrive at a system of finite type or order less than the original order, we have to explore all possible consequences of this new lower order system and check for consistency with equations already present in the system. Once again one returns to the original procedure to try to produce new zeroth or first conditions.

The fifth and final possibility is that none of the remaining four cases occur. One then proceeds as follows. One differentiates all the zeroth, first and second order equations of the augmented PDE system (containing the "new" equations but not the third order equations (2.1)). These differentiated equations are converted into second order equations where necessary by means of eq. (2.1). The algorithm terminates now if all the differentiated equations are algebraically dependent on the zeroth, first and second order equations that were used to determine them. If a new condition is produced so that the algorithm does not terminate, it is added to the system. Eventually, for any "reasonable" system of PDE (quasi-linear or real analytic systems for example) the process will stabilize and produce either no new conditions or a system of finite type of order two in which case the procedure begins all over again.

We refer to the step in case five where one differentiates to check consistency of the system as the "one more derivative phenomenon" and the reader will see it appear in Section 4 and 5. The actual existence of solutions follows from the Frobenius theorem. The "one more derivative procedure" amounts to checking the integrability conditions of that theorem, case three above being a special case. Also, in practice it is not always necessary to eliminate derivatives explicitly; it is convenient sometimes to keep derivatives present in equations and view them as purely algebraic quantities. An example of this situation occurs following eq. (4.16) in Section 4 where we do not eliminate the derivative $a_{1;2}$.

A further point to note is that a PDE system may not be of finite type as it stands. It may require several differentiations to convert it into a system of finite type. Killing's equations of degree n determine whether a homogeneous integral of degree n in velocities exists for the geodesic flow of a given symmetric connection. Killing's equations require n differentiations to appear as a system of finite type [2], [3].

Finally, as the theory of finite type systems relates to problems in differential geometry we are usually dealing there with covariant rather then ordinary partial derivatives. It is much more natural to work with these covariant derivatives the major difference being that covariant derivatives are commuted by means of Ricci's identities. Keeping this structure opens up the possibility of interpreting various algebraic conditions geometrically, tensorially or invariantly. For example in Section 5 a major dichotomy in the theory occurs according as the Ricci tensor K is or is not symmetric.

3. Parallel line distributions and vector fields

A vector field X on M is said to be *recurrent* if when X is covariantly differentiated in any direction the result is a multiple of X. A more convenient formulation is that there exists a one-form θ on M such that

$$\nabla X = \theta \otimes X. \tag{3.1}$$

From (3.1) we find the following condition on the curvature R of ∇, for all vector fields Y and Z:

$$R(Y,Z)X = \mathrm{d}\theta(Y,Z)X. \tag{3.2}$$

If θ is closed then X may be scaled by a function so as to obtain a parallel vector field in the small. Thus if we are addressing the issue of whether M possesses a parallel vector field X we see that a first necessary condition is

$$R(Y,Z)X = 0. \tag{3.3}$$

Since X is assumed to be parallel, differentiating (3.3) in the direction of a vector field W gives

$$\nabla_W R(Y,Z)X = 0. \tag{3.4}$$

The process can be iterated to yield a sequence of homogeneous linear conditions on X with coefficients that are higher covariant derivatives of R. It may now be shown that the necessary and sufficient condition for the existence of parallel X is that there should exist a positive integer r such that the conditions at order $r+1$ are consistent with the totality of conditions of order $0, 1, \ldots, r$ [4, 5]. This Theorem provides another example of the "one more derivative" phenomenon.

Let us now specialize to the case where the dimension of M is two. Then it is well known that M admits two linearly independent parallel vector fields if and only if ∇ is flat. We shall formulate conditions for M to have a single linearly independent vector field. Recall that in dimension two R and the Ricci tensor K are related by

$$R(X,Y)Z = K(Z,X)Y - K(Z,Y)X \tag{3.5}$$

for all vector fields X, Y and Z. From (3.3) it follows that

$$K(X,Y) = 0 \tag{3.6}$$

where X is parallel and Y is arbitrary. Differentiating (3.6) along the field W gives

$$(\nabla_W K)(X,Y) = 0. \tag{3.7}$$

Since we are supposing that M has just one linearly independent vector field it is unnecessary to consider further derivatives of (3.7).

The conditions for the existence of the parallel vector field may be reformulated as follows:

Proposition 3.1. *∇ has a parallel vector field X if and only if X satisfies* (3.6) *and in addition*

$$\nabla K = \varphi \otimes K + \theta \otimes \alpha \otimes \alpha \tag{3.8}$$

where φ and θ are fixed one-forms and α denotes the form $K(-,X)$.

Proof. Remark first of all that the structure equation (3.8) expresses the equality of two (0,3) tensors. Clearly (3.8) and (3.6) entail (3.7). Conversely, for any vector field W, eq. (3.7) implies that in components the matrix representing $\nabla_W K$ has a zero first row, where the first coordinate is chosen so as to provide a flowbox for X in (3.6). Furthermore, if K is symmetric then so too will be $\nabla_W K$ and hence they will be dependent. Thus (3.8) will be satisfied with $\theta = 0$. Otherwise, if K is not symmetric, K and $\alpha \otimes \alpha$ must be linearly independent and it must be possible to write ∇K in the form (3.8). □

We now turn to the situation where ∇ has a single linearly independent parallel line field.

