ON RADIALLY EXTREMAL GRAPHS AND DIGRAPHS, A SURVEY

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(Received May 12, 1998)

Abstract. The paper gives an overview of results for radially minimal, critical, maximal and stable graphs and digraphs.

Keywords: extremal graphs, extremal digraphs, radius of graphs, radius of digraphs

MSC 2000: 05C12, 05C20, 05C35

1. Introduction

The paper gives an overview of results for radially minimal, critical, maximal and stable graphs and digraphs. First we will speak about graphs and then digraphs. We consider nonempty and finite graphs and digraphs here.

Let G be a graph. Then we denote by: $V(G)$ the vertex set of G; $E(G)$ the edge set of G; $d(u,v)$ the distance in G between the vertices u, v; $e(u) = \max\{d(u,v)\colon v \in V(G)\}$ the eccentricity of u; $r(G) = \min\{e(u)\colon u \in V(G)\}$ the radius of G.

General notions and notation are according to [2] while the specific notions are defined here.

Clearly, the following three inequalities hold: $r(G-e) \geqslant r(G)$ for every edge e of G; $r(G-u) \neq r(G)$ for every vertex u of G; $r(G+e) \leqslant r(G)$ for every edge e of the complement of G;

The second inequality is illustrated in Fig. 1, where $r(G) = 3; 2 = r(G-b) < r(G)$ and $\infty = r(G-a) > r(G)$. Moreover, the paper [10] notes that $r(G) - 1 \leqslant r(G-u)$ for every vertex u of G.

Now we give an overview of radially extremal graphs.

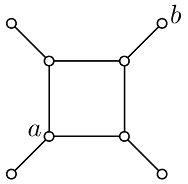

Fig. 1

2. Radially extremal graphs

We say that a graph G is *minimal* (by r), if $r(G-e) > r(G)$ for every $e \in E(G)$; *critical* (by r), if $r(G-u) \neq r(G)$ for every $u \in V(G)$; *maximal* (by r), if $r(G+e) < r(G)$ for every edge e of the complement of G (see [2]).

Extremal (minimal, critical, maximal) graphs are defined and studied using different parameters as connectivity, chromatic number, diameter, ..., see e.g. [3], [2]. In the sequel we will study radially extremal graphs only. Basic facts about these graphs are in [2, Ch. 5, 95–116].

Minimal graphs form a simple class of graphs:

Theorem 1. (see [10].) *A nontrivial graph G is minimal if and only if G is a tree.*

Critical graphs and maximal graphs are more complicated.

2.1. Critical graphs. Critical graphs were defined and studied in [10]. Examples of such graphs are the graph in Fig. 1 or the path P_4 of length 3 (and radius 2) or the graph in Fig. 2 of radius 2.

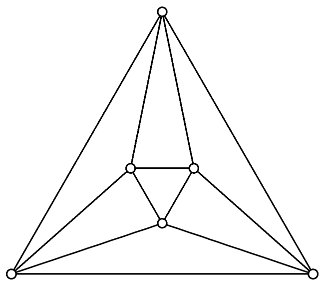

Fig. 2

We say that a vertex u of G is *peripheral*, if there exists a central vertex w such that $e(w) = r(G) = d(w,u)$. Using this notion we give the following lemma.

Lemma 2. (See [10].) *Let G be a critical graph and $u \in V(G)$. Then $r(G-u) < r(G)$ if and only if u is peripheral.*

Critical graphs of radius two can be described completely.

Theorem 3. (See [10].) *A graph G of radius 2 is critical if and only if G is either a path of length 3 or a complete n-partite graph $K_{2,2,\ldots,2}$, where $n \geqslant 2$.*

The class of critical graphs can be partitioned into three sets: *decreasing graphs* for which $r(G-u) = r(G) - 1$ for every $u \in V(G)$; *increasing graphs* for which $r(G-u) > r(G)$ for every $u \in V(G)$; *changing graphs* containing all the other critical graphs.

The following theorem reduces the study of critical graphs to decreasing graphs.

