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A DESCRIPTIVE DEFINITION OF A BV INTEGRAL IN THE REAL LINE

Diana Caponetti, Arcavacata di Rende, Valeria Marraffa, Palermo

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Abstract. A descriptive characterization of a Riemann type integral, defined by BV partition of unity, is given and the result is used to prove a version of the controlled convergence theorem.

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0. Introduction

J. Kurzweil, J. Mawhin and W. F. Pfeffer, to obtain an additive continuous integral for which a quite general formulation of Gauss-Green theorem holds, introduced in [5] a multidimensional integral (called $\mathcal{I}$-integral) defined via BV partitions of unity. In dimension one this integral falls properly in between the Lebesgue and Denjoy-Perron integrals and the integration by parts formula holds.

An integral satisfying quite the same properties but defined by using partitions with BV-sets or with *figures* (finite unions of intervals), was studied by W. F. Pfeffer in [7] and [9]. Descriptive characterizations for this integral are given in [3] and [8]. An application of the notion of absolute continuity given in [3] is contained in [1], where a version of the controlled convergence theorem for the one-dimensional Pfeffer-integral is proved.

It seems to us to be of interest to find a descriptive characterization even for the $\mathcal{I}$-integral. The aim of this paper is to solve this problem in the case of the

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one dimensional $\mathcal{I}$-integral and then to apply it to prove a controlled convergence theorem.

The main difficulty has been related to the fact that it is impossible to use the Saks-Henstock lemma, since it is not known whether it holds for the $\mathcal{I}$-integral. To solve our problem we have made use of a useful modification of the *Strong Lusin* condition introduced by P. Y. Lee in [6].

1. Preliminares

The set of all real numbers is denoted by $\mathbb{R}$. If $E \subset \mathbb{R}$, then χ_E, $d(E)$, $\operatorname{cl} E$ and $|E|$ denote the characteristic function, the diameter, the closure and the outer Lebesgue measure of E, respectively. Let $[a,b]$ be a fixed, non degenerate, compact interval of $\mathbb{R}$.

A *figure* of $[a,b]$ is a finite nonempty union of subintervals of $[a,b]$. A collection of figures is called *nonoverlapping* whenever the collection of their interiors is disjoint. The algebraic operations and convergence for functions on the same set are defined pointwise. The usual variation of a function ϑ over the interval $[a,b]$ is denoted $V(\vartheta,[a,b])$. Let θ be a function on $\mathbb{R}$, we set $S_\theta = \{x \in \mathbb{R} \colon \theta(x) \neq 0\}$. Given $\theta \in L^1(\mathbb{R})$ such that $S_\theta \subset (a,b)$ we set

$$\|\theta\| = \inf V(\vartheta,[a,b])$$

where the infimum is taken over all functions ϑ such that $S_\vartheta \subset (a,b)$ and $\vartheta = \theta$ almost everywhere with respect to the Lebesgue measure in $\mathbb{R}$ (abbreviated as a.e.). The family of all nonnegative functions θ on $[a,b]$ for which θ and S_θ are bounded and $\|\theta\| < +\infty$ is denoted by $\mathrm{BV}_+([a,b])$. The *regularity* of $\theta \in \mathrm{BV}_+([a,b])$ at a point $x \in \mathbb{R}$ is the number

$$r(\theta,x) = \begin{cases} \frac{|\theta|_1}{d(S_\theta \cup \{x\})\|\theta\|} & \text{if } d(S_\theta \cup \{x\})\|\theta\| > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $|\theta|_1$ denotes the L^1 norm of θ. Let A be a figure of $[a,b]$, then the characteristic function χ_A of A belongs to $\mathrm{BV}_+([a,b])$ and the symbols $\|A\| = \|\chi_A\|$ and $r(A,x) = r(\chi_A,x)$ coincide with those introduced in [2, Section 1].

A *partition in* $[a,b]$ is a collection $P = \{(A_1,x_1),\dots,(A_p,x_p)\}$ where $A_1,\dots,A_p$ are nonoverlapping subfigures of $[a,b]$ and $x_i \in [a,b]$ for $i = 1,\dots,p$. In particular, P is called

(i) *special* if $A_1,\dots,A_p$ are intervals;
(ii) *tight* if $x_i \in A_i$ for $i = 1,\dots,p$.

