ON ROHN'S RELATIVE SENSITIVITY COEFFICIENT OF THE OPTIMAL VALUE FOR A LINEAR-FRACTIONAL PROGRAM

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Abstract. In this note we consider a linear-fractional programming problem with equality linear constraints. Following Rohn, we define a generalized relative sensitivity coefficient measuring the sensitivity of the optimal value for a linear program and a linear-fractional minimization problem with respect to the perturbations in the problem data.

By using an extension of Rohn's result for the linear programming case, we obtain, via Charnes-Cooper variable change, the relative sensitivity coefficient for the linear-fractional problem. This coefficient involves only the measure of data perturbation, the optimal solution for the initial linear-fractional problem and the optimal solution of the dual problem of linear programming equivalent to the initial fractional problem.

Keywords: linear-fractional programming, generalized relative sensitivity coefficient

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1. Introduction

Let us consider the linear programming problem
$P(A,b,c)$. Find

$$Z(A,b,c) = \min\{cx\colon\ Ax = b,\ x \geqslant 0\} \tag{1.1}$$

where A is a given $m \times n$ real matrix, $b \in \mathbb{R}^m$ is a resource vector and $c \in \mathbb{R}^n$ is the objective vector. We suppose that the feasible set of $P(A,b,c)$ is bounded and nonempty.

Sensitivity analysis of solutions in linear programming is very important and can be realized [6] by a) parametric programming, b) perturbation analysis, c) stability analysis and d) error analysis.

Parametric linear programming consists in analyzing the effects of variations in parameters on the optimal basis. Perturbation analysis investigates conditions under which the parameters may be perturbed without affecting the choice of the optimal basis. Static stability (i.e. the parametric variations are not essentially dependent on time) of a linear programming problem consists in characterizing the neighborhood of the optimal solution with a specified set of parameters. Error analysis analyzes the sensitivity of the optimal solution due to measurement errors of the coefficients around the optimal basis.

Sensitivity analysis investigates also the behavior of the optimal value with respect to some variations of the problem data. In this sense, let $S = (A, b, c)$ be a parameter set of the linear program (1.1) and let $S' = (A', b', c')$ be a perturbation of S. We are interested in the behavior of $Z(S + hS')$ near $h = 0$. Mills [2] has proposed as a measure of sensitivity to local perturbations the concept of "marginal value" of a linear programming problem defined by

$$\lim_{h \to 0_+} \frac{Z(S + hS') - Z(S)}{h}. \tag{1.2}$$

According to Williams [9], a necessary and sufficient condition for the existence of the limit in (1.2) is that both the primal and dual optimal sets of the linear program with a coefficient S be bounded both from below and above.

The marginal value (if it exists) can be computed as

$$\max_x \max_y L(S, x, y)$$

where (x, y) is the optimal pair of the primal and dual variables and $L(\cdot)$ is the Lagrangian function

$$L(S, x, y) = cx + yb - yAx.$$

In the next section, we present an alternative to the "marginal value" for a linear program, i.e. Rohn's [5] relative sensitivity coefficient.

2. The linear programming case

Rohn [5] constructed for the linear programming problem $P(A, b, c)$ a sensitivity coefficient involving only the initial data $S = (A, b, c)$, the optimal solution x^* of the problem (1.1) and the optimal solution y^* of the dual problem

$$\max\{by\colon\ A^T y \leqslant c\}. \tag{2.1}$$

In order to define this sensitivity coefficient for a given real number $r > 0$, Rohn considered the family of perturbed problem

$P(A', b', c')$. Find

$$Z(A', b', c') = \min\{c'x\colon\ A'x = b',\ x \geqslant 0\} \tag{2.2}$$

for which the relative errors of the data do not exceed r, i.e. the inequalities

$$|A' - A| \leqslant r|A|,\ \ |b' - b| \leqslant r|b|,\ \ |c' - c| \leqslant r|c| \tag{2.3}$$

hold, where the absolute value $|A|$ of a matrix $A = (a_{ij})$ is defined by

$$|A| = (|a_{ij}|), \tag{2.4}$$

and similarly for vectors.

