THE DIRECTED DISTANCE DIMENSION OF ORIENTED GRAPHS

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Abstract. For a vertex v of a connected oriented graph D and an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices of D, the (directed distance) representation of v with respect to W is the ordered k-tuple $r(v \mid W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$, where $d(v, w_i)$ is the directed distance from v to w_i. The set W is a resolving set for D if every two distinct vertices of D have distinct representations. The minimum cardinality of a resolving set for D is the (directed distance) dimension $\dim(D)$ of D. The dimension of a connected oriented graph need not be defined. Those oriented graphs with dimension 1 are characterized. We discuss the problem of determining the largest dimension of an oriented graph with a fixed order. It is shown that if the outdegree of every vertex of a connected oriented graph D of order n is at least 2 and $\dim(D)$ is defined, then $\dim(D) \leqslant n-3$ and this bound is sharp.

Keywords: oriented graphs, directed distance, resolving sets, dimension

MSC 2000: 05C12, 05C20

1. Introduction

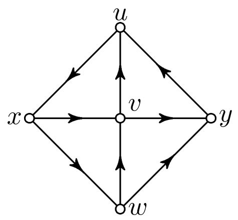
For an oriented graph D of order n, an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices of D, and a vertex v of D, the k-vector (ordered k-tuple)

$$r(v \mid W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

is referred to as the (*directed distance*) *representation of* v *with respect to* W, where $d(x, y)$ denotes the directed distance from x to y, that is, the length of a shortest directed $x - y$ path in D. Since directed $x - y$ paths need not exist in D, even if D is connected (its underlying graph is connected), the vector $r(v \mid W)$ need not exist as well. If $r(v \mid W)$ exists for every vertex v of D, then the set W is called a *resolving set* for D if every two distinct vertices of D have distinct representations. A resolving set of minimum cardinality is called a *basis* for D and this cardinality is the (*directed distance*) *dimension* $\dim(D)$ of D. Of course, not every oriented graph has a dimension. An oriented graph of dimension k is also called k-*dimensional*.

To determine whether an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices in an oriented graph D is a resolving set, we need only show that the representations of the vertices of $V(D) - W$ are distinct since $r(w_i \mid W)$ is the only representation whose ith coordinate is 0.

The directed distance dimension of an oriented graph is a natural analogue of the metric dimension of a graph that was introduced independently by Harary and Melter [2] and Slater [3], [4]. This concept was also investigated in [1] as a result of studying a problem in pharmaceutical chemistry.



D

Figure 1. An oriented graph D with dimension 2

In the oriented graph D of Figure 1, let $W_1 = \{u, v\}$. The five representations of the vertices of D with respect to W_1 are $r(u \mid W_1) = (0, 2)$, $r(v \mid W_1) = (1, 0)$, $r(w \mid W_1) = (2, 1)$, $r(x \mid W_1) = (2, 1)$, and $r(y \mid W_1) = (1, 3)$. Since x and w have the same representation, W_1 is not a resolving set for D.

The five representations of the vertices of D with respect to $W_2 = \{u, v, w\}$ are

$$r(u \mid W_2) = (0, 2, 2), \quad r(v \mid W_2) = (1, 0, 3), \quad r(w \mid W_2) = (2, 1, 0),$$
$$r(x \mid W_2) = (2, 1, 1), \quad r(y \mid W_2) = (1, 3, 3)$$

Since these five 3-vectors are distinct, W_2 is a resolving set for D. However, W_2 is not a basis for D. To see this, let $W_3 = \{x, y\}$. Then $r(u \mid W_3) = (1, 3)$, $r(v \mid W_3) = (2, 1)$, $r(w \mid W_3) = (3, 1)$, $r(x \mid W_3) = (0, 2)$, and $r(y \mid W_3) = (2, 0)$, which are distinct as well. So W_3 is a resolving set for D. Since there is no 1-element resolving set for D, it follows that W_3 is a basis and $\dim(D) = 2$.

Now let T be the tournament shown in Figure 2. Table 1 gives all 2-element choices for W and shows that for each such choice, there exist two equal 2-vectors, thus showing that $\dim(T) \geqslant 3$. However, $\dim(T) = 3$ since $\{v_1, v_3, v_6\}$ is a basis for T. Figure 3 shows an oriented graph D containing T as an induced subdigraph. The set $W = \{x, y\}$ is a basis of D, so $\dim(D) = 2$. Hence we have the possibly unexpected property that the 3-dimensional tournament T is an induced subdigraph of the 2-dimensional oriented graph D.