A *pseudopartition in* $[a,b]$ is a collection $Q = \{(\theta_1,x_1),\dots,(\theta_p,x_p)\}$ where $\theta_1,\dots,\theta_p$ are functions from $\mathrm{BV}_+([a,b])$ such that $\sum\limits_{i=1}^{p}\theta_i \leqslant \chi_{[a,b]}$ a.e. and $x_i\in[a,b]$ for $i=1,\dots,p$. We say that a pseudopartition P is *anchored* in a set $E\subset[a,b]$ if $x_i\in E$ for $i=1,\dots,p$. Let $P=\{(A_1,x_1),\dots,(A_p,x_p)\}$ be a partition in $[a,b]$, then $P^*=\{(\chi_{A_1},x_1),\dots,(\chi_{A_p},x_p)\}$ is a pseudopartition in $[a,b]$, called the pseudopartition in $[a,b]$ *induced* by P.

Let $\varepsilon>0$ and let δ be a positive function on $[a,b]$. A pseudopartition $Q = \{(\theta_1,x_1),\dots,(\theta_p,x_p)\}$ in $[a,b]$ is called

(i) a pseudopartition *of* $[a,b]$ if $\sum\limits_{i=1}^{p}\theta_i=\chi_{[a,b]}$ a.e.;

(ii) *ε-regular* if $r(\theta_i,x_i)>\varepsilon$, $i=1,\dots,p$;

(iii) *δ-fine* if $d(S_{\theta_i}\cup\{x_i\})<\delta(x_i)$, $i=1,\dots,p$.

A partition $P=\{(A_1,x_1),\dots,(A_p,x_p)\}$ in $[a,b]$ is a partition *of* $[a,b]$, or ε-regular, or δ-fine whenever the pseudopartition P^* induced by P has the respective property.

For a given function f on $[a,b]$ and a pseudopartition $P=\{(\theta_1,x_1),\dots,(\theta_p,x_p)\}$ in $[a,b]$ we set $\sigma(f,P)=\sum\limits_{i=1}^{p} f(x_i)\int_{[a,b]}\theta_i$, where the symbol $\int$ is used to denote the Lebesgue integral.

Definition 1.1. (See [4].) A function $f\colon[a,b]\to\mathbb{R}$ is said to be *integrable* in $[a,b]$ if there is a real number I with the following property: given $\varepsilon>0$, we can find a positive function δ on $[a,b]$ such that

$$\bigl|\sigma(f,P)-I\bigr|<\varepsilon$$

for each ε-regular δ-fine pseudopartition P of $[a,b]$.

We denote by $\mathcal{I}([a,b])$ the family of all integrable functions in $[a,b]$ and set $\int^*_{[a,b]}f=I$. For each $f\in\mathcal{I}([a,b])$, the function $x\mapsto\int^*_{[a,x]}f$, defined on $[a,b]$, is called the *primitive* of f.

Let $\theta\in\mathrm{BV}_+([a,b])$, then the distributional derivative $D\theta$ is a signed Borel measure in $\mathbb{R}$ whose support is contained in $\operatorname{cl}S_\theta$. For a bounded Borel function f on $[a,b]$, $\int_{[a,b]}f D\theta$ denotes the Lebesgue integral of f over $[a,b]$ with respect to $D\theta$.

Given a continuous function F on $[a,b]$ and a pseudopartition $P=\{(\theta_1,x_1),\dots,(\theta_p,x_p)\}$ in $[a,b]$ we define

$$\sum_P\int_{[a,b]}FD\theta=\sum_{i=1}^{p}\int_{[a,b]}FD\theta_i.$$

The following lemma was proved in [4, Lemma 3.1].

Lemma 1.2. *Let f be a bounded function on $[a,b]$ whose derivative $f'(x)$ exists at $x \in [a,b]$. Given $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\left| f'(x)|\theta|_1 + \int_{[a,b]} f D\theta \right| < \varepsilon |\theta|_1$$

for each $\theta \in \mathrm{BV}_+([a,b])$ satisfying $d(S_\theta \cup \{x\}) < \delta$ and $r(\theta, x) > \varepsilon$.