Proposition 3.2. *∇ has a parallel line field if and only if one of the following two situations occurs:*

(i) *There exists a vector field X on M that satisfies for all vector fields Y on M*

$$K(X,X) = 0 \tag{3.9a}$$

and

$$(\nabla_Y K)(X,X) = 0. \tag{3.9b}$$

(ii) *For all vector fields X and Y on M*

$$K(X,Y) + K(Y,X) = 0 \tag{3.10}$$

P r o o f. (i) We leave the necessity of (i) and (ii) to the reader. Conversely, we choose coordinates (x^i) so that $\frac{\partial}{\partial x^1}$ is a flow-box for X. Then (3.9a) gives that K_{11} is zero. Next, computing (3.9b) locally we find that:

$$K_{11;j} = -\Gamma^2_{ij}(K_{12} + K_{21}). \tag{3.11}$$

In view of (ii) we may assume that $K_{12} + K_{21}$ is non-zero and so Γ^2_{11} and Γ^2_{12} are zero by virtue of (3.9b). However, the latter conditions ensure precisely that $\frac{\partial}{\partial x^1}$ spans a parallel line field. In case (ii) K is skew-symmetric. But in this case according to [6] coordinates may be introduced on M relative to which the geodesic equations may be written as

$$\ddot{x} = -c_x \dot{x}^2, \qquad \ddot{y} = c_y \dot{y}^2 \tag{3.12}$$

where c is some function of x and y from which the conclusion of the Proposition is apparent. Indeed both x^1 and x^2 determine parallel line fields.

We next consider the case where ∇ has two linearly independent parallel line fields. □

Proposition 3.3. *∇ has two linearly independent parallel line fields if and only if there exist linearly independent vector fields X and Y satisfying*

$$K(X, X) = K(Y, Y) = 0$$
$$(\nabla_W K)(X, X) = \nabla_W(Y, Y) = 0.$$

P r o o f. Apply proposition 3.2 (i) to both X and Y. We do not claim that X and Y are coordinate fields. □

The last step we take in the direction of parallel line fields is as follows:

Proposition 3.4. *∇ has a parallel vector field and parallel line field if and only if there exist vector fields X and Y satisfying*

$$K(X, X) = K(X, Y) = K(Y, Y) = 0$$

and

$$\nabla K = \theta \otimes K$$

for some fixed one-form θ.

P r o o f. Again we omit the necessity. For the sufficiency note that the conditions on X and Y imply that X satisfies (3.6) and the recurrence property of K implies that X also satisfies (3.7). Thus by scaling, X may be assumed to be parallel. Similarly Y satisfies the conditions of Proposition (3.3) and so spans a parallel line field. □

4. The solution of Killing's equations of degree one

Irrespective of whether or not ∇ is engendered by a metric, one may formulate Killing's equations in the form

$$a_{i;j} + a_{j;i} = 0, \tag{4.1}$$

signifying that $a_i \dot{x}^i$ is a first integral of the geodesics of ∇.

If we covariantly differentiate (4.1) once, perform Christoffel elimination with the help of Ricci's identities we obtain

$$a_{i;jk} + a_n R^n_{kij} = 0. \tag{4.2}$$

It follows that system (3.1) is of finite type and that the most general solution of (3.1) for a given ∇ can depend on at most $n\binom{n}{2}$ constants, where M has dimension n.

Now specialize to the case where n is 2 and use eq. (3.5) to rewrite (4.2) as

$$a_{i;jk} + a_j K_{ki} - a_k K_{ji} = 0. \tag{4.3}$$

Differentiate (4.3) once more, express $a_{i;jkl} - a_{i;jlk}$ by means of Ricci's identities and use (4.3) again to eliminate second order derivatives:

$$\begin{aligned} a_{j;k}(K_{li} - K_{il}) + a_{i;k}(K_{jl} - K_{lj}) + a_{i;l}(K_{kj} - K_{jk}) \\ + a_{j;l}(K_{ik} - K_{ki}) + a_i B_{kjl} + a_j B_{lik} = 0. \end{aligned} \tag{4.4}$$

In (4.4) B_{kjl} is defined by

$$B_{kjl} = K_{kj;l} - K_{lj;k}. \tag{4.5}$$

In considering (4.4) the first possibility is that it is satisfied identically, in which case

$$K_{ij} - K_{ji} = 0, \tag{4.6}$$

$$B_{kjl} = 0. \tag{4.7}$$

Conditions (4.6) and (4.7) reproduce a well known classical result which is valid equally in dimension n: the geodesics of ∇ possess the maximum number $\binom{n+1}{2}$ of linearly independent first integrals if and only if the Ricci tensor is symmetric and ∇ is projectively flat [5].