We shall assume that the optimal value $Z(A, b, c)$ is nonzero, and there exists $t > 0$ such that for each $r \in [0, t]$, each problem (2.2) (satisfying (2.3)) has an optimal solution.

Rohn [5] introduced the sensitivity coefficient of the problem $P(A, b, c)$ as

$$E(A, b, c) = \lim_{r \to 0_+} \frac{E_r(A, b, c)}{r}, \tag{2.5}$$

where for a given r, $E_r(A, b, c)$ denotes the maximal relative error of the optimal value, i.e.

$$E_r(A, b, c) = \max\Big\{\Big|\frac{Z(A', b', c') - Z(A, b, c)}{Z(A, b, c)}\Big|\colon\ A', b', c' \text{ satisfy (2.3)}\Big\}. \tag{2.6}$$

Now we consider a more general case of the perturbed problem than that studied by Rohn [5]. We define the perturbation set for $P(A, b, c)$ as the set $H_r(A'', b'', c'')$ of all systems (A', b', c') satisfying the conditions

$$|A' - A| \leqslant rA'', \tag{2.7}$$

$$|b' - b| \leqslant rb'', \tag{2.8}$$

$$|c' - c| \leqslant rc'', \tag{2.9}$$

where A'', b'' and c'' are a given matrix and vectors with nonnegative elements having the same dimensions as A, b and c, respectively.

Now, for a given $r > 0$, we define the maximal relative error of the optimal value under relative data errors (A'', b'', c'') by

$$\delta_r(A, b, c; A'', b'', c'') = \max\left\{\left|\frac{Z(A', b', c') - Z(A, b, c)}{Z(A, b, c)}\right| : (A', b', c') \in H_r(A'', b'', c'')\right\}.$$

Definition 2.1. The generalized relative sensitivity coefficient is defined by:

$$\delta(A, b, c; A'', b'', c'') = \lim_{r \to 0_+} \frac{\delta_r(A, b, c; A'', b'', c'')}{r}. \tag{2.10}$$

When $A'' = |A|$, $b'' = |b|$, $c'' = |c|$, we obtain Rohn's sensitivity coefficient, i.e.

$$E(A, b, c) = \delta(A, b, c; |A|, |b|, |c|).$$

Lemma 2.1. *If the problem $P(A, b, c)$ has a unique non-degenerate optimal solution x^* and $cx^* \neq 0$, then there exists $r > 0$ such that, for every $(A', b', c') \in H_r(A'', b'', c'')$*

$$Z(A', b', c') = Z(A, b, c) + y^*(b' - b) + (c' - c)x^* + y^*(A - A')x^* + O(r^2)$$

where y^* is the optimal solution of the dual problem (2.1).

The proof of this lemma follows an argument similar to that used by Rohn [5] in proving his theorem (see also [8], theorem 2.4, and [4]).

We will show that the relative sensitivity coefficient $\delta(A, b, c; A'', b'', c'')$ defined by (2.10) can be expressed in terms of the optimal solutions x^*, y^* of problems (1.1) and (2.1).

Theorem 2.1. *Let the optimal solution x^* of* (1.1) *be nondegenerate, let the nonbasic relative cost coefficients be positive and let $cx^* \neq 0$. Then*

$$\delta(A, b, c; A'', b'', c'') = \frac{|c''|x^* + |b''|\,|y^*| + |y^*|\,|A''|x^*}{|cx^*|} \tag{2.11}$$

holds, where y^* is the optimal dual solution.

Proof. From Lemma 2.1 we can express the optimal value of (2.2) by

$$Z(A', b', c') = Z(A, b, c) + y^*(b' - b) + (c' - c)x^* + y^*(A - A')x^* + O(r^2).$$

Like in Rohn [5], we observe that the maximal value of $y^*(b' - b)$ for all b' satisfying (2.8) is equal to $r|y^*|\,|b''|$; similarly for the other two terms. Hence, we obtain

$$\delta_r(A, b, c; A'', b'', c'') = \frac{|c''|x^* + |b''|\,|y^*| + |y^*|\,|A''|x^*}{|cx^*|} r + O(r^2).$$

Therefore, taking the limit (see (2.10)) and using the fact that $cx^* \neq 0$, we get (2.11). $\square$