Proposition 1.3. *Let $f \in \mathcal{I}([a,b])$. If $F(x) = \int_{[a,x]}^* f$ for each $x \in [a,b]$, then the function $F\colon [a,b] \to \mathbb{R}$ is continuous. In addition, for almost all $x \in [a,b]$, F is derivable at x and $F'(x) = f(x)$.*

P r o o f. Since $\mathcal{I}([a,b])$ is a subfamily of the family $\mathcal{R}_t^*([a,b])$ introduced in [2, Section 3], the proposition follows from [8, Proposition 2.4]. □

For each figure $A \subset [a,b]$ and for each function F defined on $[a,b]$ we set

$$F(A) = \sum_{h=1}^{n} [F(b_h) - F(a_h)],$$

where $[a_1, b_1], \ldots, [a_n, b_n]$ are the connected components of A.

A function F (or a sequence $\{F_n\}$ of functions) is called AC* (see [3]) (respectively uniformly AC* (see [1])) on a set $E \subset [a,b]$ whenever for every $\varepsilon > 0$ there exist a positive number α and a positive function δ on E satisfying the condition

$$\sum_{i=1}^{p} \bigl|F(A_i)\bigr| < \varepsilon \quad \left(\sup_n \sum_{i=1}^{p} \bigl|F_n(A_i)\bigr| < \varepsilon \right)$$

for each tight ε-regular δ-fine partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ in $[a,b]$ anchored in E with $\sum_{i=1}^{p} |A_i| < \alpha$. A function F (a sequence $\{F_n\}$) is called ACG* (uniformly ACG*) on a set $E \subset [a,b]$ whenever there are sets $E_n \subset E$, $n = 1, 2, \ldots$ such that $E = \bigcup\limits_{n=1}^{\infty} E_n$ and F is AC* (uniformly AC*) on each E_n.

2. Characterization of primitives

The following condition (denoted by WSL$^\circ$) is a modification of the Strong Lusin condition, introduced by P. Y. Lee in [6].

Definition 2.1. Let $N \subset [a,b]$ be a set of measure zero. A continuous function F is said to satisfy condition WSL$^\circ$ on N if, given $\varepsilon > 0$, there exists a positive function δ on $[a,b]$ such that

$$\left| \sum_{x_i \in N} \int_{[a,b]} F D\theta_i \right| < \varepsilon$$

for each ε-regular δ-fine pseudopartition $P = \{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ of $[a,b]$.

Proposition 2.2. *Let $f \in \mathcal{I}([a,b])$ and $F(x) = \int^*_{[a,x]} f$. Then F is continuous, derivable a.e. on $[a,b]$ and satisfies condition* WSL° *on $N = \{x\colon\ F'(x)$ does not exist$\}$.*

Proof. By Proposition 1.3, F is continuous and $F'(x) = f(x)$ a.e. in $[a,b]$. Then $|N| = 0$ and by [4, Corollary 2.10] we can assume $f(x) = 0$ on N and $f(x) = F'(x)$ elsewhere. By Lemma 1.2, for each $\varepsilon > 0$ and for each $x \in [a,b] \setminus N$ we can find a $\delta_0(x) > 0$ such that

$$\left| f(x)|\theta|_1 + \int_{[a,b]} F D\theta \right| < \frac{\varepsilon}{2(b-a)} |\theta|_1$$

for every $\theta \in BV_+([a,b])$ satisfying $d(S_\theta \cup \{x\}) < \delta_0(x)$ and $r(\theta, x) > \varepsilon$. Since $f \in \mathcal{I}([a,b])$, there is a positive function δ on $[a,b]$ ($\delta \leqslant \delta_0$) such that

$$\left|\sigma(f,P) - [F(b) - F(a)]\right| < \tfrac{\varepsilon}{2}$$

for each ε-regular δ-fine pseudopartition P of $[a,b]$.

Let $P = \{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ be an ε-regular δ-fine pseudopartition of $[a,b]$. Then

$$\sigma(f,P) = \sum_{i=1}^{p} f(x_i) \int_{[a,b]} \theta_i = \sum_{x_i \in [a,b] \setminus N} f(x_i) \int_{[a,b]} \theta_i$$

and

$$-[F(b) - F(a)] = \int_{[a,b]} F D\chi_{[a,b]} = \sum_{x_i \in N} \int_{[a,b]} F D\theta_i + \sum_{x_i \in [a,b] \setminus N} \int_{[a,b]} F D\theta_i.$$

