DIRICHLET FUNCTIONS OF REFLECTED BROWNIAN MOTION

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Abstract. We give a complete analytical characterization of the functions transforming reflected Brownian motions to local Dirichlet processes.

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1. Introduction

In classical stochastic calculus semimartingales have proved to be the "right" class of stochastic integrators. It is an important issue of stochastic analysis to describe the functions that leave the class of semimartingales invariant. In the one-dimensional case, which we will restrict ourselves to throughout this paper, the Itô-Tanaka formula ([11, VI.1.5]) tells us that the transformed process $F(X)$ is a semimartingale whenever X is a semimartingale and F a function that is locally the difference of two convex functions. Conversely, in the case of Brownian motion, such functions are known to be the only semimartingale functions, i.e., the only functions to preserve the semimartingale property ([5, Theorem 5.5]). For a generalization of this result to continuous local martingales see [3, Théorème 1].

However, some natural procedures such as C^1-transformations of Brownian motion (see [2]) and the Fukushima decomposition in the theory of Dirichlet forms (see [9]) suggest the need of studying Dirichlet processes, i.e., processes admitting a decomposition into a sum of a martingale and a process of zero energy (see [2, 6, 7]). From the point of view of stochastic analysis the class of local Dirichlet processes

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appears to be a convenient and useful object for two reasons. It extends the class of semimartingales ([18, Section 4]) but, nevertheless, enjoys properties allowing to develop some elements of stochastic calculus (see [18, 19]).

Bouleau and Yor established a change of variables formula which describes transformations of one-dimensional semimartingales with absolutely continuous functions admitting a locally bounded density ([4]). In [18] the second author developed a generalized Bouleau-Yor formula. This formula includes transformations of continuous local martingales with absolutely continuous functions admitting a locally square integrable density. The transformations governed by the generalized Bouleau-Yor formula represent mappings from the class of continuous semimartingales into the class of continuous local Dirichlet processes ([18, Section 5]). Besides analogies in the theory of symmetric Dirichlet forms ([9]), for the special case of Brownian motion, this result was also obtained by H. Föllmer, P. Protter, A. N. Shiryaev in [8, 3.45].

So, in a natural way, we are led to the problem whether all functions transforming a given semimartingale into a local Dirichlet process allow to apply the generalized Bouleau-Yor formula. In the case of Brownian motion this amounts to asking whether these functions are necessarily absolutely continuous admitting a locally square integrable density.

A first result concerning this problem was established by J. Bertoin in [2, Théorème 3.2]. He showed that every function F that transforms Brownian motion into a Dirichlet process is absolutely continuous and has a density F' satisfying $\int_{\mathbb{R}} (F'(x))^2 \exp(-x^2)\,\mathrm{d}x < +\infty$.

In this paper we will study analytical properties of the functions that transform reflected Brownian motions stopped at certain passage times into local Dirichlet processes. As a result we will see that, in the case of stopped reflected Brownian motions, the answer to the above problem is yes. In a subsequent paper this will play an important role in the investigation of strong Markov continuous local Dirichlet processes (see [17, Chapter 3]) because, under weak hypotheses, every continuous strong Markov process can be reduced to a stopped reflected Brownian motion by means of spatial transformations and time changes (see [1] or [15, Lemma 4.2]).

After introducing definitions and recalling basic facts on reflected Brownian motion in Section 2, we deal with Dirichlet functions in Section 3. We call a function F a Dirichlet function for the local Dirichlet process Y if the transformed process $F(Y)$ is again a local Dirichlet process. First we show that all Dirichlet functions of Brownian motion stopped when leaving (a, b) are absolutely continuous and admit a density that is locally square integrable on (a, b). As a main ingredient we use local time in our proofs. Then we develop necessary conditions for Dirichlet functions of reflected Brownian motion stopped at certain passage times. Finally, we deduce a complete analytical characterization of the Dirichlet functions for reflected Brownian motions.

2. Definitions and prerequisites

Throughout, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ satisfying the usual hypotheses. For any process X and $\mathbb{F}$-stopping time T, the notation X^T is used for the process X stopped at T. Let $(X, \mathbb{F})$ be a continuous semimartingale. Then $\langle X \rangle$ always denotes the square variation process associated with X. Furthermore, $L^X(t, a)$ denotes the (right) local time of X spent in a up to time t. This is a process which is $\mathbb{P}$-a.s. continuous in t and right-continuous with left hand limits in a such that the occupation times formula

$$\int_0^t g(X_s)\,\mathrm{d}\langle X \rangle_s = \int_{\mathbb{R}} g(a) L^X(t, a)\,\mathrm{d}a \quad \mathbb{P}\text{-a.s.} \tag{1}$$

holds for every nonnegative Borel function g and $t \geqslant 0$ ([11, Chapter VI]).

We consider local Dirichlet processes in the framework of the approach to stochastic integration by Russo and Vallois (see [12, 13, 14]). Let us recall some basic notions (see [18, Section 4]).