If (4.4) is not satisfied identically there are two subcases according as (4.6) is or is not valid. Suppose first of all that (4.6) does not hold. Then (4.4) is a first order condition. Note that since (4.4) is skew-symmetric in both i and j and k and l, respectively, it constitutes a single condition. Thus (4.1) and (4.4) imply that all four first order derivatives are determined uniquely. Again we differentiate (4.4) and use (4.3) to eliminate second order derivatives. We write out the result, as well as (4.4) itself, but this time the indices assuming the numerical values 1 and 2:

$$B_{221}a_1 - B_{211}a_2 - 4Q_{12}a_{1;2} = 0 \tag{4.8}$$

$$\lceil B_{2211} - 4Q_{12}K_{21}\rceil a_1 - \lceil B_{2111} - 4Q_{12}K_{11}\rceil a_2 + \lceil B_{211} - 4Q_{12;1}\rceil a_{1;2} = 0 \tag{4.9}$$

$$[B_{1222} + 4Q_{12}K_{22}]a_1 - [B_{1122} + 4Q_{12}K_{12}]a_2 + [B_{122} + 4Q_{12;2}]a_{1;2} = 0 \tag{4.10}$$

In the above equations Q_{ij} denotes the skew-symmetric part of K_{ij} and we are presently assuming that Q_{ij} is non-zero. If (4.8) is not satisfied identically but (4.9) and (4.10) are each proportional to (4.8) then there exists a one-form θ that satisfies:

$$B_{ijk;l} + 4Q_{ik}K_{jl} = \theta_1 B_{ijk} \tag{4.11}$$

$$B_{ijk} + 4Q_{ik;j} = 4\theta_j Q_{ik}. \tag{4.12}$$

The existence of the one-form θ satisfying (4.11) and (4.12) is equivalent to ∇ having two linearly independent first integrals. The conditions may also be formulated without reference to the auxiliary one-form θ as

$$C_{ijklmn} = 0 \tag{4.13}$$

where the tensor C is defined by

$$C_{ijklmn} = 16Q_{ik}Q_{mn}K_{jl} + 4Q_{mn}B_{ijk;l} - 4Q_{mn;l}B_{ijk} - B_{ijk}B_{mln} \tag{4.14}$$

If (4.13) does not hold there are several other possibilities. The first of these possibilities is that the 3×3 matrix of coefficients formed from (4.10)–(4.12) is non-singular. In this case there is no non-zero linear first integral associated to ∇. The fact that the matrix of coefficients is singular, an obvious necessary condition for the existence of a non-zero linear first integral, may be expressed in tensorial form as

$$B_{i[j\widehat{k}}\overline{B}_{\widehat{p}q]r;[s}B_{\widehat{l}m]n} + B_{iqk}Q_{ln;[m}B_{\widehat{p}\widehat{j}\widehat{r};s]} + Q_{ln}\overline{B}_{ijk;[m}\overline{B}_{\widehat{p}\widehat{q}\widehat{r},s]} = 0 \tag{4.15}$$

where $\overline{B}$ is defined by

$$\overline{B}_{ijk;l} = B_{ijk;l} + Q_{ik}K_{jl}. \tag{4.16}$$

A point to be noted about (4.8) is that we could use it in conjunction with (4.1) to eliminate all first order derivatives. However, it appears easier to formulate conditions on a_1, a_2 and $a_{1;2}$. Because of (4.1) and (4.3) any derivative of (4.8) will yield a similar linear homogeneous expression in these three variables.

The most difficult case arising from (4.8)–(4.10) is where the matrix of coefficients has rank two. In this case there can only be one linearly independent linear first integral for ∇. To see if there is one, we obtain the purely algebraic equation from (4.8)–(4.10):

$$C_{jklmnp}a_i - C_{iklmnp}a_j = 0 \tag{4.17}$$

Note that (4.17) comprises two conditions. We are assuming that the linear homogeneous system has rank one. We covariantly differentiate (4.17) and (4.8) to eliminate first order derivatives thereby obtaining

$$\begin{aligned}(4Q_{rs}C_{jklmnp;q} + C_{jklmnp}B_{sqr})a_i - (4Q_{rs}C_{iklmnp;q} + C_{iklmnp}B_{sqr})a_j\\ + (C_{jklmnp}B_{ris} - C_{iklmnp}B_{rjs})a_q = 0.\end{aligned} \tag{4.18}$$

In order that there should exist one linearly independent solution to (4.1) it is necessary and sufficient that the linear system consisting of (4.17) and (4.18) should have rank one. Consequently the $\binom{6}{2} = 15$ determinants of all possible 2×2 matrices formed from the coefficients in (4.17) and (4.18) must be zero.

Let us now return to the case where (4.6) is satisfied. Then (4.4) reduces to a purely algebraic equation. If we covariantly differentiate (4.4) then again we obtain (4.9) and (4.10) with Q set to zero. Thus the simplified form of (4.15) is again a necessary condition for the existence of a linear integral. Notice that it is no longer possible to have two linearly independent first integrals.

Assuming then that (4.15) holds we must differentiate once more to see whether an integral indeed exists. In principle there result new linear conditions. However, it turns out that the derivative of (4.9) in the $\frac{\partial}{\partial x^1}$-direction yields the same equation as (4.10) differentiated in the $\frac{\partial}{\partial x^2}$-direction, apart from a sign. The two may be consolidated into a single more symmetric condition that may be simplified by using (4.4) and written as

$$B_{122;21}a_1 + B_{211;12}a_2 + (B_{221;1} - B_{112;2})a_{1;2} = 0. \tag{4.19}$$

The other conditions arise as the "one" derivative of (4.9) and the "two" derivative of (4.10).