Hence

$$\begin{aligned}
\left| \sum_{x_i \in N} \int_{[a,b]} F D\theta_i \right| &\leqslant \left| \sum_{i=1}^{p} f(x_i) \int_{[a,b]} \theta_i + \int_{[a,b]} F D\chi_{[a,b]} \right| \\
&\quad + \left| \sum_{x_i \in [a,b] \setminus N} \left(f(x_i)|\theta_i|_1 + \int_{[a,b]} F D\theta_i \right) \right| \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)} \sum_{x_i \in [a,b] \setminus N} |\theta_i|_1 \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon.
\end{aligned}$$

Thus the claim is proved. □

Proposition 2.3. *A function f on $[a,b]$ belongs to $\mathcal{I}([a,b])$ if and only if there exists a continuous function F such that for almost all $x \in [a,b]$ F is derivable at x with $F'(x) = f(x)$ and satisfies condition* WSL° *on the set $N = \{x\colon$ $F'(x)$ does not exist$\}$. In particular, $F(x) = \int^*_{[a,x]} f$.*

Proof. The necessity is given in Proposition 2.2. Now suppose that there exists a function F on $[a,b]$ satisfying the hypotheses of the theorem. Then $|N|=0$. Assume $f(x)=0$ on N and $f(x)=F'(x)$ elsewhere. Since F satisfies condition WSL$^\circ$ on N, given $\varepsilon>0$, there exists a positive function δ on $[a,b]$ such that

$$\Bigl|\sum_{x_i\in N}\int_{[a,b]} FD\theta_i\Bigr| < \frac{\varepsilon}{2}$$

for each ε-regular δ-fine pseudopartition $P=\{(\theta_1,x_1),\ldots,(\theta_p,x_p)\}$ of $[a,b]$. By Lemma 1.2, to each $x\in[a,b]\setminus N$ such a $\delta_0(x)>0$ corresponds that

$$\Bigl|f(x)|\theta|_1+\int_{[a,b]} FD\theta\Bigr| < \frac{\varepsilon}{2(b-a)}|\theta|_1$$

for each $\theta\in \mathrm{BV}_+([a,b])$ satisfying $d(S_\theta\cup\{x\})<\delta_0(x)$ and $r(\theta,x)>\varepsilon$. Define

$$\delta^*(x)=\begin{cases}\min\{\delta(x),\delta_0(x)\} & \text{if } x\in[a,b]\setminus N,\\ \delta(x) & \text{if } x\in N.\end{cases}$$

Then for each ε-regular δ^*-fine pseudopartition $P=\{(\theta_1,x_1),\ldots,(\theta_p,x_p)\}$ of $[a,b]$ we have

$$\begin{aligned}\Bigl|\sum_{i=1}^p f(x_i)\int_{[a,b]}\theta_i-[F(b)-F(a)]\Bigr|\\ &=\Bigl|\sum_{i=1}^p f(x_i)\int_{[a,b]}\theta_i+\int_{[a,b]}FD\chi_{[a,b]}\Bigr|\\ &\leqslant\Bigl|\sum_{x_i\in N}\int_{[a,b]}FD\theta_i\Bigr|+\sum_{x_i\in[a,b]\setminus N}\Bigl|f(x_i)\int_{[a,b]}\theta_i+\int_{[a,b]}FD\theta_i\Bigr|<\varepsilon.\end{aligned}$$

Hence $f\in\mathcal{I}([a,b])$. □

Remark 2.4. Let $F\colon [a,b]\to\mathbb{R}$ be a continuous function. If F is differentiable a.e. on $[a,b]$ and satisfies condition WSL$^\circ$ on the set $N=\{x\colon F'(x) \text{ does not exist}\}$ then F is ACG* on $[a,b]$.

Indeed, by the previous theorem $F'=f$ belongs to $\mathcal{I}([a,b])$, thus $f\in\mathcal{R}_t^*([a,b])$. By [3, Proposition 3.4] it follows that F is ACG* on $[a,b]$.