Let $Q = (Q_t)_{t \geqslant 0}$ be an adapted right-continuous process having left limits at every $t > 0$. Then $Q = (Q_t)_{t \geqslant 0}$ has *zero energy* if

$$\lim_{\varepsilon \to 0} \mathbb{E} \frac{1}{\varepsilon} \int_0^\infty (Q_{s+\varepsilon} - Q_s)^2\,\mathrm{d}s = 0.$$

We say that Q has *zero quadratic variation* if there exists a non-decreasing sequence of stopping times $(T_n)_{n \in \mathbb{N}}$ with $\lim\limits_{n \to \infty} T_n = \infty$ a.s. such that, for each $n \in \mathbb{N}$, the stopped process Q^{T_n} has zero energy.

R e m a r k 2.1. Every process Q of zero quadratic variation is automatically $\mathbb{P}$-a.s. continuous. This immediately follows from [13, (1.16)].

A *Dirichlet process* Y is defined to be a process admitting a decomposition $Y = Y_0 + M + Q$, where $(M, \mathbb{F})$ is a right-continuous martingale with $M_0 = 0$ and Q is a process of zero energy with $Q_0 = 0$.

A process Y is a *local Dirichlet process* if there exists a non-decreasing sequence of stopping times $(T_n)_{n \in \mathbb{N}}$ with $\lim\limits_{n \to \infty} T_n = \infty$ a.s. such that, for each $n \in \mathbb{N}$, the stopped process Y^{T_n} is a Dirichlet process. We say that $(T_n)_{n \in \mathbb{N}}$ reduces the local Dirichlet process.

Lemma 2.2.

(i) *A process Y is a local Dirichlet process if and only if it admits a decomposition $Y = Y_0 + M + Q$, where $(M, \mathbb{F})$ is a right-continuous local martingale with $M_0 = 0$ and Q is a process of zero quadratic variation with $Q_0 = 0$.*

(ii) *If $Y = Y_0 + M + Q$ is a continuous local Dirichlet process then there exists a sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ reducing Y and satisfying $\langle M^{T_n}\rangle \leqslant n$ and $|Q^{T_n}| \leqslant n$.*

(iii) *If Y is a local Dirichlet process and T a stopping time then Y^T is also a local Dirichlet process.*

Proof. Using Remark 2.1, the proof is exactly the same as that of [18, 4.5] where only the case of *continuous* local Dirichlet processes was considered. □

Clearly, the class of local Dirichlet processes extends the class of continuous semimartingales.

Let Y be a local Dirichlet process. A universally measurable real function F is said to be a *Dirichlet function* for Y if $F(Y)$ is a local Dirichlet process.

We need the following information on reflected Brownian motion.

Let $W = W_0 + M + V$ be a continuous semimartingale, T a stopping time and $r_1 \in \mathbb{R}\cup\{-\infty\}$, $r_2 \in \mathbb{R}\cup\{+\infty\}$ with $r_1 < r_2$. We call W a Brownian motion stopped at T with reflecting barriers r_1, r_2 if

(i) $W_0 \in [r_1, r_2] \cap \mathbb{R}$,

(ii) $\langle M\rangle_t = t \wedge T$, $t \geqslant 0$, a.s. and

(iii) $V_t = \frac{1}{2}L^W(t, r_1) - \frac{1}{2}L^W(t, r_2)$, $t \geqslant 0$, a.s., where, by convention, $L^W(\cdot, -\infty) = L^W(\cdot, +\infty) = 0$. In the case $T = \infty$ we briefly call W a Brownian motion with reflecting barriers r_1, r_2 or a reflected Brownian motion.

Obviously, in the special case $T = \infty$, $r_1 = -\infty$, $r_2 = \infty$ in 2.4, W is a Brownian motion.

Moreover, if W is a Brownian motion stopped at T with reflecting barriers r_1, r_2 then, on a possibly extended probability space, there exists a Brownian motion $\widetilde{W}$ with reflecting barriers r_1, r_2 such that $W = \widetilde{W}^T$.

Furthermore, Brownian motion W with reflecting barriers r_1, r_2 can be characterized as the pathwise unique solution to the stochastic differential equation

$$W_t = W_0 + B_t + \tfrac{1}{2}L^W(t, r_1) - \tfrac{1}{2}L^W(t, r_2),\ t \geqslant 0, \quad W_0 \in [r_1, r_2] \cap \mathbb{R}$$

(see [16, Section 1]).

Finally, it is known that a version of Brownian motion with reflecting barriers r_1, r_2 can be obtained by transforming a Brownian motion B with

(i) $f(x) := r_1 + |x - r_1|, \quad -\infty < r_1 < r_2 = +\infty,$

(ii) $f(x) := r_2 - |x - r_2|, \quad -\infty = r_1 < r_2 < +\infty,$

(iii) $f(x) := r_1 + |x - r_1 + 2n(r_2 - r_1)|,$
if $x \in [r_1 - (2n+1)(r_2 - r_1), r_1 - (2n-1)(r_2 - r_1)]$ $(n \in \mathbb{Z})$,
$-\infty < r_1 < r_2 < +\infty.$