3. The linear-fractional case

The above result can be also extended to the following linear-fractional program: $F(A,b,c,d)$. Find

$$f(A,b,c,d) = \min\Big\{\frac{c_1x+c_0}{d_1x+d_0}:\ Ax=b,\ x\geqslant 0\Big\}, \tag{3.1}$$

where we denote

$$c=(c_1,c_0)\in\mathbb{R}^{n+1},\ d=(d_1,d_0)\in\mathbb{R}^{n+1}.$$

We suppose that

$$d_1x+d_0>0,\ \forall x\in X\equiv\{x\in\mathbb{R}^n:\ Ax=b,\ x\geqslant 0\}, \tag{3.2}$$

$$\text{the feasible set } X \text{ is bounded and nonempty set.} \tag{3.3}$$

Like in the previous linear case, let A'',b'',c'',d'' be the perturbation bounds (having nonnegative elements). For a positive real number r, we define the perturbation set for $F(A,b,c,d)$ as the set $H_r(A'',b'',c'',d'')$ of all systems (A',b',c',d') satisfying the conditions which are similar to (2.7)–(2.9):

$$\begin{aligned} H_r(A'',b'',c'',d'') = \{(A',b',c',d'):\ & |A'-A|\leqslant rA'',\ |b'-b|\leqslant rb'',\\ & |c'-c|\leqslant rc'',\ |d'-d|\leqslant rd''\}. \end{aligned} \tag{3.4}$$

Let us suppose that the linear-fractional problem $F(A,b,c,d)$ has the optimal value $f(A,b,c,d)\neq 0$.

Definition 3.1. The generalized relative sensitivity coefficient of the linear-fractional programming problem $F(A,b,c,d)$ is the number $S(A,b,c,d;A'',b'',c'',d'')$ given by

$$S(A,b,c,d;A'',b'',c'',d'') = \lim_{r\to 0_+}\frac{S_r(A,b,c,d;A'',b'',c'',d'')}{r},$$

where the maximal relative error of the optimal value is expressed by

$$\begin{aligned} &S_r(A,b,c,d;A'',b'',c'',d'') =\\ &= \max\Big\{\Big|\frac{f(A',b',c',d')-f(A,b,c,d)}{f(A,b,c,d)}\Big|:\ (A',b',c',d')\in H_r(A'',b'',c'',d'')\Big\}. \end{aligned}$$

Definition 3.2. We say that the problem $F(A,b,c,d)$ is regular if there exists a real positive number r' such that the problem $F(A',b',c',d')$ satisfies the conditions (3.2), (3.3), for every (A',b',c',d') in $H_{r'}(A'',b'',c'',d'')$.

Employing the Charnes-Cooper [1] (see also [7]) change of variables

$$u = tx,\ t = \frac{1}{d_1x + d_0} > 0,$$

we can associate with the fractional problem $F(A, b, c, d)$ the following linear program:

$P(B, e, c)$. Find

$$Z(B, e, c) = \min_{u,t}(c_1u + c_0t)$$

subject to

$$B\begin{pmatrix} u \\ t \end{pmatrix} = e,$$

$$\begin{pmatrix} u \\ t \end{pmatrix} \geqslant 0,$$

where

$$B = \begin{pmatrix} A & -b \\ d_1 & d_0 \end{pmatrix},\ e = (0, \dots, 0, 1) \in \mathbb{R}^{m+1}.$$

Since in problem $P(B, e, c)$ the right hand side $e = (0, \dots, 0, 1)$ is not perturbed, we can consider a partial sensitivity coefficient of this linear program defined by

$$G(B, c) = \delta(B, e, c; B'', \theta, c''), \tag{3.5}$$

where

$$B'' = \begin{pmatrix} A'' & b'' \\ d_1'' & d_0'' \end{pmatrix},\ \theta = (0, 0, \dots, 0) \in \mathbb{R}^{n+1}. \tag{3.6}$$

Theorem 3.1. *If $F(A, b, c, d)$ is regular, if $P(B, e, c)$ has a single nondegenerate optimal solution $(u'', t'') \in \mathbb{R}^{n+1}$ and if $c_1u'' + c_0t'' \neq 0$, then*

$$S(A, b, c, d; A'', b'', c'', d'') = \frac{c_1''x'' + c_0'' + |v''|A''x'' + |v_{m+1}''|(d_1''x'' + d_0'') + |v''|b''}{|c_1x'' + c_0|} \tag{3.7}$$

where $v^* = (v'', v_{m+1}'') \in \mathbb{R}^{m+1}$ is the dual optimal solution for the dual problem of $P(B, e, c)$.