Definition 2.5. Let F be a continuous function on $[a,b]$ and let $E\subset[a,b]$. The function F is called AC$^\circ$ on E if, given $\varepsilon>0$, there exist a positive number α and a positive function δ on E such that

$$\sum_{i=1}^p\Bigl|\int_{[a,b]}FD\theta_i\Bigr|<\varepsilon$$

for each ε-regular δ-fine pseudopartition $P=\{(\theta_1,x_1),\dots,(\theta_p,x_p)\}$ in $[a,b]$ anchored in E with $\sum\limits_{i=1}^{p}|\theta_i|_1<\alpha$. The function F is called ACG$^\circ$ on E if there are measurable sets $E_n\subset E$, $n=1,2,\dots$ such that $E=\bigcup\limits_{i=1}^{\infty}E_n$ and F is AC$^\circ$ on each E_n.

Remark 2.6. If F is ACG$^\circ$ on $X\subset[a,b]$, then F is ACG* on X. In particular, F is differentiable a.e. on X ([3], Corollary 3.3).

The following lemma is a straightforward modification of [3, Lemma 2.2].

Lemma 2.7. *Let $X\subset[a,b]$ and let F be an ACG$^\circ$ function on X. If E is a subset of X of measure zero, given $\varepsilon>0$, there exists a positive function δ on $[a,b]$ such that $\sum\limits_{i=1}^{p}|\int_{[a,b]}FD\theta_i|<\varepsilon$ for each ε-regular δ-fine pseudopartition $P=\{(\theta_1,x_1),\dots,(\theta_p,x_p)\}$ in $[a,b]$ anchored in E.*

Proposition 2.8. *Let F be a continuous function on $[a,b]$. Then F is differentiable a.e. in $[a,b]$ and satisfies condition* WSL$^\circ$ *on the set $N=\{x\colon F'(x)$ does not exist$\}$ if and only if there exists a set X with $|[a,b]\setminus X|=0$ such that the function F is* ACG$^\circ$ *on X and satisfies condition* WSL$^\circ$ *on $[a,b]\setminus X$.*

Proof. Assume first that F is differentiable a.e. in $[a,b]$ and satisfies condition WSL$^\circ$ on the set $N=\{x\colon F'(x)$ does not exist$\}$. We show that F is ACG$^\circ$ on $X=[a,b]\setminus N$. For $n=1,2,\dots$, let $E_n=\{x\notin N\colon n-1\leqslant|F'(x)|<n\}$, then $X=\bigcup\limits_{i=1}^{\infty}E_n$. By Lemma 1.2, for each $\varepsilon>0$ and for each $x\in E_n$ there is a $\delta_n(x)>0$ such that

$$\left|F'(x)|\theta|_1+\int_{[a,b]}FD\theta\right|<\frac{\varepsilon}{2(b-a)}|\theta|_1$$

for all $\theta\in\mathrm{BV}_+([a,b])$ satisfying $d(S_\theta\cup\{x\})<\delta_n(x)$ and $r(\theta,x)>\varepsilon$. Now let $\alpha_n=\frac{\varepsilon}{2n}$. Then, for each ε-regular δ_n-fine pseudopartition $P=\{(\theta_1,x_1),\dots,(\theta_p,x_p)\}$ in $[a,b]$ anchored in E_n with $\sum\limits_{i=1}^{p}|\theta_i|_1<\alpha_n$ it follows

$$\left|\sum_{i=1}^{p}\int_{[a,b]}FD\theta_i\right|\leqslant\sum_{i=1}^{p}\left|F'(x_i)|\theta_i|_1+\int_{[a,b]}FD\theta_i\right|+\sum_{i=1}^{p}|F'(x_i)||\theta_i|_1<\frac{\varepsilon}{2}+n\alpha_n=\varepsilon$$

Hence F is AC$^\circ$ on E_n. □

Conversely, let $T=[a,b]\setminus X$ and fix $\varepsilon>0$. By Remark 2.6 F is differentiable a.e. on X. Let $N=\{x\colon F'(x)$ does not exist$\}$, then $N=N_1\cup N_2$ where $N_1\subset X$ and $N_2\subset T$. By Lemma 2.7 there exists a positive function δ_1 on $[a,b]$ such that

$\sum_P |\int_{[a,b]} FD\theta| < \frac{\varepsilon}{4}$ for each ε-regular δ_1-fine pseudopartition P in $[a,b]$ anchored in N_1.