We outline the proof restricting ourselves to case iii). The Itô-Tanaka formula ([11, VI.1.5]) yields

$$\begin{aligned} W_t = f(B_t) &= f(B_0) + \int_0^t f'_-(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_{\mathbb{R}} L^B(t,x)\,\mathrm{d}f'(x) \\ &= f(B_0) + \int_0^t f'_-(B_s)\,\mathrm{d}B_s \\ &\quad + \sum_{n\in\mathbb{Z}} \Big(L^B\big(t, r_1 + 2n(r_2 - r_1)\big) - L^B\big(t, r_2 + 2n(r_2 - r_1)\big)\Big). \end{aligned}$$

Rewriting the last line in terms of L^W by [11, VI.1.9] we obtain

$$W_t = f(B_0) + \int_0^t f'_-(B_s)\,\mathrm{d}B_s + \tfrac{1}{2}L^W(t,r_1) - \tfrac{1}{2}L^W(t,r_2).$$

Since $(f'_-(x))^2 = 1$ for every $x \in \mathbb{R}$, P. Lévy's characterization theorem ([11, IV.3.6]) shows that $\int_0^\cdot f'_-(B_s)\,\mathrm{d}B_s$ is a Brownian motion. Thus W is a Brownian motion with reflecting barriers.

3. Dirichlet functions

First we study the analytical properties of Dirichlet functions for Brownian motion stopped at passage times. Looking at the construction in Section 2 we then derive necessary conditions for Dirichlet functions of reflected Brownian motion stopped at certain passage times in 3.6. In 3.7 we state a complete analytical characterization of Dirichlet functions for reflected Brownian motion.

We need some preparatory lemmas.

Lemma 3.1. *Let B be a Brownian motion with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ and $a \in \mathbb{R}$. Furthermore, suppose H is a continuous bounded process such that, for every $t \geqslant 0$, H_t is independent of $\mathcal{F}_t$. Then*

$$\mathbb{E}\bigg(\int_0^T H_s\,\mathrm{d}_s L^B(s,a)\bigg) = \int_0^\infty \mathbb{E}H_s\,\mathrm{d}_s\big(\mathbb{E}L^B(T\wedge s, a)\big)$$

holds for every $\mathbb{F}$-stopping time T with $\mathbb{E}L^B(T,a) < \infty$.

Proof. Since H is bounded and $L^B(T,a) < \infty$ a.s. we can compute

$$\int_0^T H_s\,\mathrm{d}_s L^B(s,a) \quad \text{a.s.}$$

pathwise as a Riemann-Stieltjes integral. Given a sequence of partitions

$$\pi_m : (0 = t_0^{(m)} < t_1^{(m)} < \ldots < t_{n_m}^{(m)}),\ m \in \mathbb{N},$$

satisfying $\lim_{m\to\infty} t_{n_m}^{(m)} = \infty$ and $\lim_{m\to\infty} \sup\{|t_{i+1}^{(m)} - t_i^{(m)}|;\ i = 0, 1, \ldots, n_m - 1\} = 0$ we therefore have

$$\int_0^T H_s \,\mathrm{d}_s L^B(s,a) = \lim_{m\to\infty} \sum_{i=1}^{n_m-1} H_{t_{i+1}^{(m)}} \big(L^B(t_{i+1}^{(m)} \wedge T, a) - L^B(t_i^{(m)} \wedge T, a) \big) \quad \text{a.s.}$$

Since H is bounded and $\mathbb{E}L^B(T,a) < \infty$ these Riemann sums converge in $L^1(P)$ by the dominated convergence theorem. Using the hypothesis that H_t is independent of $\mathcal{F}_t$ we conclude

$$\begin{aligned}\mathbb{E}\int_0^T H_s \,\mathrm{d}_s L^B(s,a) &= \lim_{m\to\infty} \mathbb{E}\Big[\sum_{i=1}^{n_m-1} H_{t_{i+1}^{(m)}} \big(L^B(t_{i+1}^{(m)} \wedge T, a) - L^B(t_i^{(m)} \wedge T, a) \big) \Big] \\ &= \lim_{m\to\infty} \sum_{i=1}^{n_m-1} \mathbb{E}H_{t_{i+1}^{(m)}} \big(\mathbb{E}L^B(t_{i+1}^{(m)} \wedge T, a) - \mathbb{E}L^B(t_i^{(m)} \wedge T, a) \big).\end{aligned}$$

Since $(\mathbb{E}H_t)_{t\geqslant 0}$ is continuous and bounded and $\mathbb{E}L^B(T,a) < \infty$ these Riemann sums converge and we obtain

$$\mathbb{E}\int_0^T H_s \,\mathrm{d}_s L^B(s,a) = \int_0^\infty \mathbb{E}H_s \,\mathrm{d}_s\Big(\mathbb{E}L^B(s \wedge T, a) \Big).$$

$\square$

Lemma 3.2. *Let B be a Brownian motion with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ and T an $\mathbb{F}$-stopping time with $\mathbb{E}T < \infty$. Then*

$$\mathbb{E}\bigg(\int_0^T h(B_s, B_{s+\varepsilon} - B_s) \,\mathrm{d}s \bigg) = \int_{\mathbb{R}} \int_{\mathbb{R}} h(a,x) \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{x^2}{2\varepsilon} \right) \mathrm{d}x \,\mathbb{E}L^B(T,a) \,\mathrm{d}a$$

holds for all measurable and bounded functions $h\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and every $\varepsilon > 0$.