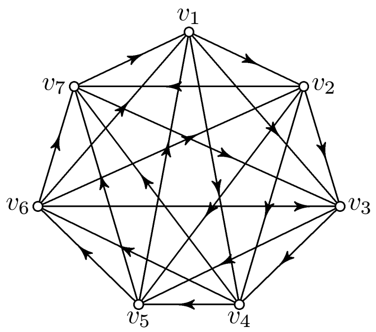


Figure 2. The tournament T

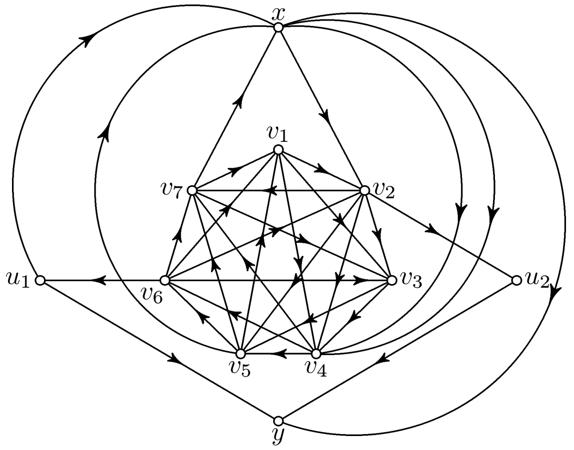


Figure 3. The digraph D

There is a fundamental question here—one whose answer is not known to us, but one which deserves further study. What is a necessary and sufficient condition for the dimension of a digraph D to be defined? Certainly, if D is strong, then $\dim(D)$ is defined. Also, if D is connected and contains a vertex such that $D - v$ is strong, then $\dim(D)$ is defined. This last statement follows because if $\operatorname{od} v > 0$, then $V(D) - \{v\}$ is a resolving set; while if $\operatorname{id} v > 0$, then $V(D)$ is a resolving set. There are numerous other sufficient conditions for $\dim(D)$ to be defined.

W	equivalent vectors
$\{v_1, v_2\}$	$r(v_5 \mid W) = r(v_7 \mid W) = (1, 2)$
$\{v_1, v_3\}$	$r(v_6 \mid W) = r(v_7 \mid W) = (1, 1)$
$\{v_1, v_4\}$	$r(v_5 \mid W) = r(v_7 \mid W) = (1, 2)$
$\{v_1, v_5\}$	$r(v_6 \mid W) = r(v_7 \mid W) = (1, 2)$
$\{v_1, v_6\}$	$r(v_2 \mid W) = r(v_3 \mid W) = (2, 2)$
$\{v_1, v_7\}$	$r(v_5 \mid W) = r(v_6 \mid W) = (1, 1)$
$\{v_2, v_3\}$	$r(v_1 \mid W) = r(v_6 \mid W) = (1, 1)$
$\{v_2, v_4\}$	$r(v_5 \mid W) = r(v_7 \mid W) = (2, 2)$
$\{v_2, v_5\}$	$r(v_1 \mid W) = r(v_5 \mid W) = (1, 2)$
$\{v_2, v_6\}$	$r(v_4 \mid W) = r(v_5 \mid W) = (2, 1)$
$\{v_2, v_7\}$	$r(v_4 \mid W) = r(v_5 \mid W) = (2, 1)$
$\{v_3, v_4\}$	$r(v_1 \mid W) = r(v_2 \mid W) = (1, 1)$
$\{v_3, v_5\}$	$r(v_6 \mid W) = r(v_7 \mid W) = (1, 2)$
$\{v_3, v_6\}$	$r(v_1 \mid W) = r(v_2 \mid W) = (1, 2)$
$\{v_3, v_7\}$	$r(v_2 \mid W) = r(v_6 \mid W) = (1, 1)$
$\{v_4, v_5\}$	$r(v_2 \mid W) = r(v_3 \mid W) = (1, 1)$
$\{v_4, v_6\}$	$r(v_1 \mid W) = r(v_2 \mid W) = (1, 2)$
$\{v_4, v_7\}$	$r(v_1 \mid W) = r(v_3 \mid W) = (1, 2)$
$\{v_5, v_6\}$	$r(v_2 \mid W) = r(v_3 \mid W) = (1, 2)$
$\{v_5, v_7\}$	$r(v_2 \mid W) = r(v_4 \mid W) = (1, 1)$
$\{v_6, v_7\}$	$r(v_4 \mid W) = r(v_5 \mid W) = (1, 1)$

Table 1.

2. 1-dimensional oriented graphs

In this section we characterize those oriented graphs having dimension 1. We also describe some properties of bases for 1-dimensional oriented graphs.

Theorem 2.1. *Let D be a nontrivial oriented graph of order n. Then $\dim(D) = 1$ if and only if there exists a vertex v in D such that*

(i) *D contains a hamiltonian path P with terminal vertex v such that $\operatorname{id}_D v = 1$; and*

(ii) *if the hamiltonian path P in* (i) *is of the form*

$$v_{n-1}, v_{n-2}, \ldots, v_1, v,$$

then, for each pair i, j of integers with $1 \leqslant i < j \leqslant n-1$, the digraph $D - E(P)$ contains no arc of the form (v_j, v_i).