We finally arrive at a system of six homogeneous linear equations for the three unknowns a_1, a_2 and $a_{1;2}$. The necessary and sufficient condition for the existence of a linear integral for ∇ is that all $20 = \binom{6}{3}$ of the 3×3 determinants of the system should be zero. However, it turns out that these 20 conditions can be reduced to just two, one of which is (4.15) with Q set to zero!

To understand the latter remarks note first of all that (4.15) is the only one of the 20 conditions that involves just first order derivatives of B. After deleting redundant equations that are merely linear combinations of others and using (4.15) to eliminate other equations algebraically from the list one may reduce to just four conditions one of which is (4.15) and the remaining three are:

$$\begin{aligned}(B_{112}B_{112;2} + B_{112;1}B_{221})B'_{221;21} &- (B_{112}B_{221;2} + B_{221}B_{221;1})B'_{112;12} \\ &- (B_{112;2}B_{221;1} - B_{112;1}B_{221;2})(B_{112;2} - B_{221;1}) = 0\end{aligned} \tag{4.20}$$

$$\begin{aligned}(B_{112}B_{112;2} + B_{221}B_{112;1})B'_{221;11} &- (B_{112}B_{221;2} + B_{221}B_{221;1})B'_{112;11} \\ &+ 2B_{112;1}(B_{112;1}B_{221;2} - B_{112;2}B_{221;1}) = 0\end{aligned} \tag{4.21}$$

$$\begin{aligned}(B_{221}B_{221;1} + B_{112}B_{221;2})B'_{112;22} &- (B_{221}B_{1112;1} + B_{112}B_{112;2})B'_{221;22} \\ &+ 2B_{221;2}(B_{112;1}B_{221;2} - B_{112;1}B_{221;1}) = 0\end{aligned} \tag{4.22}$$

where B'_{ijklm} is defined as $B_{ijklm} + B_{ijk}K_{lm}$.

Observe however that if (4.20) holds then (4.21) and (4.22) follow by differentiating (4.15). It is in this sense that the twenty conditions can be reduced to just (4.15) and (4.20).

Conditions (4.20)–(4.22) can be written in tensorial form as

$$\text{SKEW}(i \leftrightarrow n, l \leftrightarrow q) \quad B_{hij}B_{klm;n}[B_{pqr;(st)} + B_{p(s\hat{r}}K_{t)q}] + B_{p(s\hat{r},t)}B_{hij;q}B_{knm;l} = 0 \tag{4.23}$$

where the round parentheses denote symmetrization over the indices s and t and SKEW denotes skew-symmetrization over the indices i and n and l and q, respectively.

The following Theorem summarizes the various possibilities that can occur when ∇ possesses a linear first integral.

Theorem 4.1. *∇ possesses the following number of linearly independent integrals:*

(i) 3 *if and only if K_{ij} is symmetric and B_{ijk} is zero*
(ii) 2 *if and only if K_{ij} is not symmetric but C_{ijklmn} is zero*
(iii) 1 *if and only if either* (a) *or* (b) *occurs where*

(a) K_{ij} *is not symmetric,* C_{ijklmn} *is not zero but the linear system consisting of* (4.17) *and* (4.18) *has rank* 1

(b) K_{ij} *is symmetric,* B_{ijk} *is not zero but* (4.15) *with* Q_{ij} *set to zero and also* (4.23) *hold, where only the component corresponding to* (4.20) *need be checked.*

The above Theorem may be compared with a theorem of Levine [7] whose investigations begin in the same way. Levine introduces many relative tensors whose vanishing characterizes connections that have a linear integral of motion. The invariant meaning of all these relative tensors is, however, not entirely transparent.

5. The solution of Killing's equations of degree 2

In this section we enquire whether ∇ has a homogeneous quadratic integral $a_{ij}\dot{x}^i\dot{x}^j$. In this case Killings's equations assume the form

$$a_{ij;k} + a_{ki;j} + a_{jk;i} = 0. \tag{5.1}$$

It is possible to show that all third order derivatives of a_{ij} may be solved for explicitly in terms of lower order derivatives, thus showing that (5.1) is of finite type. Indeed this result holds true when M is of dimension n [2, 3]. We shall give the details assuming that n is 2.

Writing out (5.1) explicitly for $n = 2$ gives:

$$a_{111} = 0 \tag{5.2}$$

$$a_{112} + 2a_{121} = 0 \tag{5.3}$$

$$a_{221} + 2a_{212} = 0 \tag{5.4}$$

$$a_{222} = 0 \tag{5.5}$$

Note that (5.2)–(5.5) comprises four equations for the first derivatives of a_{ij}. In (5.2)–(5.5) we suppress the semi-colon for the covariant derivative of a_{ij} and shall use it only when it is absolutely necessary. We shall adhere to this convention for higher order derivatives of a and K throughout this section. Covariantly differentiating (5.2)–(5.5) once and making use of Ricci's identities gives:

$$a_{111i} = a_{222i} = 0 \tag{5.6}$$

$$a_{1121} = -2a_{1211} = 2K_{12}a_{11} - 2K_{11}a_{12} \tag{5.7}$$

$$a_{2212} = -2a_{2122} = 2K_{21}a_{22} - 2K_{22}a_{12} \tag{5.8}$$

$$a_{1122} + 2a_{1212} = 0 \tag{5.9}$$

$$a_{2211} = 3a_{1221} = 0 \tag{5.10}$$

In (5.7) and (5.8) K_{ij} denotes the Ricci tensor of ∇ and collectively (5.6)–(5.10) comprise eight equations for the nine possible second order derivatives of a_{ij}.