Theorem 4. (See [11].) *Every connected critical graph G is either decreasing or consists of a decreasing subgraph H and endpaths so that one endpath of length $r(G) - r(H)$ is joined to each vertex of H.*

The following characterization is given for decreasing graphs.

Theorem 5. (See [10].) *The following statements are equivalent:*

1. *G is a decreasing graph;*
2. *for every $u \in V(G)$ there exists exactly one $\bar{u} \in V(G) - \{u\}$ such that*

$$e(u) = e(\bar{u});$$

3. *there exists a decomposition of $V(G)$ into pairs $\{u, \bar{u}\}$ such that*

$$d(u, \bar{u}) = r(G) > \max_{x \in V(G) - \{u, \bar{u}\}} \{d(x, u), d(x, \bar{u})\}.$$

The following theorem has a constructive proof for the existence of decreasing and maximal graphs of radius $r \geqslant 3$.

Theorem 6. (See [10].) *Let G be a graph with $p \geqslant 3$ vertices. Let $r \geqslant 3$ and $k \geqslant 2p + 2$ be given integers. Then there exists a k-regular, decreasing and maximal graph Q of radius r which has G as an induced subgraph.*

Directly from Theorem 6 a problem follows.

P r o b l e m 1. Let G be a graph and $r \geqslant 3$ a given integer. What is the minimum number of vertices needed to add to G in order to receive an overgraph H of radius r containing G as an induced subgraph such that H is either a decreasing graph or a maximal graph.

Fajtlowicz [8] defines the following class of radially critical graphs:

A graph is α-critical if it has radius $r \geqslant 2$ and every proper induced connected subgraph has radius strictly smaller than r.

It is clear that α-critical graphs are critical. If we take any connected graph G of radius greater than r, $r \geqslant 3$, then the proof of Theorem 6 gives us a critical graph Q of radius r. So α-critical graphs form a special class of critical graphs.

Paper [8] defines certain graphs of radius $r \geqslant 2$, called r-ciliates, and then proves the following:

Theorem 7. (See [8].) *A graph G of radius $r \geqslant 2$ is α-critical if and only if G is an r-ciliate.*

Segawa [28] proved the following estimation.

Theorem 8. (See [28].) *Let G be a connected graph and let a subset $F \subset E(G)$ be such that $G - F$ is connected. Then $r(G - F) \leqslant (|F| + 1)r(G) - \lfloor |F|/2 \rfloor$.*

2.2. Maximal graphs. The first result concerning maximal graphs was the following estimation proved by Vizing in 1967.

Theorem 9. (See [31].) *The maximum number of edges in a graph with p vertices and radius r is $p(p-1)/2$, if $r = 1$; $\lfloor p(p-2)/2 \rfloor$, if $r = 2$; $(p^2 - 4pr + 5p + 4r^2 - 6r)/2$, if $r \geqslant 3$.*

These estimations are the best ones possible.

Maximal graphs were studied by different authors, see e.g. Nishanov [23–26], Harary and Thomassen [17], Gliviak, Knor and Šoltés [12–13], and Du, Shi and Zhao [5]. We note that papers [23–26] used the non-standard notions and notation defined e.g. in [32].

The paper [23] does not give any bounds for the diameter of maximal graphs, but it gives some conditions for the maximal graphs with radius r and maximal diameter. Paper [17] studies maximal graphs with respect to various graph parameters, including the radius. This paper shows the following result.

Theorem 10. (See [17].) *Let G be a maximal graph of radius at least two. Then $r(G) \leqslant d(G) \leqslant 2r(G) - 2$.*

The paper also describes maximal graphs of radius two.

Theorem 11. (See [17].) *A graph G with radius two is maximal if and only if the complement of G, $\overline{G}$, is disconnected and each component of $\overline{G}$ is a star $K_{1,s}$ for $s \geqslant 1$.*

According to Theorem 6, given a graph G and a natural number $r \geqslant 3$, we can construct a maximal graph of radius r containing G as an induced subgraph.

Maximal graphs of radius two were described later constructively in [5]. They were studied also in [13], where it is shown that the central subgraph of any maximal graphs of radius two contains an edge and shows that those of them that have a star as the central subgraph are sequential joins of complete graphs. Paper [25] describes maximal graphs of radius two independently and in the same way as in Theorem 11.