Proof. Assume that $\dim(D) = 1$. Let $W = \{v\}$, $v \in V(D)$, be a basis of D. Then the distance $d(u,v)$ from u to v is defined for each vertex u in D and the set $\{d(u,v);\ u \in V(D)\} = \{0,1,\ldots,n-1\}$. Thus, we may assume that $V(D) = \{v, v_1, v_2, \ldots, v_{n-1}\}$ where $d(v_i, v) = i$ $(1 \leqslant i \leqslant n-1)$. Clearly, $\operatorname{id} v = 1$. Since $d(v_{n-1}, v) = n-1$, there exists a hamiltonian path in D, namely P: $v_{n-1}, v_{n-2}, \ldots, v_1, v$, so (i) holds. Furthermore, if there exists a pair i, j of integers $(1 \leqslant i < j \leqslant n-1)$ such that the arc (v_j, v_i) is in $D - E(P)$, then $j \neq i+1$ and $d(v_j, v) = d(v_{i+1}, v)$ (shown in Figure 4). This contradicts the fact that $\{d(u,v);\ u \in V(D)\}$ consists of n distinct integers, so (ii) holds.

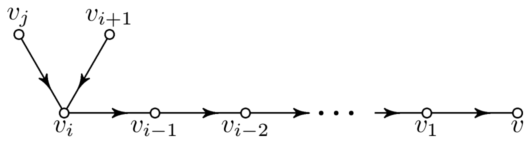


Figure 4.

Conversely, assume that there is a vertex v in D such that (i) and (ii) hold. We show that $W = \{v\}$ is a resolving set of D. Since $d(u,v)$ is defined for each $u \in V(D)$, it suffices to show that the set $\{d(v_i, v);\ 1 \leqslant i \leqslant n-1\}$ consists of $n-1$ distinct integers. Suppose that this is not the case. Then there exist integers i, j $(1 \leqslant i < j \leqslant n-1)$ such that $d(v_j, v) = d(v_i, v) = \ell$. Let P_1 be a $v_i - v$ path and P_2 a $v_j - v$ path in D such that P_1 and P_2 have the same length ℓ. Since $\operatorname{id} v = 1$, there exists a vertex $v_k \neq v$ in D that belongs to both P_1 and P_2. Assume that v_k is the vertex with largest index k such that the path $v_k, v_{k-1}, \ldots, v_1, v$ is on both P_1 and P_2 (see Figure 5).

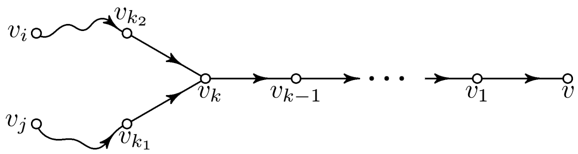


Figure 5.

Let $(v_{k_1}, v_k) \in E(P_1)$ and $(v_{k_2}, v_k) \in E(P_2)$ where $(v_{k_1}, v_k) \neq (v_{k_2}, v_k)$. Clearly, $k_1 > k$ and $k_2 > k$. It follows that at least one of these arcs is in $D - E(P)$, but this is a contradiction to (ii). $\square$

We now present some facts concerning bases in 1-dimensional oriented graphs.

Theorem 2.2. *Let D be a digraph of order n with $\dim(D) = 1$. Furthermore, let v_1 and v_2 be distinct vertices of D with $d(v_1, v_2) = 2$ such that both $\{v_1\}$ and*

$\{v_2\}$ *are bases of* D. *If* v *is a vertex of* D *such that* $(v_1, v), (v, v_2) \in E(D)$, *then* $\{v\}$ *is also a basis of* D.

Proof. To show that $\{v\}$ is a basis of D, we show that for each $u \in V(D)$, the distance $d(u, v)$ is defined and the set $\{d(u, v);\ u \in V(D)\}$ consists of n distinct integers.

First notice that $\operatorname{id} v = 1$, for otherwise there exist distinct vertices x and y of D such that $d(x, v) = d(y, v) = 1$. Since $\operatorname{id} v_2 = 1$, by Theorem 2.1, we have

$$d(x, v_2) = d(y, v_2) = d(x, v) + 1 = 2$$

This contradicts the fact that $\{v_2\}$ is a basis of D.

Furthermore, suppose that there exist vertices u, w in D such that $d(u, v) = d(w, v)$. Since $\operatorname{id} v = 1$, each $u - v$ path contains the arc (v_1, v) as its terminal arc, as does each $w - v$ path, so

$$d(u, v_1) = d(w, v_1) = d(u, v) - 1$$

Again, this contradicts the fact that $\{v_1\}$ is a basis of D. □

We now have an immediate consequence of Theorem 2.2.