If we differentiate (5.6)–(5.10) again and employ Ricci's identities then all third order derivatives of a may be solved for as

(5.11) $a_{111jk} = 0$

(5.12) $a_{11211} = -2a_{12111} = 2K_{121}a_{11} - 2K_{111}a_{12} - 2K_{11}a_{121}$

(5.13) $a_{11212} = -2a_{12112} = 2K_{122}a_{11} - 2K_{112}a_{12} + 2K_{12}a_{112} - 2K_{11}a_{122}$

(5.14) $a_{11221} = -2a_{12121} = 2[K_{122}a_{11} - K_{112}a_{21}] + [2K_{21} - 8K_{12}]a_{121} - 4K_{11}a_{212}$

(5.15) $$\begin{aligned} a_{11222} = -2a_{12122} = 2[&K_{222}a_{11} + (K_{122} - K_{221} - K_{212})a_{12} \\ &+ (K_{211} - K_{112})a_{22} - (5K_{21} - 2K_{12})a_{122} - 4K_{22}a_{211}] \end{aligned}$$

and all the remaining derivatives maybe obtained by transposing the coordinates.

Following the general theory for systems of finite type we covariantly differentiate each third order derivative and "equate mixed partials" by means of the Ricci identities. In principle we have to construct equations corresponding to $a_{ijkl[12]}$ where $ijkl$ assume the following values 1111, 1112, 1121, 1122, 1211, 1212 and values obtained by transposing coordinates. Of these values one easily checks that the first two do not lead to new conditions but are identities. Corresponding to the value 1121 we obtain the following condition:

$$(K_{12} - K_{21})(a_{1121} - 2K_{12}a_{11} + 2K_{11}a_{12}) = 0, \tag{5.16}$$

and likewise for the value 1211

$$(K_{12} - K_{21})(a_{1211} + K_{12}a_{11} - K_{11}a_{12}) = 0. \tag{5.17}$$

Thus by virtue of (5.7), both (5.16) and (5.17) are identities.

Turning next to the value 1212 we obtain the condition:

$$\begin{aligned} &[K_{2212} - K_{1222}]a_{11} + [K_{1122} - K_{2112} + K_{1221} - K_{2211}]a_{12} \\ &-[K_{1121} - K_{2111}]a_{22} + [7K_{122} - 5K_{221} - 2K_{212}]a_{121} \\ &-[7K_{211} - 5K_{112} - 2K_{121}]a_{122} + 8[K_{12} - K_{21}]a_{1212} \\ &+2[K_{12} - K_{21}][3K_{22}a_{11} + (K_{12} - 3K_{21}))a_{12} - K_{11}a_{22}] = 0. \end{aligned} \tag{5.18}$$

Essentially the same condition arises from the values 1122 and 2121 and 2211. In fact the only asymmetry in (5.18) derives from the term involving a_{1212}. Thus (5.18) is the only new integrability condition obtained by equating mixed partial derivatives of order three. Notice also that (5.18) is satisfied identically if (4.6) and (4.7) hold. In this case we obtain a special case of a well-known result [3] and the degree two Killing tensors consist of sums of symmetrized Killing vectors of degree one.

Notice that only one second order derivative is undetermined by virtue of (5.6–5.10). Thus if K_{ij} is not symmetric (5.18) entails that all second order derivatives are determined. On the other hand if K_{ij} is symmetric, then (5.18) reduces to a first order condition provided that (4.7) is not satisfied.

Let us proceed by assuming that K_{ij} is symmetric. Then (5.18) reduces, on introducing B_{ijk} defined by (4.5), to

$$B_{221;2}a_{11} + (B_{112;2} - B_{221;1})a_{12} - B_{112'1}a_{22} + 5B_{122}a_{121} - 5B_{211}a_{212} = 0. \tag{5.19}$$

According to the general theory we now form the "1" and "2" derivatives of (5.19). We find that

$$\begin{aligned}\widetilde{B}_{112;12}a_{22} + [B_{221;12} + \widetilde{B}_{21122}]a_{12} - B_{22122}a_{11} + 7B_{221;2}a_{12;1}\\ + [6B_{211;2} + B_{221;1}]a_{12;2} - 5B_{122}a_{21;12} = 0\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}\widetilde{B}_{221;21}a_{11} + [B_{112;21} + \widetilde{B}_{122;11}]a_{12} - B_{11211}a_{22} + 7B_{112;1}a_{21;2}\\ + [6B_{122;1} + B_{112;2}]a_{21;1} - 5B_{211}a_{12;21} = 0,\end{aligned} \tag{5.21}$$

where $\widetilde{B}_{ijk;lm}$ is defined to be $B_{ijk;lm} + 5B_{ijk}K_{lm}$.