Proof. First we observe that $\infty > \mathbb{E}T = \mathbb{E}\int_0^T \mathrm{d}\langle B\rangle_s = \mathbb{E}\int_{\mathbb{R}} L^B(T,a)\,\mathrm{d}a = \int_{\mathbb{R}} \mathbb{E}L^B(T,a)\,\mathrm{d}a$ and conclude that $\mathbb{E}L^B(T,a) < \infty$ for Lebesgue-almost every $a \in \mathbb{R}$. Let g be a bounded and measurable function defined on $([0,+\infty)\times\Omega\times\mathbb{R}, \mathcal{B}([0,+\infty))\times\mathcal{F}\times\mathcal{B}(\mathbb{R}))$. Here $\mathcal{B}(E)$ denotes the σ-algebra of Borel subsets of a topological space E. We then have

$$\int_0^T g(s,\omega,B_s(\omega))\,\mathrm{d}s = \int_{\mathbb{R}} \int_0^T g(s,\omega,a)\,\mathrm{d}_s L^B(s,\omega,a)\,\mathrm{d}a \quad \mathbb{P}\text{-a.s.} \tag{2}$$

Indeed, by a monotone class argument, it suffices to verify this equality for indicator functions $g = 1_{[u,v]\times C\times D}$ with $0 \leqslant u < v < +\infty$, $C \in \mathcal{F}$ and $D \in \mathcal{B}(\mathbb{R})$. But this is a simple consequence of the occupation times formula (1).

Now, applying formula (2) to g defined by $g(s,\omega,a) = h(a, B_{s+\varepsilon}(\omega) - B_s(\omega))$, $(s,\omega,a) \in [0,+\infty)\times\Omega\times\mathbb{R}$, and using Fubini's theorem and Lemma 3.1 we calculate

$$\begin{aligned}\mathbb{E}\int_0^T h(B_s, B_{s+\varepsilon} - B_s)\,\mathrm{d}s &= \mathbb{E}\biggl(\int_{\mathbb{R}}\int_0^T h(a, B_{s+\varepsilon} - B_s)\,\mathrm{d}_s L^B(s,a)\,\mathrm{d}a\biggr)\\ &= \int_{\mathbb{R}}\mathbb{E}\int_0^T h(a, B_{s+\varepsilon} - B_s)\,\mathrm{d}_s L^B(s,a)\,\mathrm{d}a\\ &= \int_{\mathbb{R}}\int_0^\infty \mathbb{E}h(a, B_{s+\varepsilon} - B_s)\,\mathrm{d}_s\bigl(\mathbb{E}L^B(s\wedge T, a)\bigr)\,\mathrm{d}a.\end{aligned}$$

Since, for every s, $B_{s+\varepsilon} - B_s$ is normally distributed with mean zero and variance ε and, in particular, the inner integrand does not depend on s, the assertion now follows immediately. □

Lemma 3.3. *Suppose that $-\infty \leqslant a < b \leqslant \infty$, B is a Brownian motion starting in $x_0 \in (a,b)$ and $\tau := \inf\{t \geqslant 0\colon\ B_t \notin (a,b)\}$. Let F be a universally measurable function such that the process $F(B)$ is right-continuous on $[0,\tau)$ $\mathbb{P}$-a.s. Then F is continuous on (a,b).*

P r o o f. In the context of Markov processes, F would be finely continuous on (a,b) and, since the fine topology for Brownian motion coincides with the usual topology, the result would follow. However, in our situation the initial state x_0 is fixed and we need another argument.

In a first step we only assume that $F(B)$ is right-continuous on $[0,\tau)$ on a set $A \in \mathcal{F}$ of *strictly positive probability* and show that then F is continuous at x_0. To this end, let $\varepsilon > 0$ and define $\varrho = \inf\{t \geqslant 0\colon\ |F(B_t) - F(x_0)| \geqslant \varepsilon\} \wedge \tau$ on A and τ otherwise. Then ϱ is $\mathcal{F}$-measurable and $\varrho > 0$ $\mathbb{P}$-a.s. We consider

$$I(\omega) = \{B_t(\omega)\colon\ t < \varrho(\omega)\}.$$

Since B is continuous, $I(\omega)$ is an interval which, obviously, contains x_0. Furthermore, by the martingale property of B (or by the law of iterated logarithm), $I(\omega)\cap(x_0,+\infty) \neq \emptyset$ and $I(\omega)\cap(-\infty,x_0) \neq \emptyset$ $\mathbb{P}$-a.s. This yields that $I(\omega)$ is a neighbourhood of x_0 $\mathbb{P}$-a.s. But, for $\omega \in A$, $x \in I(\omega)$ implies $|F(x) - F(x_0)| < \varepsilon$. Since $\varepsilon > 0$ was chosen arbitrarily, this means that F is continuous at x_0.