Corollary 2.3. *If* D *is a* 1*-dimensional oriented graph of order* $n \geqslant 3$ *such that* $\{v\}$ *is a basis of* D *for every vertex* v *in* D, *then* D *is a directed cycle.*

Proof. Let $V(D) = \{v_1, v_2, \ldots, v_n\}$. By Theorem 2.2, $\operatorname{id} v = 1$ for every vertex v of D. Moreover, D contains a hamiltonian path P. We can assume that

$$P\colon\ v_n, v_{n-1}, \ldots, v_2, v_1$$

Next, we show that D contains the cycle

$$C_n\colon\ v_n, v_{n-1}, \ldots, v_2, v_1, v_n$$

Since $\operatorname{id} v_n = 1$, there exists a unique vertex v such that $(v, v_n) \in E(D)$. If $v \neq v_1$, then $(v_i, v_n) \in E(D)$ for some i $(2 \leqslant i \leqslant n - 1)$. Since $\{v_n\}$ is a basis of D, there exists a hamiltonian path in D with terminal vertex v_n. However, since every vertex has indegree 1, the only possible path in D with v_n as its terminal vertex is

$$P'\colon\ v_{n-1}, v_{n-2}, \ldots, v_{i+1}, v_i, v_n$$

Since P' has length $n - i$, it is not a hamiltonian path. This contradicts the fact that $\{v_n\}$ is a basis. So D contains the cycle C_n. Furthermore, since $\operatorname{id} v = 1$, D cannot contain any arc except those in C_n. So $D = C_n$. □

We can improve Corollary 2.3 slightly.

Corollary 2.4. *If D is a 1-dimensional oriented graph of order $n \geqslant 3$ such that*

$$|\{v;\ \{v\} \text{ is a basis of } D\}| \geqslant n-1$$

then D is a directed cycle.

Proof. Let $V(D) = \{v, v_1, v_2, \ldots, v_{n-1}\}$. Without loss of generality, we assume that $\{v_i\}$ is a basis of D for $1 \leqslant i \leqslant n-1$. By Corollary 2.3, it suffices to show that $\{v\}$ is a basis as well.

We claim that $\operatorname{od} v > 0$. Suppose that this is not the case. Then for each vertex u $(\neq v)$, the distance $d(v,u)$ is not defined, which contradicts the fact that $\{u\}$ is a basis of D. Hence, there is a vertex x $(\neq v)$ such that $(v,x) \in E(D)$. Since $\{x\}$ is also a basis of D, then by Theorem 2.1(i), D contains a hamiltonian path with terminal vertex x and $\operatorname{id} x = 1$. This implies that there exists a vertex y distinct from x and v such that $(y,v) \in E(D)$. It follows that $d(y,x) = 2$ and by Theorem 2.2, $\{v\}$ is also a basis of D. □

The bound in Corollary 2.4 cannot be improved in general. For example, consider the oriented graph D of order n in Figure 6. Since $\{v_i\}$ is a basis for D for $1 \leqslant i \leqslant n-2$, $\dim(D) = 1$. However, neither $\{v_{n-1}\}$ nor $\{v_n\}$ is a basis D. So $|\{v;\ \{v\} \text{ is a basis of } D\}| = n-2$ and D is not a directed cycle.

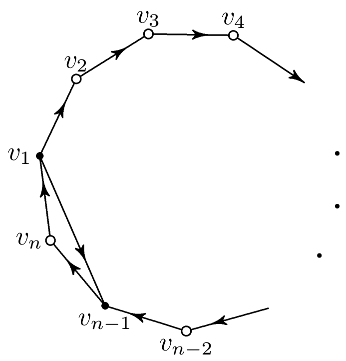


Figure 6. An oriented graph with $(n-2)$ 1-element bases

There is only one 1-dimensional oriented tree of every order.

Theorem 2.5. *For every oriented tree T, $\dim(T) = 1$ or $\dim(T)$ is undefined. Furthermore, if $\dim(T) = 1$, then T is a directed hamiltonian path.*

P r o o f. There are certainly oriented trees whose dimension is undefined, for example, any orientation of a star $K_{1,t}$, where $t \geqslant 3$. Now let T be an oriented tree whose dimension is defined. Since T contains no cycles, for every pair x, y of vertices, whenever $d(x, y)$ is defined, $d(y, x)$ is undefined. Thus $\dim(T) = 1$.

If $\dim(T) = 1$, then, by Theorem 2.1, T contains a hamiltonian path P and so $T = P$. $\square$

3. On oriented graphs with large dimension

We have characterized those oriented graphs with dimension 1. But how large can the dimension of an oriented graph of order n be? In this section, we describe upper bounds for the dimension of a connected oriented graph in terms of lower bounds for the outdegrees of its vertices. The outdegree of every vertex in the oriented graph D of Figure 7 is 2, yet $\dim(D)$ is undefined. Such examples exist regardless of the outdegrees.