One class of graphs belonging to the intersection of the classes of maximal and critical graphs of radius r was studied in [24]. This paper also shows that the line graph of the cartesian product of circuits $C_{2k}.C_{2r}$, where $k \geqslant 2$, $s \geqslant 2$, is a critical graph.

Paper [26] proved that a graph G is a maximal graph of radius $r \geqslant 2$ with exactly one cycle if and only if certain five conditions hold simultaneously. As to maximal graphs having a cut-vertex the following two results were proved:

Theorem 12. (See [12].) *Let G be a maximal graph of radius $r \geqslant 3$ containing a cut-vertex y. Then the graph $G - y$ has exactly two components, say A', B'. Let A and B be the subgraphs of G induced on $V(A') \cup \{y\}$ and $V(B') \cup \{y\}$, respectively. Let the eccentricity satisfy $e_A(y) \geqslant e_B(y)$. Then:*

1. $e_A(y) + e_B(y) \leqslant 2r - 2$,
2. *B is a diametrically maximal graph of diameter $e_B(y)$.*

Theorem 13. (See [12].)

1. *Each maximal graph of radius three contains at most two cut-vertices.*
2. *Let k and r be given integers such that $r \geqslant 4$, $k \geqslant 1$. Then there exists an infinite number of maximal graphs of radius r, containing k cut-vertices.*

We can formulate two problems about cut-vertices and centers in maximal graphs.

Problem 2. To study maximal graphs with small number of cut-vertices.

Problem 3. Study the center $C(G)$ of either a decreasing graph G or a maximal graph G. Estimate this center and describe some properties of the induced subgraph $\langle C(G) \rangle$ generated by $C(G)$.

Paper [7] gives asymptotically sharp upper bounds for the maximum diameter and radius of (i) a connected graph, (ii) a connected triangle-free graph, (iii) a connected C_4-free graph with n vertices and with minimum degree δ, where n tends to infinity. Now we give these results only for the radius and a connected graph or a connected triangle-free graph.

Theorem 14. (See [7].)

1. *Let G be a connected graph with n vertices and with minimum degree $\delta \geqslant 2$. Then $r(G) \leqslant \frac{3}{2} \cdot \frac{n-3}{\delta+1} + 5$ and this bound is tight apart from the exact value of the aditive constant.*
2. *Let G be a connected triangle-free graph with n vertices and with minimum degree $\delta \geqslant 2$. Then $r(G) \leqslant \frac{n-2}{\delta} + 12$ and this bound is tight apart from the exact value of the aditive constant.*

The following theorem gives an upper bound the for radius of a 3-connected graph containing n vertices.

Theorem 15. *Let G be a 3-connected graph containing n vertices. Then*

1. (see [20])*:* $r(G) < \frac{n}{4} + O(\log n)$,
2. (see [19])*:* $r(G) \leqslant \frac{n}{4} + 8$,
3. (see [21])*:* $r(G) \leqslant \frac{n+15}{4}$.

For self-centered connected graphs (see [2]) the following result holds.

Theorem 16. (See [1].) *Let $n \geqslant 2r > 2$, except the case $n = 2r = 4$. Then there exists a self-centered connected graph with n vertices, k edges and radius r if and only if $\lfloor (nr - 2r - 1)/(r-1) \rfloor \leqslant k \leqslant (n^2 - 4rn + 5n + 4r^2 - 6r)/2$. If $n = 2r = 4$, then k must equal* 4.

The following problem concerns some estimations of basic parameters of decreasing or maximal graphs.

Problem 4. Estimate the number of edges, the maximum degree and the minimum degree of either a decreasing graph or a maximal graph.

2.3. Radius stable graphs. Let f be any graph invariant, e.g. radius, diameter, ... F. Harary [16] defines three changing invariants by f and three unchanging invariants by f. Three changing invariants by f are usually used in the same sense as graphs minimal by f, critical by f and maximal by f—defined here for radius—see [6], [18] and [15]. Three unchanging invariants are new and we will call them edge-stable (by f), vertex-stable (by f) and adding-stable (by f). Now we will define them for radius.