For a general $x \in (a,b)$ we define the $\mathbb{P}$-a.s. finite stopping time $\sigma_x = \inf\{t \geqslant 0\colon\ B_t = x\}$. Then B^x defined by $B^x_t = B_{\sigma_x + t}$, $t \geqslant 0$, is again a Brownian motion, now starting from x. Because of $\mathbb{P}(\sigma_x < \tau) > 0$, the hypothesis of the lemma entails that $F(B^x)$ satisfies the assumption of the first step. Hence F is continuous at x and the proof of the lemma is completed. □

In the above proof it would be sufficient to know that F is only Lebesgue measurable. We also notice that Lemma 3.3 remains true for continuous local martingales M with $\langle M\rangle_\infty = +\infty$ instead of the Brownian motion B.

In the following we say that a real function F is locally square integrable on a Borel set B if, for every compact set K with $K \subseteq B$, the function $F\mathbf{1}_K$ is square integrable. For every interval $I \subseteq \mathbb{R}$ let $W^{1,2}(I)$ (or $W^{1,2}_{\mathrm{loc}}(I)$) denote the Sobolev space of all absolutely continuous functions on I admitting a density that is square integrable (respectively, locally square integrable) on I.

Theorem 3.4. *Suppose that $-\infty \leqslant a < b \leqslant \infty$, B is a Brownian motion starting in $x_0 \in (a,b)$ and $\tau := \inf\{t \geqslant 0\colon\ B_t \notin (a,b)\}$. If F is a Dirichlet function for the stopped process B^τ then $F_{|(a,b)} \in W^{1,2}_{\mathrm{loc}}\big((a,b)\big)$.*

P r o o f. The definition of a Dirichlet function implies that $F(B^\tau)$ is right-continuous. Hence the assumptions of Lemma 3.3 are satisfied. By Lemma 3.3, F is continuous on (a,b). Let now c and d be real numbers such that $c < x_0 < d$ and $[c,d] \subseteq (a,b)$. It is sufficient to show that the restriction of F to (c,d) belongs to $W^{1,2}_{\mathrm{loc}}\big((c,d)\big)$. But F is bounded and continuous on $[c,d]$. Furthermore, setting $\varrho := \inf\{t \geqslant 0\colon\ B_t \notin (c,d)\}$, the stopped process $F(B^\varrho)$ is again a local Dirichlet process by Lemma 2.2, (iii). Consequently, without loss of generality we may assume that a and b are finite and that F is continuous and hence bounded on $[a,b]$.

1) By Lemma 2.2, (ii), we find a sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times reducing the continuous local Dirichlet process $F(B^\tau)$ such that

$$\lim_{\varepsilon\to 0} \mathbb{E}\biggl(\frac{1}{\varepsilon}\int_0^\infty \Bigl(F(B^{\tau\wedge T_n}_{s+\varepsilon}) - F(B^{\tau\wedge T_n}_s)\Bigr)^2 \,\mathrm{d}s\biggr)\ (\leqslant n)$$

exists for each $n \in \mathbb{N}$. Since F is continuous and bounded we observe that

$$\lim_{\varepsilon\to 0} \mathbb{E}\biggl(\frac{1}{\varepsilon}\int_0^\infty \Bigl(F(B^{\tau\wedge T_n}_{s+\varepsilon}) - F(B^{\tau\wedge T_n}_s)\Bigr)^2 \,\mathrm{d}s\biggr) = \lim_{\varepsilon\to 0} \mathbb{E}\biggl(\frac{1}{\varepsilon}\int_0^{\tau\wedge T_n} \Bigl(F(B_{s+\varepsilon}) - F(B_s)\Bigr)^2 \,\mathrm{d}s\biggr)$$

by the dominated convergence theorem. From the well-known property $\mathbb{E}\tau < \infty$ (cf., e.g., [11, VI.2.8, 2°)] for $f = 1$) we now see that $\mathbb{E}(\tau \wedge T_n) < \infty$ and using $B_{s+\varepsilon} = B_s + (B_{s+\varepsilon} - B_s)$ we conclude by 3.2 that

$$(3)\qquad \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{\mathbb{R}} \bigl(F(a+x) - F(a)\bigr)^2 \frac{1}{\sqrt{2\pi\varepsilon}} \exp\biggl(-\frac{x^2}{2\varepsilon}\biggr)\,\mathrm{d}x\,\mathbb{E}L^B(\tau\wedge T_n, a)\,\mathrm{d}a$$

exists for each $n \in \mathbb{N}$.

2) It is well known ([11, VI.2.8]) that $f(x) := \frac{1}{2}\,\mathbb{E}L^B(\tau, x)$, $x \in \mathbb{R}$, has the form

$$f(x) = \begin{cases} (x-a)(b-x_0)/(b-a) & a \leqslant x \leqslant x_0 \leqslant b, \\ (x_0-a)(b-x)/(b-a) & a \leqslant x_0 \leqslant x \leqslant b, \\ 0 & \text{otherwise.} \end{cases}$$

Thus f is a continuous bounded function with $A := \{x \in \mathbb{R}\colon\ f(x) > 0\} = (a,b)$. The functions $f_n(x) := \mathbb{E}L^B(\tau \wedge T_n, x)$, $x \in \mathbb{R}$, form a non-decreasing sequence converging pointwise to f. Looking at the Tanaka formula we verify that f_n, $n \in \mathbb{N}$, are continuous. Thus the functions f_n converge uniformly to f.