D:

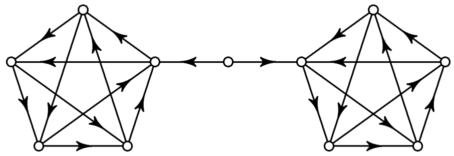


Figure 7. The oriented graph D

Theorem 3.1. *If D is a connected oriented graph of order $n \geqslant 3$ with $\operatorname{od} v \geqslant 1$ for all $v \in V(D)$ such that $\dim(D)$ is defined, then $\dim(D) \leqslant n - 2$.*

P r o o f. Let D be an oriented graph satisfying the hypothesis of the theorem. Certainly $\dim(D) \leqslant n - 1$. Assume, to the contrary, that $\dim(D) = n - 1$. Let $W = \{v_1, v_2, \ldots, v_{n-1}\}$ be a basis for D and let $V(D) - W = \{x\}$. Since $\operatorname{od} x \geqslant 1$, assume, without loss of generality, that x is adjacent to v_1. Also, since $\operatorname{od} v_1 \geqslant 1$, we may assume that v_1 is adjacent to v_2. Since $\dim(D) = n - 1$, $r(v_i \mid W - \{v_i\}) = r(x \mid W - \{v_i\})$ for $1 \leqslant i \leqslant n - 1$. Since x is adjacent to v_1, it follows that v_2 is adjacent to v_1, but this contradicts the fact that D is an oriented graph. $\square$

We now describe a class of oriented graphs. For $k \geqslant 2$, let D_k be an oriented graph with vertex set

$$V(D_k) = \{u, v, w_1, w_2, \ldots, w_k\}$$

and let $E(D_k)$ consist of the arc (u, v) and the arcs (v, w_j) and (w_j, u) for $1 \leqslant j \leqslant k$. The oriented graph D_k is shown in Figure 8. Then D_k has order $n = k + 2$ and $\operatorname{od} v \geqslant 1$ for all $v \in V(D_k)$. We claim that $\dim(D_k) = n - 3$.

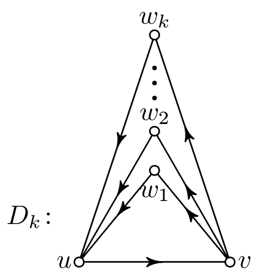


Figure 8. The oriented graph D_k with minimum outdegree 1

First we show that $\dim(D_k) \leqslant n-3$. Let $W = \{w_2, w_3, \ldots, w_k\}$, where then $|W| = n-3$. The distances $d(u, w_2) = 2$, $d(v, w_2) = 1$, and $d(w_1, w_2) = 3$ show that W is a resolving set for D_k and so $\dim(D_k) \leqslant n-3$. On the other hand, at least $k-1$ of the vertices $w_1, w_2, \ldots, w_k$ must belong to every resolving set of D_k since the distance from any two of these vertices to every other vertex of D_k is the same. Hence $\dim(D_k) \geqslant n-3$ and so $\dim(D) = n-3$. Of course, this does not show that sharpness of the bound in Theorem 3.1, except that if D_1 is the directed 3-cycle, then $\dim(D_1) = 1 = n-2$.

We can, however, improve the bound in Theorem 3.1 if we require that the outdegree of every vertex is at least 2.

Theorem 3.2. *If D is a connected oriented graph of order $n \geqslant 5$ with $\operatorname{od} v \geqslant 2$ for all $v \in V(D)$ such that $\dim(D)$ is defined, then $\dim(D) \leqslant n-3$.*

Proof. Suppose, to the contrary, that D contains a basis $\mathcal{B}$ of cardinality $n-2$. Let $\mathcal{B} = \{v_1, v_2, \ldots, v_{n-2}\}$, and $V(D) - \mathcal{B} = \{x, y\}$. For each i $(1 \leqslant i \leqslant n-2)$, $\mathcal{B} - \{v_i\}$ is not a resolving set. Hence for each such i, some two of the three vertices x, y, v_i have the same representations with respect to $\mathcal{B} - \{v_i\}$. We consider two cases.

Case 1: For some i $(1 \leqslant i \leqslant n-2)$, x and y have the same representations with respect to $\mathcal{B} - \{v_i\}$. Assume, without loss of generality, that x and y have the same representations with respect to $W = \mathcal{B} - \{v_{n-2}\}$. Then x and y have the same out-neighbors in W. Since x and y have distinct representations with respect to $\mathcal{B}$, exactly one of x and y is adjacent to v_{n-2}; for if neither x nor y is adjacent to v_{n-2}, then $d(x, v_{n-2}) = d(y, v_{n-2})$. Therefore, we may assume that y is adjacent to v_{n-2}.

Let $W' = \{v_1, v_2, \ldots, v_{n-4}, v_{n-2}\}$. Two of x, y, and v_{n-3} have the same representations with respect to W'. However, y is adjacent to v_{n-2} and x is not, so x and y do not have the same representations with respect to W'. Thus there are two possibilities.