Now since we are assuming that (4.7) does not hold, not both of (5.20) and (5.21) can be satisfied identically. In fact all second order derivatives of a_{ij} are determined. We must now differentiate each of (5.20) and (5.21) in the "1" and "2" directions. However, it turns out that the "1" derivative of (5.20) gives the same condition as the "2" derivative of (5.21). Thus (5.20) and (5.21) produce only the following three new conditions at the next stage:

$$\begin{aligned}&- [B_{221;222} + 5B_{221}K_{222}]a_{11} + [B_{221122} - \widetilde{B}_{1112222}(B_{2211} - 6B_{1122})K_{22}\\ &+ 5B_{221}K_{221}]a_{12} + [\widetilde{B}_{112122} + 5B_{112}B_{221} + (6B_{1122} - B_{2211})K_{12}]a_{22}\\ &+ [9B_{22122} + 20B_{221}K_{22}]a_{121} + [2B_{22112} - 7B_{11222} + 15B_{221}K_{12}\\ &- 5B_{112}K_{22}]a_{122} + 12B_{2212}a_{1212} = 0\end{aligned} \tag{5.22}$$

$$
\begin{aligned}
&[B_{221211} + 5B_{112}B_{221} + (6B_{2211} - B_{1122})K_{12} + 12B_{1121}K_{22]}]a_{11} \\
&+ [B_{112211} - B_{221111} + 5B_{112}K_{112} + (B_{1122} - 6B_{2211})K_{11}]a_{12} \\
&- [B_{112111} + 5B_{112}K_{111} + 12B_{1121}K_{11}]a_{22} + [2B_{11221} - 7B_{22111} \\
&+ 15B_{112}K_{12} - 5B_{221}K_{11}]a_{121} + [9B_{11211} + 20B_{112}K_{11}]a_{12} \\
&+ 12B_{1121}a_{1212} = 0
\end{aligned} \tag{5.23}
$$

$$
\begin{aligned}
&[B_{221221} + 7B_{2212}K_{12} + 5B_{221}K_{212} + (6B_{1122} - B_{2211})K_{22}]a_{11} \\
&+ [B_{112212} - B_{221121} + 5B_{112}K_{221} - 5B_{221}K_{112} + 6B_{1121}K_{22} \\
&- 6B_{2212}K_{11}]a_{12} - [B_{112112} + 7B_{1121}K_{12} + 5B_{112}K_{121} \\
&+ 5B_{112}K_{11}]a_{22} + [B_{11222} - 8B_{22121} + 4B_{112}K_{22} \\
&- 16B_{221}K_{12}]a_{121} - [B_{22111} - 8B_{11212} + 4_{221}K_{11} \\
&- 16B_{112}K_{12}]a_{122} + 6(B_{1122} - B_{2211})a_{1212} = 0.
\end{aligned} \tag{5.24}
$$

Equations (5.19)–(5.24) constitute a system of six equations for six unknowns. Thus a necessary condition for the existence of a quadratic integral is that the associated 6×6 matrix should be singular. Furthermore it is clear just from (5.19)–(5.21), that if (4.7) is not satisfied that the 6×6 matrix has rank at least two. However, we claim that it actually must have rank of at least three if a nonzero quadratic integral is to exist.

The proof of the preceding remark proceeds as follows. Suppose that the matrix of coefficients of the unknowns a_{11}, a_{12}, a_{22}, a_{121}, a_{122}, a_{1212} has rank two. Use the top three rows and last three columns to write down the determinants of various 3×3 minors. The upper right 3×3 block gives a condition on just B_{ijk} and B_{ijkl} which is found to be the same as (4.15) with Q_{ij} set to zero. From the remaining 3×3 minors one finds the following conditions that are linear in B_{ijklm}: in each case we have used either the last two columns or top two rows of the upper right 3×3 block in the 6×6 matrix in question:

Accordingly

$$
\begin{aligned}
&5B_{112}B_{221}B_{11211} + 5(B_{112})^2 B_{11212} \\
&\qquad + B_{1121}[B_{112}(B_{2211} - 6B_{1122}) - 7B_{1121}B_{221}] = 0
\end{aligned} \tag{5.25}
$$

$$
\begin{aligned}
&5B_{112}B_{221}B_{22121} + 5(B_{112})^2 B_{22122} \\
&\qquad + B_{2212}[B_{112}(B_{221} - 6B_{1122}) - 7B_{1121}B_{221}] = 0
\end{aligned} \tag{5.26}
$$

$$
\begin{aligned}
&5B_{112}(B_{221})^2B_{11212} - 5B_{112}(B_{221})^2B_{22111} + 5(B_{112})^2B_{221}B_{11211}\\
&\qquad - 5(B_{112})^2B_{221}B_{22121} + 5B_{112}(B_{221})^3K_{11} - 5(B_{112})B_{221}K_{22}\\
&\qquad + (B_{1122} - B_{2211})(B_{112}(B_{2211} - 6B_{1122}) - 7B_{1121}(B_{221})) = 0
\end{aligned} \tag{5.27}
$$