Proof. Indeed, since $F(A, b, c, d)$ is regular, hence

$$Z(B', e, c') = f(A', b', c', d') \text{ for all } (A', b', c', d') \in H_r(A'', b'', c'', d'').$$

Consequently,

$$\delta(B, e, c; B'', \theta, c'') = S(A, b, c, d; A'', b'', c'', d''). \tag{3.8}$$

But, by applying Theorem 2.1 to the linear program $P(B,e,c)$, we have the maximal relative error of the optimal value of $P(B,e,c)$ given by

$$\delta(B,e,c;B'',\theta,c'') = \frac{c_1''u'' + c_0''t'' + |(v'',v_{m+1}'')|B''(u'',t'')}{|c_1u'' + c_0t''|}, \tag{3.9}$$

where $(u'',t'') \in \mathbb{R}^{n+1}$ is an optimal solution of $P(B,e,c)$.

However, (3.9) and the fact that $t'' > 0$ yield

$$\delta(B,e,c;B'',\theta,c'') = \frac{c_1''\frac{u''}{t''} + c_0'' + |(v'',v_{m+1}'')|B''(\frac{u''}{t''},1)}{|c_1\frac{u''}{t''} + c_0|} \tag{3.10}$$

Then by a simple calculation, taking $x'' = u''/t''$, from (3.6), (3.8) and (3.10) we obtain (3.7). □

For $d_1 = 0$, $d_0 = 1$, $c_0 = 0$, $c_0'' = 0$, $d_1'' = 0$ and $d_0'' = 0$ the sensitivity coefficient reduces to (2.11).

4. Conclusions

In the paper we have obtained a generalized relative sensitivity coefficient for the linear-fractional problem, involving only the measure of data perturbation, the optimal solution for the initial problem and the optimal solution of the dual problem of linear programming equivalent to the initial fractional problem. These results are related to those obtained by Podkaminer [3] concerning the partial derivatives of the optimal value function of fractional-linear programming.

References

[1] *A. Charnes, W. W. Cooper*: Programming with linear fractional programming. Naval Res. Logist. Quart. *9* (1962), 3–4, 181–186.

[2] *H. D. Mills*: Marginal values of matrix games and linear programs. Linear inequalities and related systems (H. W. Kuhn, A. W. Tucker, eds.). Princeton University Press, Princeton, 1956, pp. 183–193.

[3] *L. Podkaminer*: The dual price and other parameters in the fractional programming problem. Przeglad Statyst. *18* (1971), 3–4, 333–338. (In Polish.)

[4] *A. Prékopa*: On the probability distribution of the optimum of a random linear program. J. SIAM Control *4* (1966), 1, 211–222.

[5] *J. Rohn*: On sensitivity of the optimal value of a linear program. Ekonom. -Mat. Obzor *25* (1989), 1, 105–107.

[6] *J. K. Sengupta; K. A. Fox*: Economic Analysis and Operations Research: Optimization Techniques in Quantitative Economic Models. Studies in Mathematical and Managerial Economics, vol. 10, North-Holland Publishing Co., Amsterdam-London, American Elsevier Publishing Co., Inc., New York, 1969.

[7] *I. M. Stancu-Minasian*: Fractional Programming: Theory, Methods and Applications. Kluwer Academic Publishers, Dordrecht, 1997.

[8] *I. M. Stancu-Minasian*: Stochastic Programming with Multiple Objective Functions. Editura Academiei Române, Bucuresti and D. Reidel Publishing Company, Dordrecht, Boston, Lancester, 1984.

[9] *A. C. Williams*: Marginal values in linear programming. Journal of Society of Industrial and Applied Mathematics *11* (1963), 1, 82–94.

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