Subcase 1.1: $r(x \mid W') = r(v_{n-3} \mid W')$. We claim that x is adjacent to at most one of $v_1, v_2, \ldots, v_{n-2}$. Suppose that this is not the case. Then we can assume without loss of generality that x is adjacent to v_1 and v_2. Then $r(v_1 \mid \mathcal{B} - \{v_1\}) = r(x \mid \mathcal{B} - \{v_1\})$ or $r(v_1 \mid \mathcal{B} - \{v_1\}) = r(y \mid \mathcal{B} - \{v_1\})$. Similarly, $r(v_2 \mid \mathcal{B} - \{v_2\}) = r(x \mid \mathcal{B} - \{v_2\})$ or $r(v_2 \mid \mathcal{B} - \{v_2\}) = r(y \mid \mathcal{B} - \{v_2\})$. Since the out-neighbors of y in W are the same as the out-neighbors of x in W, we have that v_2 is an out-neighbor of v_1 and that v_1 is an out-neighbor of v_2. Since D is an oriented graph, this is impossible, so, as claimed, x is adjacent to at most one of $v_1, v_2, \ldots, v_{n-2}$. Now, since $\operatorname{od} x \geqslant 2$, it follows that x is adjacent to y and exactly one vertex from $v_1, v_2, \ldots, v_{n-2}$, say v_1. However, since for $1 \leqslant i \leqslant n-3$, $r(v_i \mid \mathcal{B} - \{v_i\}) = r(x \mid \mathcal{B} - \{v_i\})$ or $r(v_i \mid \mathcal{B} - \{v_i\}) = r(y \mid \mathcal{B} - \{v_i\})$, it follows that v_1 is an out-neighbor of every vertex in the set $\{x, y, v_2, v_3, \ldots, v_{n-3}\}$, so $\operatorname{od} v_1 \leqslant 1$, which contradicts the assumption that every vertex in D has out-degree at least 2.

Subcase 1.2: $r(y \mid W') = r(v_{n-3} \mid W')$. We first suppose that x is adjacent to some vertex in W', say v_1. Because of the assumptions in Case 1 and Subcase 1.2, it follows that y and v_{n-3} are also adjacent to v_1. However, since for $2 \leqslant i \leqslant n-3$, $r(v_i \mid \mathcal{B} - \{v_i\}) = r(x \mid \mathcal{B} - \{v_i\})$ or $r(v_i \mid \mathcal{B} - \{v_i\}) = r(y \mid \mathcal{B} - \{v_i\})$, it follows that v_1 is an out-neighbor of every vertex in the set $\{x, y, v_2, v_3, \ldots, v_{n-3}, v_{n-2}\}$, so $\operatorname{od} v_1 = 0$, which is a contradiction. Therefore, x is not adjacent to any of $v_1, v_2, \ldots, v_{n-4}, v_{n-2}$. Thus, since $\operatorname{od} x \geqslant 2$, it follows that x must be adjacent to both y and v_{n-3}. But y is adjacent to v_{n-3} as well, because x and y have the same representations with respect to W. Since x is not adjacent to any of $v_1, v_2, \ldots, v_{n-4}$, it follows that y is not adjacent to any of $v_1, v_2, \ldots, v_{n-4}$. Now $r(y \mid W') = r(v_{n-3} \mid W')$, so it follows that v_{n-3} is not adjacent to any of $v_1, v_2, \ldots, v_{n-4}$. All of this implies that $\operatorname{od} v_{n-3} = 1$, which is a contradiction.

Case 2: For every i $(1 \leqslant i \leqslant n-2)$, x and y have distinct representations with respect to $\mathcal{B} - \{v_i\}$. We next prove that every vertex of $\mathcal{B}$ is an out-neighbor of x or y but at most one vertex of $\mathcal{B}$ is an out-neighbor of both x and y. To prove this, we first show that among the out-neighbors $y_1, y_2, \ldots, y_k$ of y in $\mathcal{B}$, at most one y_i has the same representation as y with respect to $\mathcal{B} - \{y_i\}$. Suppose that this is not the case. Then we may assume that $r(y_1 \mid \mathcal{B} - \{y_1\}) = r(y \mid \mathcal{B} - \{y_1\})$ and that $r(y_2 \mid \mathcal{B} - \{y_2\}) = r(y \mid \mathcal{B} - \{y_2\})$. The first equality tells us that y_2 is an out-neighbor of y_1 and the second equality tells us that y_1 is an out-neighbor of y_2, contradicting the fact that D is an oriented graph. Similarly, among the out-neighbors $x_1, x_2, \ldots, x_\ell$ of x in $\mathcal{B}$, at most one x_j has the same representation as x with respect to $\mathcal{B} - \{x_j\}$.

Next, we show that for each i $(1 \leqslant i \leqslant n-2)$, at least one of x and y is adjacent to v_i. This follows from the fact that if neither x nor y is adjacent to v_i, then no other vertex v_j from $\mathcal{B} - \{v_i\}$ can be adjacent to v_i since $r(v_j \mid \mathcal{B} - \{v_j\}) = r(x \mid \mathcal{B} - \{v_j\})$ or $r(v_j \mid \mathcal{B} - \{v_j\}) = r(y \mid \mathcal{B} - \{v_j\})$. Thus $\operatorname{id} v_i = 0$, which is impossible since $d(z, v_i)$ must be defined for all $z \in V(D)$. Finally, x and y are simultaneously adjacent to at most one vertex v_i $(1 \leqslant i \leqslant n-2)$, for if v_a and v_b are distinct out-neighbors of both x and y, then v_a and v_b are out-neighbors of each other, which is impossible.