$$
\begin{aligned}
&35(B_{221})^2B_{11222} - 10(B_{221})^2B_{22121} - 45B_{112}B_{221}B_{22122}\\
&\quad + 12B_{2212}B_{2211}B_{221} - 72B_{2212}B_{1122}B_{221} + 84B_{112}(B_{2212})^2\\
&\quad - 35(B_{221})^3K_{12} - 35B_{112}(B_{221})^3K_{22} = 0
\end{aligned} \tag{5.28}
$$

$$
\begin{aligned}
&-45(B_{221})^2B_{11211} - 10B_{112}B_{221}B_{11212} + 35B_{112}B_{221}B_{22111}\\
&\quad + 12B_{1121}B_{2211}B_{221} - 72B_{1121}B_{1122}B_{221} + 84B_{112}B_{1121}B_{2212}\\
&\quad - 35B_{112}(B_{221})^2K_{11} - 35B_{112}B_{221}K_{12} = 0
\end{aligned} \tag{5.29}
$$

$$
\begin{aligned}
&-40(B_{221})^2B_{11212} - 5B_{112}B_{221}B_{11222} + (B_{221})^2B_{22111}\\
&\quad + 40B_{112}B_{221}B_{22121} - 5(B_{221})^2K_{11} + 5(B_{221})^2B_{221}K_{22}\\
&\quad + 6(B_{1122} - B_{2211})(7B_{112}B_{2212} + B_{221}(B_{2211} - 6B_{1122})) = 0.
\end{aligned} \tag{5.30}
$$

In fact (5.25)–(5.30) are not independent because $(2B_{112} - 9B_{221})9B_{221}(5.25) + (9(B_{112})^2 - 2(B_{221})^2)9B_{221}(5.26) + 63B_{112}B_{221}(5.27) + (9(B_{112})^3 - 2B_{112}(B_{221})^2)(5.28) + 2((B_{112})^3 - 9B_{112}(B_{221})^2)(5.29) + 14(B_{112})^2B_{221}(5.30)$ is zero.

Now (5.27) and (5.30) together with (4.15) imply (4.20). Similarly (4.21) and (4.22) may be obtained. Thus the assumption that the 6×6 matrix corresponding to equations (5.19)–(5.24) has rank two entails that (4.15) and (4.23) are satisfied.

Accordingly in this case there must exist a linear integral of motion. The argument to show that rank two of the 6×6 matrix is impossible will be completed in a new section.

6. Connections with degree one integrals again

We saw above that if the 6×6 matrix corresponding to eqs. (5.19–24) has rank two and if a quadratic integral exists then necessarily the connection has a linear integral that may be identified with a one-form on M. There are two local coordinate normal forms for ∇ depending on whether the one-form is of form $\mathrm{d}y$ or $x\,\mathrm{d}y$, namely

$$\ddot{x} = -(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2), \quad \ddot{y} = 0 \tag{6.1}$$

or

$$\ddot{x} = -(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2), \quad \ddot{y} = -\frac{\dot{x}\dot{y}}{x} \tag{6.2}$$

Indeed in both (6.1) and (6.2) the form maybe further simplified by the requirement that K_{ij} be symmetric which entails the existence of a function φ such that a and b are given by φ_x and φ_y respectively. However, the connection in (6.2) is projectively related in the sense that

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \varrho_k + \delta^i_k \varrho_j \tag{6.3}$$

to a connection with the geodesics

$$\ddot{x} = -(\varphi - \ln(x))_x \dot{x}^2 - 2\varphi_y \dot{x}\dot{y} - c\dot{y}^2, \quad \ddot{y} = 0, \tag{6.4}$$

where ϱ is the closed one-form with components $(-\frac{1}{2x}, 0)$. This projective change preserves all properties of first integrals: for example a homogeneous polynomial integral of degree n with symmetric tensor a corresponds to an integral with tensor $a\mathrm{e}^{2np}$ where p is the function whose differential is ϱ. Thus the projective change takes the connection determined by (6.2) to one of the type given by (6.1) with φ replaced by $\varphi - \ln x$. Thus it is sufficient to consider connections of the type given by (6.1). Furthermore by a transformation that keeps y fixed we even can assume and will that c is transformed to zero in (6.1). Of course the preceding argument is a short cut and in no way conforms to the procedure of section 2. However, Killing's equations are of considerable interest in their own right and there may be situations in practice where one can eliminate some steps in the algorithm.

Consider then the connection whose geodesics are given by

$$\ddot{x} = -\varphi_x \dot{x}^2 - 2\varphi_y \dot{x}\dot{y}, \quad \ddot{y} = 0. \tag{6.5}$$

Its Ricci tensor is easily found to be

$$K_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & \varphi_{yy} + (\varphi_y)^2 \end{bmatrix}. \tag{6.6}$$

One may then check that all the components B_{1i2}, B_{1i2j}. $B_{1i2jk}, B_{1i2jkl}, \ldots$ are zero.