Since F satisfies condition WSL° on T, there exists a positive function δ_0 ($\delta_0 \leqslant \delta_1$) such that

$$\left| \sum_{x_i \in T} \int_{[a,b]} FD\theta_i \right| < \frac{\varepsilon}{4}$$

for each ε-regular δ_0-fine pseudopartition $P = \{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ of $[a,b]$. Use Lemma 1.2 to find a positive function δ_2 in $T \setminus N_2$ such that

$$\left| F'(x)|\theta|_1 + \int_{[a,b]} FD\theta \right| < \frac{\varepsilon}{4(b-a)} |\theta|_1$$

for each $\theta \in BV_+([a,b])$ satisfying $d(S_\theta \cup \{x\}) < \delta_2(x)$ and $r(\theta, x) > \varepsilon$. For $n = 1, 2, \ldots$, set $T_n = E_n \cap (T \setminus N_2)$, E_n being the sets defined above. Since $|T_n| = 0$ there exists an open set O_n such that $T_n \subset O_n$ and $|O_n| < \varepsilon/n2^{n+2}$. Now define a positive function δ on $[a,b]$ by setting

$$\delta(x) = \begin{cases} \min\{\delta_0(x), \delta_2(x), \varepsilon/n2^{n+2}\} & \text{if } x \in T_n,\ n = 1, 2, \ldots, \\ \delta_0(x) & \text{elsewhere.} \end{cases}$$

Let $P = \{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ be an ε-regular δ-fine pseudopartition of $[a,b]$. It follows that

$$\begin{aligned} \left| \sum_{x_i \in N} \int_{[a,b]} FD\theta_i \right| &\leqslant \left| \sum_{x_i \in N_1} \int_{[a,b]} FD\theta_i \right| + \left| \sum_{x_i \in N_2} \int_{[a,b]} FD\theta_i \right| \\ &\leqslant \frac{\varepsilon}{4} + \left| \sum_{x_i \in N_2} \int_{[a,b]} FD\theta_i + \sum_{x_i \in T \setminus N_2} \int_{[a,b]} FD\theta_i \right| \\ &\quad + \left| \sum_{x_i \in T \setminus N_2} \int_{[a,b]} FD\theta_i \right| \\ &\leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \sum_{x_i \in T \setminus N_2} \left| F'(x_i)|\theta_i|_1 + \int_{[a,b]} FD\theta_i \right| \\ &\quad + \sum_{x_i \in T \setminus N_2} |F'(x_i)||\theta_i|_1 \\ &\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \sum_{n=1}^{\infty} \sum_{x_i \in E_n} |F'(x_i)||\theta_i|_1 \leqslant \frac{3}{4}\varepsilon + \sum_{n=1}^{\infty} n \frac{\varepsilon}{n2^{n+2}} = \varepsilon. \end{aligned}$$

Combining Theorem 2.3 and Proposition 2.8 we get the following theorem.

Theorem 2.9. *A function f on $[a,b]$ belongs to $\mathcal{I}([a,b])$ if and only if there exist a subset X of $[a,b]$ and a continuous function $F\colon [a,b] \to \mathbb{R}$ such that*

(i) $|[a,b] \setminus X| = 0$,
(ii) *F is* ACG° *on* X,
(iii) *F satisfies condition* WSL° *on* $[a,b] \setminus X$,
(iv) $F' = f$ *a.e. on* $[a,b]$.

In particular, $F(x) = \int_{[a,x]}^{*} f$.

It is interesting to point out that the Saks-Henstock lemma for the $\mathcal{I}$-integral has not been proved nor a counterexample has been produced. The validity of the Saks-Henstock lemma would allow us to improve the formulation of the above descriptive characterization. More precisely, in the formulation of condition WSL° the expression $|\sum_{x_i \in N} \int_{[a,b]} F D\theta_i| < \varepsilon$ would be replaced by $\sum_{x_i \in N} |\int_{[a,b]} F D\theta_i| < \varepsilon$. Thus in Proposition 2.3 the function F would satisfy such condition on every set of measure zero, moreover the statement of Theorem 2.9 would be:

A function f on $[a,b]$ belongs to $\mathcal{I}([a,b])$ if and only if there exists a continuous function F such that F is ACG° *on $[a,b]$ and $F' = f$ a.e. on $[a,b]$.*

3. Controlled convergence

In this section we give a definition of *uniform generalized absolute continuity* and use it to prove a controlled convergence theorem for sequences of $\mathcal{I}$-integrable functions.