This creates a natural partition of the vertices of $\mathcal{B}$ into either two or three subsets, depending on whether there exists a vertex to which x and y are simultaneously adjacent. We now consider these two subcases.

Subcase 2.1: There exists a unique common out-neighbor of x and y.

We assume, without loss of generality, that v_{n-2} is an out-neighbor of both x and y. Furthermore, we can assume, without loss of generality, that the set $X = \{v_1, v_2, \ldots, v_k\}$ consists of the out-neighbors of x and not y, and that the set $Y = \{v_{k+1}, v_{k+2}, \ldots, v_{n-3}\}$ consists of the out-neighbors of y and not x. We further assume, without loss of generality, that the representations of y and v_{n-2} with respect to $\mathcal{B} - \{v_{n-2}\}$ are the same. Therefore, there is no vertex in $v_j \in Y$ for which the representations of y and v_j with respect to $\mathcal{B} - \{v_j\}$ are the same. Therefore, for every $v_j \in Y$, the representations of x and v_j with respect to $\mathcal{B} - \{v_j\}$ are the same.

Since x is adjacent to every vertex in X, every vertex in Y is adjacent to every vertex in $X \cup \{v_{n-2}\}$. Now, there is at most one $v_i \in X$ for which the representations of x and v_i are the same with respect to $\mathcal{B} - \{v_i\}$. Therefore, if $|X| \geqslant 2$, there exists at least one vertex $v_i \in X$ for which the representations of y and v_i with respect to $\mathcal{B} - \{v_i\}$ are the same. Hence, such a vertex v_i is adjacent to every vertex in Y, but this implies that D is not an oriented graph since for any $v_j \in Y$, there is an arc from v_i to v_j and an arc from v_j to v_i. Therefore, $|X| \leqslant 1$. But if $|X| = 1$, then v_1 is the only vertex that could possibly be an out-neighbor of v_{n-2}. This contradicts the assumption that the out-degree of every vertex in D is at least 2, so $|X| = 0$. We have already seen that every vertex in $Y \cup \{x\}$ is adjacent to vertex v_{n-2}, so even if $|X| = 0$, we have that $\operatorname{od} v_{n-2} = 0$, which cannot occur.

Subcase 2.2: No vertex is a common out-neighbor of x and y.

We assume, without loss of generality, that the set $X = \{v_1, v_2, \ldots, v_k\}$ consists of the out-neighbors of x and not y, and that the set $Y = \{v_{k+1}, v_{k+2}, \ldots, v_{n-2}\}$ consists of the out-neighbors of y and not x. Recall that there is at most one $v_i \in X$ such that the representations of v_i and x with respect to $\mathcal{B} - \{v_i\}$ are equal and at most one $v_j \in Y$ such that the representations of v_j and y with respect to $\mathcal{B} - \{v_j\}$ are equal. This produces three possibilities to consider.

Subcase 2.2.1: For every $v_i \in X$ and $v_j \in Y$, the representations of v_i and y with respect to $\mathcal{B} - \{v_i\}$ are the same and the representations of v_j and x with respect to

$\mathcal{B} - \{v_j\}$ *are the same.* Then every vertex in Y is adjacent to every vertex in X, and every vertex in X is adjacent to every vertex in Y. This contradicts the fact that D is an oriented graph as long as X and Y are both nonempty. However, if X or Y is empty, then $\operatorname{od} x \leqslant 1$ or $\operatorname{od} y \leqslant 1$, respectively, which is a contradiction.

Subcase 2.2.2: There is exactly one $v_i \in X$ for which the representations of v_i and x with respect to $\mathcal{B} - \{v_i\}$ are equal and there is no $v_j \in Y$ for which v_j and y have the same representations with respect to $\mathcal{B} - \{v_j\}$. (Note that this subcase is symmetric to the case when there is exactly one $v_j \in Y$ for which the representations of v_j and y with respect to $\mathcal{B} - \{v_j\}$ are equal and for which there is no $v_i \in X$ such that v_i and x have the same representations with respect to $\mathcal{B} - \{v_i\}$.) Now every vertex in Y has the same out-neighbors as x, namely the vertices in the set X. So if $Y \neq \emptyset$, then every vertex in Y is adjacent to every vertex in X. Furthermore, every vertex in $X - \{v_i\}$ has the same out-neighbors as y. So if $|X| \geqslant 2$, then there is at least one vertex in X which is adjacent to every vertex in Y. But this produces a contradiction since D is an oriented graph. Note that if $Y = \emptyset$, then y is adjacent to at most one vertex, namely x, and this is a contradiction.