3) In order to show $F_{|(a,b)} \in W^{1,2}_{\mathrm{loc}}\big((a,b)\big)$ it suffices to prove $F_{|I_m} \in W^{1,2}(I_m)$, $m \in \mathbb{N}$, where $I_m := \{x \in \mathbb{R}\colon\ f(x) \geqslant \frac{1}{m}\}$, since I_m, $m \in \mathbb{N}$, are compact intervals with $(a,b) = \bigcup\limits_m I_m$ and $I_m \subseteq I_{m+1}$. Let $I_m = [r,s]$. Since (f_n) converges uniformly to f by 2) we find some n such that $\mathbb{E}L^B(\tau \wedge T_n, \cdot) > \frac{1}{2m}$ on I_m. Consequently, the existence of the limit (3) entails that

$$\left\{ l \int_s^r \int_{\mathbb{R}} \big(F(a+x) - F(a)\big)^2 \sqrt{\frac{l}{2\pi}} \exp\left(-\frac{l}{2}x^2\right) \mathrm{d}x\,\mathrm{d}a;\ l \in \mathbb{N} \right\}$$

is bounded. We set

$$\Delta_l(x) := x^2 l \sqrt{\frac{l}{2\pi}} \exp\left(-\frac{l}{2}x^2\right), \quad x \in \mathbb{R},\ l \in \mathbb{N},$$

and

$$h_l(a) := \int_{\mathbb{R}} \frac{F(a+x) - F(a)}{x} \Delta_l(x)\,\mathrm{d}x, \quad a \in [r,s],\ l \in \mathbb{N}.$$

Since $\Delta_l(x)\,\mathrm{d}x$ is a probability measure on $\mathbb{R}$ the Cauchy-Schwarz inequality yields

$$h_l^2(a) \leqslant \int_{\mathbb{R}} \left(\frac{F(a+x) - F(a)}{x}\right)^2 \Delta_l(x)\,\mathrm{d}x, \quad a \in [r,s].$$

Thus we conclude that $(h_l)_{l\in\mathbb{N}}$ is bounded in the Hilbert space $L^2([r,s], \mathrm{d}a)$. By passing to a subsequence if necessary we may therefore assume that $(h_l)_{l\in\mathbb{N}}$ converges weakly to some $h \in L^2([r,s], \mathrm{d}a)$. As a consequence we have

$$\int_r^u h(a)\,\mathrm{d}a = \lim_{l\to\infty} \int_r^u \int_{\mathbb{R}} \frac{F(a+x) - F(a)}{x} \Delta_l(x)\,\mathrm{d}x\,\mathrm{d}a$$

for every $u \in [r,s]$. We set $G(u) := \int_r^u F(y)\,\mathrm{d}y$. Since $\big((F(a+x)-F(a))/x\big)\Delta_l(x)$ is bounded we obtain

$$\begin{aligned}\int_r^u h(a)\,\mathrm{d}a &= \lim_{l\to\infty}\int_{\mathbb{R}}\int_r^u (F(a+x)-F(a))\,\mathrm{d}a\,\frac{\Delta_l(x)}{x}\,\mathrm{d}x\\ &= \lim_{l\to\infty}\int_{\mathbb{R}}\Big(\frac{G(u+x)-G(u)}{x}-\frac{G(r+x)}{x}\Big)\Delta_l(x)\,\mathrm{d}x,\ \ u\in[r,s],\end{aligned}$$

by Fubini's theorem. As the measures $\Delta_l(x)\,\mathrm{d}x$ converge weakly to the Dirac measure δ_0 we conclude

$$\int_r^u h(a)\,\mathrm{d}a = F(u)-F(r),\ \ u\in[r,s].$$

Thus $F_{|I_m}$ lies in $W^{1,2}(I_m)$. This completes the proof of Theorem 3.4. □

We remark that, in step 3) of the above proof, we use some analytical arguments due to J. Bertoin (see the proof of [2, Théorème 3.4]).

By [18, Corollary 5.8] (see also [8, 3.45]), we know that, in fact, every absolutely continuous function with locally square integrable density is a Dirichlet function for Brownian motion. Thus, combining Theorem 3.4 and [18, Corollary 5.8] we get

Theorem 3.5. *Let B be a Brownian motion starting in $x_0 \in \mathbb{R}$. A real function is a Dirichlet function for B if and only if it is absolutely continuous with locally square integrable density.*

It is easy to show that, in the case of Brownian motion, $W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ is the set of all functions for which the generalized Bouleau-Yor formula ([18, Theorem 2.2]) can be stated. Thus, we have characterized the Dirichlet functions of Brownian motion as the functions inducing a transformation according to the generalized Bouleau-Yor formula. We emphasize the complete analogy to the well-known fact that a real function transforms Brownian motion into a semimartingale if and only if it allows to apply the Itô-Tanaka formula.