Definition 3.1. Let $\{F_n\}$ be a sequence of functions defined on $[a,b]$. We say that $\{F_n\}$ is *uniformly* AC° on $E \subset [a,b]$ if, given $\varepsilon > 0$, there exist a positive function δ on $[a,b]$ and a positive number α such that

$$\sup_n \sum_{i=1}^{p} \left| \int_{[a,b]} F_n D\theta_i \right| < \varepsilon$$

for each ε-regular δ-fine pseudopartition $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ in $[a,b]$ anchored in E with $\sum_{i=1}^{p} |\theta_i|_1 < \alpha$. A sequence $\{F_n\}$ of functions is said to be *uniformly* ACG° on E if there are disjoint sets $E_k \subset E$, $k = 1, 2, \dots$ such that $E = \bigcup_{k=1}^{\infty} E_k$ and every F_n is uniformly AC° on each E_k.

Definition 3.2. Let N be a set of measure zero. A sequence of functions $\{F_n\}$ defined on $[a,b]$ is said to satisfy *uniformly* condition WSL° on N if, given $\varepsilon > 0$, there exists a positive function δ on $[a,b]$ such that

$$\sup_n \left| \sum_{x_i \in N} \int_{[a,b]} F D\theta_i \right| < \varepsilon$$

for each ε-regular δ-fine pseudopartition $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ of $[a,b]$.

Lemma 3.3. *Let $\{F_n\}$ be a sequence of functions and X a subset of $[a,b]$ such that*

(i) $|[a,b]\setminus X| = 0$,

(ii) *$\{F_n\}$ is uniformly* ACG$^\circ$ *on X,*

(iii) *$\{F_n\}$ satisfies uniformly condition* WSL$^\circ$ *on $[a,b]\setminus X$.*

Then $\{F_n\}$ is uniformly ACG* *on $[a,b]$.*

Proof. Let $X = \bigcup\limits_{k=1}^{\infty} E_k$, where the E_k's are disjoint and the sequence $\{F_n\}$ is uniformly AC$^\circ$ on each E_k. Clearly the sequence $\{F_n\}$ is uniformly AC* on E_k for $k = 1, 2, \ldots$. We have to prove that the sequence $\{F_n\}$ is uniformly AC* on $[a,b]\setminus X$. Given $\varepsilon > 0$, there is a positive function δ on $[a,b]$ such that

$$\sup_n \Big| \sum_{x_i \in [a,b]\setminus X} F_n(A_i) \Big| = \sup_n \Big| \sum_{x_i \in [a,b]\setminus X} \int_{[a,b]} F_n D\chi_{A_i} \Big| < \frac{\varepsilon}{2}$$

for each ε-regular δ-fine partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ of $[a,b]$. Fix $n \geqslant 1$. By Theorem 2.9 the function f_n belongs to $\mathcal{I}([a,b])$, hence (see Remark 2.6) its primitive F_n is ACG* on $[a,b]$. Thus, by [3, Lemma 2.2] there is a positive function δ_n on $[a,b]$ ($\delta_n \leqslant \delta$) such that

$$\Big| \sum_{i=1}^{s} F_n(A_i) \Big| < \frac{\varepsilon}{2}$$

for each ε-regular δ_n-fine partition $\{(A_1, x_1), \ldots, (A_s, x_s)\}$ in $[a,b]$ anchored in $[a,b]\setminus X$. Choose an ε-regular δ-fine partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ anchored in $[a,b]\setminus X$. By Cousin's lemma there exists a special and tight δ_n-fine partition $P_1 = \{(B_1, y_1), \ldots, (B_r, y_r)\}$ of $[a,b]\setminus \cup P$. Then $P \cup P_1$ is an ε-regular δ-fine partition of $[a,b]$. Thus we obtain

$$\Big| \sum_{i=1}^{p} F_n(A_i) \Big| = \Big| \sum_{i=1}^{p} F_n(A_i) + \sum_{y_r \in [a,b]\setminus X} F_n(B_j) \Big| + \Big| \sum_{y_r \in [a,b]\setminus X} F_n(B_j) \Big| < \varepsilon.$$