Assume now that $|X| \leqslant 1$ (so $|Y| \geqslant 2$). If $|X| = 1$, then $v_i = v_1$ and since every vertex in Y is adjacent to v_i, the vertex v_i is adjacent to no vertex except possibly y. Hence, $\operatorname{od} v_i \leqslant 1$, which is a contradiction. If $X = \emptyset$, then x has no out-neighbors except possibly for y, but this contradicts the assumption that the out-degree of x is at least 2.

Subcase 2.2.3: There exists exactly one $v_i \in X$ for which the representations of v_i and x with respect to $\mathcal{B} - \{v_i\}$ are the same and exactly one $v_j \in Y$ for which the representations of v_j and y with respect to $\mathcal{B} - \{v_j\}$ are the same. First, suppose that $|X| \geqslant 2$ and $|Y| \geqslant 2$. Then there exists at least one vertex $v \in X$ for which the representations of v and y with respect to $\mathcal{B} - \{v\}$ are the same. Therefore, v is adjacent to every vertex in Y. Similarly, there is at least one vertex $w \in Y$ for which the representations of w and x with respect to $\mathcal{B} - \{w\}$ are the same. Therefore, w is adjacent to every vertex in X. However, since $v \in X$ and $w \in Y$, it follows that v is adjacent to w and w is adjacent to v. This contradicts the fact that D is an oriented graph.

Next suppose that $|X| = 1$, that $|Y| \geqslant 2$, and that $X = \{v_1\}$. Then the out-neighbors of x are y and v_1. Furthermore, v_1 is an out-neighbor of every vertex in $Y - \{v_j\}$. The only possible out-neighbors of v_1 are y and v_j. However, if v_i is adjacent to v_j, then x is adjacent to v_j, which contradicts the fact that $v_j \notin X$. Therefore, $\operatorname{od} v_i \leqslant 1$, contradicting the fact that every vertex in D has out-degree at least 2. The case where $|Y| = 1$ and $|X| \geqslant 2$ is similar. $\square$

The sharpness of the bound in Theorem 3.1 is not illustrated by the digraph D_k shown in Figure 8 since the outdegrees of most vertices of D_k are 1. We can, however, show that the upper bound in Theorem 3.2 is sharp. Let F_k be an oriented graph with vertex set

$$V(F_k) = \{u_1, u_2, v_1, v_2, w_1, w_2, \ldots, w_k\}$$

and let $E(F_k)$ consist of (1) the arcs (u_i, v_j) for $1 \leqslant i, j \leqslant 2$ and (2) the arcs (v_i, w_j) and (w_j, u_i) for $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant k$. The oriented graph F_k is shown in Figure 9. Then F_k has order $n = k + 4$ and the property that $\operatorname{od} v \geqslant 2$ for all $v \in V(F_k)$. We claim that $\dim(F_k) = n - 3$.

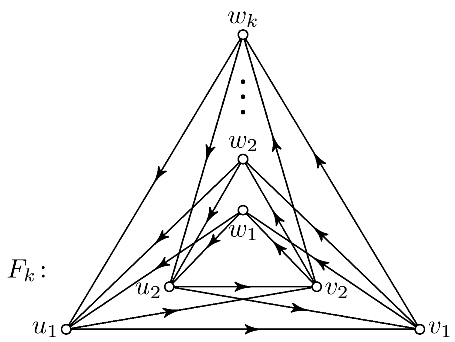


Figure 9. The oriented graph F_k with minimum outdegree 2

First we show that $\dim(F_k) \leqslant n - 3$. Let $W = \{u_1, v_1, w_2, w_3, \ldots, w_k\}$, where then $|W| = n - 3$. The distances $d(u_2, w_2) = 2$, $d(v_2, w_2) = 1$, and $d(w_1, w_2) = 3$ show that W is a resolving set for F_k and so $\dim(F_k) \leqslant n - 3$. Next we show that $\dim(F_k) \geqslant n - 3$. Let W be a resolving set for F_k. Certainly at least $k - 1$ of the vertices $w_1, w_2, \ldots, w_k$ must belong to W since the distance from any two of these vertices to every other vertex of F_k is the same. Moreover, at least one of u_1 and u_2 must belong to W since the distance from u_1 and u_2 every other vertex of F_k is the same. For the same reason, at least one of v_1 and v_2 must belong to W. Hence $\dim(F_k) \geqslant n - 3$ and so $\dim(F_k) = n - 3$.

No additional restriction on the outdegrees of the vertices of an oriented graph yields an improved bound, however. Let $r \geqslant 2$ be an integer. In the oriented graph of Figure 8, replace u_1, u_2 by the r vertices $u_1, u_2, \ldots, u_r$ and v_1, v_2 by the r vertices $v_1, v_2, \ldots, v_r$ and add the appropriate arcs. The resulting oriented graph H_k has $\operatorname{od} v \geqslant r$ for all $v \in V(H_k)$, but $\dim(H_k) = n - 3$.

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