Theorem 3.6. *Suppose that $-\infty \leqslant r_1 < r_2 \leqslant +\infty$ and W is a Brownian motion with reflecting barriers r_1, r_2, starting in $x_0 \in [r_1,r_2]\cap\mathbb{R}$, i.e.,*

$$(*)\begin{cases} W_t \in [r_1,r_2]\cap\mathbb{R},\ \ t\geqslant 0,\\ W_t = x_0 + B_t + \tfrac12 L^W(t,r_1) - \tfrac12 L^W(t,r_2) \quad \text{a.s.,}\end{cases}$$

where B is a Brownian motion with $B_0 = 0$. We consider the situations

(i) $\tau = \infty$,

(ii) $-\infty < r_1$, $x_0 < r_2$ *and* $\tau := \inf\{t \geqslant 0\colon W_t = c\}$ *for some* $c \in (x_0, r_2)$,

(iii) $r_2 < +\infty$, $x_0 > r_1$ *and* $\tau := \inf\{t \geqslant 0\colon W_t = c\}$ *for some* $c \in (r_1, x_0)$.

If F is a Dirichlet function for the stopped process W^τ we have in the respective situation

(i) $F_{|[r_1,r_2]\cap\mathbb{R}}$ *is absolutely continuous with density in* $L^2_{\mathrm{loc}}([r_1,r_2]\cap\mathbb{R})$,

(ii) $F_{|[r_1,c)}$ *is absolutely continuous with density in* $L^2_{\mathrm{loc}}([r_1,c))$,

(iii) $F_{|(c,r_2]}$ *is absolutely continuous with density in* $L^2_{\mathrm{loc}}((c,r_2])$.

P r o o f. The stochastic differential equation $(*)$ being unique in the sense of probability law it suffices to show the assertion for a given Brownian motion with reflecting barriers r_1, r_2. We choose the reflected Brownian motion constructed in Section 2, i.e., we assume that $W = f(\widetilde{B})$ is obtained from a Brownian motion $\widetilde{B}$ by a transformation f as described in Section 2. (Note that the case $r_1 = -\infty$, $r_2 = +\infty$ is treated in Theorem 3.5 since W then is a Brownian motion.) We set in the respective situation

(i) $\varrho := \infty$, $s_1 := r_1 - 1$, $s_2 := r_2 + 1$ (using the usual conventions for $\pm\infty$),

(ii) $\varrho := \inf\{t \geqslant 0\colon \widetilde{B}_t = c\} \wedge \inf\{t \geqslant 0\colon \widetilde{B}_t = s_1\}$ for some $s_1 \in (-\infty, r_1)$, $s_2 := c$,

(iii) $\varrho := \inf\{t \geqslant 0\colon \widetilde{B}_t = c\} \wedge \inf\{t \geqslant 0\colon \widetilde{B}_t = s_2\}$ for some $s_2 \in (r_2, +\infty)$, $s_1 := c$.

Always ϱ is a stopping time satisfying $\varrho \leqslant \tau$. By hypothesis and Lemma 2.2, (iii), $\big(F(W^\tau)\big)^\varrho = (F \circ f)(\widetilde{B}^\varrho)$ is a local Dirichlet process. Theorem 3.4 now gives $F \circ f_{|(s_1,s_2)} \in W^{1,2}_{\mathrm{loc}}\big((s_1,s_2)\big)$. Since $f_{|[r_1,r_2]\cap\mathbb{R}}$ is always the identical mapping the theorem follows immediately. □

In the case of reflected Brownian motions the necessary condition for Dirichlet functions in Theorem 3.6 even turns out to be sufficient.

Theorem 3.7. *Suppose $-\infty \leqslant r_1 < r_2 \leqslant +\infty$ and let W be a Brownian motion with reflecting barriers r_1, r_2, starting in $x_0 \in [r_1,r_2]\cap\mathbb{R}$. A function F is a Dirichlet function for W if and only if $F_{|[r_1,r_2]\cap\mathbb{R}}$ is absolutely continuous and admits a density in $L^2_{\mathrm{loc}}([r_1,r_2]\cap\mathbb{R})$.*

S k e t c h o f p r o o f. The necessity of the condition follows from Theorem 3.6. Now suppose F is a function such that $F_{|[r_1,r_2]\cap\mathbb{R}}$ is absolutely continuous with density in $L^2_{\mathrm{loc}}([r_1,r_2]\cap\mathbb{R})$. Without loss of generality we may assume that F is constant on $\mathbb{R} \setminus (r_1,r_2)$ thus ensuring $F \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$.