Considering separately the subfigures A_i of P for which $F_n(A_i) \geqslant 0$ and those for which $F_n(A_i) < 0$ it follows that the inequality $|\sum\limits_{i=1}^{s} F_n(A_i)| < \varepsilon$ can be replaced by $\sum\limits_{i=1}^{s} |F_n(A_i)| < \varepsilon$. Thus we get

$$\sup_n \sum_{i=1}^{s} \Big| F_n(A_i) \Big| < 2\varepsilon$$

and this completes the proof. $\square$

Definition 3.4. A sequence $\{f_n\} \in \mathcal{I}([a,b])$ is called *$\mathcal{I}$-control convergent* to f on $[a,b]$ if $f_n \to f$ a.e. in $[a,b]$, $\{\int_{[a,x]}^* f_n\}$ is uniformly ACG$^\circ$ on X, where $[a,b]\setminus X$ is of measure zero, and $\{\int_{[a,x]}^* f_n\}$ satisfies uniformly condition WSL$^\circ$ on $[a,b]\setminus X$.

Theorem 3.5. *If $\{f_n\} \in \mathcal{I}([a,b])$ is $\mathcal{I}$-control convergent to f on $[a,b]$, then $f \in \mathcal{I}([a,b])$ and*

$$\lim_n \int_{[a,b]}^* f_n = \int_{[a,b]}^* f.$$

P r o o f. By Lemma 3.3 the primitives $F_n(x) = \int_{[a,x]}^* f_n$ of f_n are uniformly ACG*. Thus by [1, Theorem 4.3] we get that $\lim\limits_n \int_{[a,b]}^* f_n = (\mathcal{R}_t)\int_{[a,b]}^* f$ and $F(x) = (\mathcal{R}_t)\int_{[a,x]}^* f$ is ACG* on $[a,b]$. It remains to show that there exists a set X with $|[a,b]\setminus X| = 0$ such that F is ACG$^\circ$ on X and satisfies condition WSL$^\circ$ on $[a,b]\setminus X$. We note that the sequence $\{F_n\}$ is equicontinuous and since $F_n(a) = 0$, it is also equibounded. Then, by Ascoli's theorem, there is a subsequence $\{F_{n(j)}\}$ of $\{F_n\}$ that converges uniformly to F on $[a,b]$. Given $\varepsilon > 0$ and a fixed k, choose δ_k and δ on $[a,b]$ and α_k according to Definition 3.1 and Definition 3.2. Then the uniform convergence of $\{F_{n(j)}\}$ to F implies that

$$\sum_P \left| \int_{[a,b]} F D\theta_i \right| \leqslant \sup_{n(j)} \sum_P \left| \int_{[a,b]} F_{n(j)} D\theta_i \right| < \varepsilon$$

for each ε-regular δ_k-fine pseudopartition P in $[a,b]$ anchored in E_k with $\sum\limits_{i=1}^{p} |\theta_i|_1 < \alpha_k$ and also

$$\left| \sum_{x_i \in N} \int_{[a,b]} F D\theta_i \right| \leqslant \sup_{n(j)} \left| \sum_{x_i \in N} \int_{[a,b]} F_{n(j)} D\theta_i \right| < \varepsilon$$

for each ε-regular δ-fine pseudopartition P of $[a,b]$.
Hence F is ACG$^\circ$ on $X = \bigcup\limits_{k=1}^{\infty} E_k$ with $|[a,b]\setminus X| = 0$ and F satisfies condition WSL$^\circ$ on $[a,b]\setminus X$. Thus by Theorem 2.9 we conclude that $f \in \mathcal{I}([a,b])$ and $F(x) = \int_{[a,x]}^* f$. $\square$

R e m a r k 3.6. Let g be a function of bounded variation on $[a,b]$ and let $\{f_n\} \in \mathcal{I}([a,b])$ be $\mathcal{I}$-control convergent to f on $[a,b]$. Then, by the integration by parts formula [4, Proposition 3.3], we get

$$\lim_n \int_{[a,b]}^* f_n g = \lim_n \left[F_n(b)g(b) - \int_{[a,b]} F_n \,\mathrm{d}g \right] = F(b)g(b) - \int_{[a,b]} F \,\mathrm{d}g = \int_{[a,b]}^* fg.$$

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Authors' addresses: *Diana Caponetti*, Dipartimento di Matematica, Università degli Studi della Calabria, 87036 Arcavacata di Rende (CS), Italy, *Valeria Marraffa*, Dipartimento di Matematica ed Applicazioni, Via Archirafi 34, 90123 Palermo, Italy.