We shall arrive at a contradiction to the hypothesis that the above-mentioned 6×6 matrix has rank two by assuming that B_{221} is not zero. For the connection corresponding to (6.5) the third column of the 6×6 matrix is zero and since B_{221} is non-zero, the third and fifth rows must each be multiples of the first which necessitates that

$$6(B_{2211})^2 - 5B_{221}B_{22111} = 0 \tag{6.7}$$

$$6B_{2212}B_{2211} - 5B_{221}B_{22121} = 0 \tag{6.8}$$

$$7B_{2211}B_{22111} - 5B_{221}B_{221111} = 0 \tag{6.9}$$

$$7B_{2212}B_{22111} - 5B_{221}B_{221211} = 0 \tag{6.10}$$

Now (6.7) and (6.8) imply that the fifth and six columns of the 6×6 matrix are proportional. If we delete from it the third and fifth rows and third and fifth columns, we obtain the following 4×4 matrix that will have rank two:

(6.11)
$$\begin{bmatrix} B_{2212} & -B_{221} & -5B_{2212} & 0 \\ -B_{22122} & -B_{22112} & 7B_{2212} & 5B_{221} \\ -(B_{221222}+5B_{221}K_{222}) & B_{221122}+B_{2211}K_{22}+5B_{221}K_{221} & 9B_{22122}+20B_{221}K_{22} & 12B_{2212} \\ B_{221221}-B_{2211}K_{22} & -B_{221121} & -8B_{22121} & -6B_{2211} \end{bmatrix}.$$

If we delete the third row and first column from (6.11), set the resulting determinant of the 3×3 matrix to zero and make use of (6.7) and (6.8) we eventually obtain

$$(B_{2211})^2 B_{2212} = 0. \tag{6.12}$$

Hence from (6.8)

$$B_{22121} = 0 \tag{6.13}$$

The argument now bifurcates assuming first that B_{2211} is zero and secondly that B_{2212} is zero but B_{221} is non-zero. In the former case we find that B_{221221} hence B_{221122} is zero and hence from row three and column two of (6.11) that K_{221} is zero. Considering now (6.5) it follows that B_{221} is zero. Hence (6.11) is the zero matrix.

In the case where B_{2212} is zero since B_{22121} is also zero we write out the covariant derivative B_{2211} in the x^1-direction using (6.5). Since we are now assuming that B_{2211} is non-zero we conclude that Γ^1_{12} is zero, hence the connection of (6.5) is flat. This completes the proof of the fact that the 6×6 matrix we are studying cannot have rank two.

We summarize the preceding discussion as follows.

Theorem 6.1. *The dimension of the space of quadratic Killing tensors for a two-dimensional non-flat symmetric connection whose Ricci tensor is symmetric is less than or equal to three.*

We remark finally that this Theorem strengthens a result of Kalnins and Miller [8] in two ways. First of all in [8] it is shown only that four is the upper bound for the dimension of the space Killing tensors. Secondly the argumentation given here applies to symmetric connections whose Ricci tensor i symmetric. In [8] the argument applies to metric tensors and depends on using isothermal coordinates for the metric.

In this Section we give some explicit examples which are obtained by using the theory of the previous Sections. We omit the details and invite the reader to verify the calculations.

1) Choose A to be a function of y and B and C functions of x. Define the geodesics of a connection by

$$\ddot{x} = -\frac{AB'' + C''}{AB' + C'}\dot{x}^2 - \frac{2A'B'}{AB' + C'} - \frac{A''}{A'}\dot{x}\dot{y}, \quad \ddot{y} = 0. \tag{7.1}$$

Then one may check that Ricci is symmetric and that the quantities $\dot{y}$, $\frac{(AB'+C')\dot{x}}{A'} + B\dot{y}$, $\frac{A(AB'+C')\dot{x}}{A'} - C\dot{y}$ are three independent integrals.

2) Choose φ to be a function of x and y subject only to the inequation

$$(\varphi_{yy} + \varphi_y^2)_x \neq 0. \tag{7.2}$$

Define the goedesics of a connection by

$$\ddot{x} = -\varphi_x\dot{x}^2 - 2\varphi_y\dot{x}\dot{y}, \quad \ddot{y} = 0. \tag{7.3}$$

Then the unique linear integral is $\dot{y}$ up to scaling by a constant.

3) Let φ be a function of x and y such that φ_{xy} is non-zero and consider the system

$$\ddot{x} = -\varphi_x\dot{x}^2, \quad \ddot{y} = \varphi_y\dot{y}^2. \tag{7.4}$$

Then one may verify that there is at most one linearly independent Killing covector. An example of a system with one is given by supposing that φ satisfies

$$\mathrm{e}^{2\varphi}\varphi_y = \varphi_x \tag{7.5}$$

in which case $\mathrm{e}^{\varphi}\dot{x} + \mathrm{e}^{-\varphi}\dot{y}$ is a first integral.

4) Let s be a function of x and t be a function of y. Consider the two-dimensional Lorentzian metric given by

$$y = (s + t)((\,\mathrm{d}x)^2 - (\,\mathrm{d}y)^2). \tag{7.6}$$

Then the Levi-Civita connection of g possesses the quadratic integral of motion $(s + t)(t\dot{x}^2 + s\dot{y}^2)$. This example is essentially the only class of Lorentzian metrics that have genuine quadratic integrals besides the metrics themselves.

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Author's address: *G. Thompson*, Department of Mathematics, The University of Toledo, 2801 W. Bancroft St., Toledo, Ohio 43606, USA.