In order to show that F satisfies the requirements of the generalized Bouleau-Yor formula ([18, 2.2]) we choose

$$B := \{r_1, r_2\} \cap \mathbb{R},$$

$$S_n := \inf\{t \geqslant 0\colon \langle W\rangle_t > n\} \wedge \inf\{t \geqslant 0\colon L^W(t,r_1) > n \text{ or } L^W(t,r_2) > n\},\ n \in \mathbb{N},$$

$$T_n := S_n \wedge \inf\{t \geqslant 0\colon W_t \notin (-n,n)\},\ n \in \mathbb{N},$$

in the notation of [18, 2.2], and a density F' vanishing on B. We have $\lim_{n\to\infty} T_n = \infty$ a.s. and $L^W(T_n, a) = 0$ for every $a \notin (-n, n)$. Furthermore $\mathbb{E}L^W(T_n, a) \leqslant 3n$ holds in view of the Tanaka formula. Thus we verify

$$F' \in L^2_{\mathrm{loc}}(\mathbb{R}) \subseteq \bigcap_{n\in\mathbb{N}} L^2(\mathbb{R}, \mathbb{E}L^W(T_n, a)\,\mathrm{d}a).$$

Therefore we can apply [18, Theorem 2.2]. We now see that [18, Corollary 5.8] remains valid without any changes in the proofs of the underlying statements [18, 5.5–5.7]. Thus, $F(W)$ is a local Dirichlet process. $\square$

Thus, the functions transforming a reflected Brownian motion into a local Dirichlet process are exactly the functions inducing a transformation according to the generalized Bouleau-Yor formula.

Remark 3.8. Let W and τ be as in Theorem 3.6.(ii) (or (iii)). Suppose that, in the respective case,

(ii) $F_{|[r_1,c)}$ is absolutely continuous with density in $L^2_{\mathrm{loc}}([r_1, c))$,
(iii) $F_{|(c,r_2]}$ is absolutely continuous with density in $L^2_{\mathrm{loc}}((c, r_2])$.

By using respectively the modified stopping times
(ii) $T_n := S_n \wedge \inf\{t \geqslant 0\colon\ W_t \notin (-n, c - \frac{1}{n})\}$,
(iii) $T_n := S_n \wedge \inf\{t \geqslant 0\colon\ W_t \notin (c + \frac{1}{n}, n)\}$
in the above proof it is possible to show that $F(W^\tau)$ is a local Dirichlet process up to the stopping time $\tau = \lim_{n\to\infty} T_n$. To this end, we have to apply a slight modification of [18, Corollary 5.8], where the state space $\mathbb{R}$ is replaced by $(-\infty, c)$ or $(c, +\infty)$, respectively.

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References

[1] *Assing, S., Schmidt, W.*: Continuous Strong Markov Processes in Dimension 1—A Stochastic Calculus Approach. Lecture Notes in Math. 1688, Springer, Berlin, 1998.
[2] *Bertoin, J.*: Les processus de Dirichlet et tant qu' Espace de Banach. Stochastics *18* (1988), 155–168.
[3] *Bertoin, J.*: Une application du calcul du nombre de montées et de descentes aux fonctions de martingales locales continues. Ann. Inst. H. Poincaré Probab. Statist. *24* (1988), 201–207.
[4] *Bouleau, N., Yor, M.*: Sur la variation quadratique des temps locaux de certaines semimartingales. C. R. Acad. Sci. Paris Sér. I Math. *292* (1981), 491–494.

[5] *Çinlar, E., Jacod, J., Protter, P., Sharpe, M.J.*: Semimartingales and Markov processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete *54* (1980), 161–219.
[6] *Föllmer, H.*: Calcul d'Itô sans probabilités. Sém. Prob. XV, Lecture Notes in Math. 850. Springer, Berlin, 1981, pp. 143–150.
[7] *Föllmer, H.*: Dirichlet processes. Lecture Notes in Math. 851. Springer, Berlin, 1981, pp. 476–478.
[8] *Föllmer, H., Protter, P., Shiryaev, A.N.*: Quadratic covariation and an extension of Itô's formula. Bernoulli *1* (1995), 149–169.
[9] *Fukushima, M. Oshima, Y. Takeda, M.*: Dirichlet Forms and Symmetric Markov Processes. Walter de Gruyter, Berlin, 1994.
[10] *Protter, P.*: Stochastic Integration and Differential Equations. Springer, Berlin, 1992.
[11] *Revuz, D., Yor, M.*: Continuous Martingales and Brownian Motion. 2nd edition, Springer, Berlin, 1994.
[12] *Russo, F., Vallois, P.*: Forward, backward and symmetric stochastic integration. Probab. Theory Related Fields *97* (1993), 403–421.
[13] *Russo, F., Vallois, P.*: The generalized covariation process and Itô formula. Stochastic Process. Appl. *59* (1995), 81–104.
[14] *Russo, F., Vallois, P.*: Itô formula for C^1-functions of semimartingales. Probab. Theory Related Fields *104* (1996), 27–41.
[15] *Schmidt, W.*: Über streng Markovsche stetige Semimartingale. Habilitationsschrift. Friedrich-Schiller-Universität, Jena.
[16] *Schmidt, W.*: On stochastic differential equations with reflecting barriers. Math. Nachr. *142* (1989), 135–148.
[17] *Wolf, J.*: Zur stochastischen Analysis stetiger lokaler Dirichletprozesse. Dissertation. Friedrich-Schiller-Universität, Jena.
[18] *Wolf, J.*: Transformations of semimartingales and local Dirichlet processes. Stochastics Stochastics Rep. *62* (1997), 65–101.
[19] *Wolf, J.*: An Itô formula for local Dirichlet processes. Stochastics Stochastics Rep. *62* (1997), 103–115.

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