A graph G is *e-stable*, if $r(G - e) = r(G)$ for every edge e of G; *v-stable*, if $r(G - v) = r(G)$ for every vertex v of G; *a-stable*, if $r(G + e) = r(G)$ for every edge e of the complement of G.

These three types of stable graphs may have some applications, but until now they have been studied only from the mathematical point of view. Paper [6] gives several basic facts about these graphs and one general result.

Theorem 17. (See [6].) *Every self-centered graph G with at least 3 vertices is e-stable.*

We do not know another paper about e-stable, v-stable and a-stable graphs and so we give the following problem.

P r o b l e m 5. To receive further mathematical results concerning e-stable, v-stable and a-stable graphs.

3. Basic notions from digraphs

We consider here nonempty and finite digraphs D without loops and multiple arcs. If D is a digraph, then $V(D)$ will be the node set of D and $E(D)$ the arc set of D. The (standard) *distance* $d(u,v)$ between the nodes u and v in a digraph D is the length of the shortest (directed) $u-v$ path in D if such a path exists; otherwise $d(u,v)=\infty$.

Paper [4] defines three other distances $d_{\min}, d_{\max}, d_{\text{sum}}$ for a strong digraph D, but we give these notions in general:

$$
\begin{aligned}
d_{\min}(u,v) &= \min(d(u,v), d(v,u)),\\
d_{\max}(u,v) &= \max(d(u,v), d(v,u)),\\
d_{\text{sum}}(u,v) &= d(u,v) + d(v,u),
\end{aligned}
$$

where the symbol ∞ is greater than any number and the symbol $\infty + x = \infty$ for any x.

The *detour distance* $d^*(u,v)$ from u to v in a digraph D was defined in [30] as the length of a longest directed $u-v$ path P^* (called detour path) for which the subdigraph induced by the vertices of P^* contains no shorter directed $u-v$ path. If no directed $u-v$ path exists in D, then the detour distance is infinite.

Let $k(u,v)$ be a distance defined in a digraph D (e.g. $d, d_{\min}, d_{\max}, d_{\text{sum}}$ and d^*). Then we say that the *eccentricity* of a node u is $e_k(u) = \max\limits_{v\in V(D)} k(u,v)$. So we have defined the eccentricities $e_d, e_{\min}, e_{\max}, e_{\text{sum}}$ and e^* for the distances d, $d_{\min}$, $d_{\max}$, d_{sum} and d^*, respectively. (We note that these eccentricities can be infinite.)

We define the *radius* $r_k(D) = \min\limits_{u\in V(D)} e_k(u)$. So we have defined the radii $r_d(D)$, $r_{\min}(D)$, $r_{\max}(D)$, $r_{\text{sum}}(D)$ and $r^*(D)$. We note that instead of the symbols e_d and r_d only e and r are usually used.

Several papers, e.g. [27], denote the standard eccentricity $e_d(u)$ as the *out-eccentricity* $e^+(u) = \max\limits_{v\in V(D)} d(u,v)$ and they also define the *in-eccentricity* $e^-(u) = \max\limits_{v\in V(D)} d(v,u)$, and the *eccentricity* $e^0(u) = \max(e^+(u), e^-(u))$.

Then one can define the *out-radius* $r^+(D) = \min\limits_{u\in V(D)} e^+(u) = r_d(D)$, the *in-radius* $r^-(D) = \min\limits_{u\in V(D)} e^-(u)$, and the *radius* $r^0(D) = \min\limits_{u\in V(D)} e^0(u)$. We note that it is enough to study $r^+(D)$ and $r^0(D)$, because if D is a digraph and D' results from D by reversing the orientation of all arcs, then $r^+(D) = r^-(D')$.

The distance $d_{\min}$, the eccentricity $e_{\min}$, and the radius $r_{\min}$ were defined independently in [9]. Moreover, the properties of distances $d, d_{\min}, d_{\max}, d_{\text{sum}}$ and d^* were studied in several papers, e.g. [4], [29], [30].

4. Radially extremal digraphs

First of all we define minimal, critical and maximal digraphs and then give results achieved in literature.

Let D be a digraph and $f(D)$ any radius of D defined earlier, e.g. $r(D) = r_d(D) = r^+(D), r_{\min}(D), r_{\max}(D), r_{\text{sum}}(D), r^*(D), r^-(D)$ and $r^0(D)$.

Deleting an arc e from D we cannot decrease any distance k between any two nodes of D. Hence $f(D-e) \geqslant f(D)$. Analogously, adding a new arc e to D we have $f(D+e) \leqslant f(D)$. These inequalities enable us to say that a digraph D is: *minimal by* f, if $f(D-e) > f(D)$ for every arc e of D; *critical by* f, if $f(D-u) \neq f(D)$ for every node u of D; *maximal by* f, if $f(D+e) < f(D)$ for every arc e of the complement of D.

Although there exist several radii of digraphs and then several classes of minimal, critical and maximal digraphs, we have found only three papers about these graphs: Fridman 1973, [9]; Ismailov 1971, [22] and Gliviak and Knor 1995, [14].

Paper [9] describes two classes of non-connected digraphs D such that for every arc $e \notin E(D)$ we have:

1. either $r(D+e) < r(D)$ or $D+e$ has less strong components than D;
2. either $r_{\min}(D+e) < r_{\min}(D)$ or $D+e$ has less strong components than D.

Moreover this paper proves the following estimation.

Theorem 18. (See [9].) *Let D be a digraph with p nodes, q arcs and a finite radius $r \geqslant 2$. Then $q \leqslant p(p-r) + (r^2 - r - 2)/2$.*

Let $D(n,k,r)$ be a digraph with n nodes, k strong components and a radius r. Paper [22] gives the exact lower bound for the number of arcs in such a digraph and describes all extremal digraphs.

Now we give several results from [14]. P. Kyš (oral communication, [14]) showed the following characterisation of digraphs minimal by $r^+ = r_d = r$.

Theorem 19. *A digraph D is minimal by r if and only if D is a directed rooted out-tree (i.e. acyclic digraph with $\mathrm{id}_D(x) = 1$ for all $x \in V(D)$ except the root u for which $\mathrm{id}_D(u) = 0$).*

Critical digraphs by r or r^0 with radius r or r^0 equal to ∞ and 1 are described in [14]. Then three results concerning radius $r = 2$ or $r^0 = 2$ are proved.

Theorem 20. (See [14].) *Let D be a digraph such that $r(D) = 2$ and $|V(D)| \geqslant 5$. Then D is critical by r if and only if the complement of D consists of a collection of oriented cycles.*

Theorem 21. (See [14].) *Let D be a digraph on an even number of nodes such that $r^0(D) = 2$ and $|V(D)| \geqslant 6$. Then D is critical by r^0 if and only if the complement of D consists of a collection of independent arcs and oriented two-cycles.*

Theorem 22. (See [14].) *Let D be a digraph. Then there are infinitely many digraphs critical by r^0 with radius two on an odd number of nodes, containing D as an induced subgraph.*

Maximal digraphs D by r or r^0 with radius $r(D)$ or $r^0(D)$ equal to $\infty, 1, 2$ are described in [14]. Then the following two general existence theorems for critical and maximal digraphs are proved.

Theorem 23. (See [14].) *Let D be a digraph, and let t satisfy $3 \leqslant t < \infty$. Then there exists an infinite number of digraphs H such that:*

1. *D is an induced subgraph of H;*
2. *$r(H) = r^-(H) = r^0(H) = t$;*
3. *H is critical by r, r^- and r^0;*
4. *H is maximal by r and r^-.*

Theorem 24. (See [14].) *Let D be a digraph, and let t satisfy $3 \leqslant t < \infty$. Then there is an infinite number of digraphs maximal by r^0 with radius $r^0 = t$, which contain D as an induced subgraph.*

The following problem is analogous to Theorem 4.

Problem 6. Is it possible to reduce the study of digraphs critical by r to any subclass of these graphs?

At the end we give the following practical problem.

Problem 7. What can be said about results concerning analogous problems for theory of nets (i.e. digraphs with given natural estimation of edges and a source and sink).

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