EXAMPLES FROM THE CALCULUS OF VARIATIONS II. A DEGENERATE PROBLEM

Jan Chrastina, Brno

(Received February 5, 1998)

Abstract. Continuing the previous Part I, the degenerate first order variational integrals depending on two functions of one independent variable are investigated.

Keywords: Poincaré-Cartan form, degenerate variational integral, realization problem

MSC 2000: 49-01, 49K17, 58A10, 49N45

Degenerate variational integrals have been (with the only exception of the parametrical case) entirely neglected in all monographs and we should like to discover the reasons here. To this aim, the simplest possible degenerate density

$$\alpha = f(x, w_0^1, w_0^2, w_1^1, w_1^2)\,\mathrm{d}x, \quad f_{11}^{11} f_{11}^{22} = \left(f_{11}^{12}\right)^2 \tag{1}$$

in the underlying space $\mathbf{M}(2)$ equipped with the contact diffiety $\Omega(2)$ will be analyzed. (In elementary terms, we shall deal with the first order degenerate integrals depending on two variable functions w_0^1, w_0^2 of one independent variable x. Recall that the subscripts denote the order of derivatives.) We shall see that in spite of some quite explicit results, too many rather discomposing events may occur and a complete discussion of them is hardly possible at the present time. In this sense, the difficulties that appear might bring some new stimuli into the development of a somewhat uniform calculus of variations. Concerning the notation and terminology, we refer to Part I.

Determined extremals

1. First order problems, see I 3. We recall the space $\mathbf{M}(m)$ with diffiety $\Omega(m)$. Let us consider a density $\alpha = f(x, w_0^1, \dots, w_0^m, w_1^1, \dots, w_1^m)\,\mathrm{d}x$. Owing to I (8_1), there are initial forms $\pi^i \equiv \omega_0^i$ $(i = 1, \dots, m)$ and one can find the (well-known) classical $\mathscr{PC}$ form $\breve{\alpha} = f\,\mathrm{d}x + \sum f_1^i \omega_0^i$. Then

$$\mathrm{d}\breve{\alpha} = \sum e^i \omega_0^i \wedge \mathrm{d}x + \sum a^{ij} \omega_0^i \wedge \omega_0^j + \sum f_{11}^{ij} \omega_1^i \wedge \omega_0^j \tag{2}$$

with $\mathscr{EL}$ coefficients $e^i \equiv f_0^i - Xf_1^i$ and $a^{ij} \equiv \frac{1}{2}(f_{01}^{ij} - f_{10}^{ij})$. Recall that $f_{01}^{ij} \equiv \partial^2 f / \partial w_0^i \partial w_1^j$ and $f_{11}^{ij} = \partial^2 f / \partial w_1^i \partial w_1^j$ in accordance with the notation I 3 for the contact diffieties. One can observe that the forms

$$\sum e^i \omega_0^i, \quad \sum f_{11}^{ij} \omega_0^j, \quad e^i\,\mathrm{d}x + \sum a^{ij} \omega_0^j - \sum f_{11}^{ij} \omega_1^j \tag{3}$$

generate the submodule $\operatorname{Adj} \mathrm{d}\breve{\alpha} \subset \Phi(\mathbf{M}(m))$ defined (in full generality) in I (4).

In this article, we will be interested in the degenerate case when $\det(f_{11}^{ij}) = 0$ and $m = 2$. So we shall deal with density (1). Since the particular f linear in variables w_1^1, w_1^2 seems to be quite easy to investigate, we shall moreover suppose $f_{11}^{11} \neq 0$ from now on.

2. The generic degenerate problem. One can verify the formula

$$\mathrm{d}\breve{\alpha} = e\omega_0^2 \wedge \mathrm{d}x + \omega_0 \wedge \xi \tag{4}$$

with abbreviations

$$\begin{aligned} e &= e^2 - be^1, \\ \omega_s &\equiv \omega_s^1 + b\omega_s^2, \\ \xi &= e^1\,\mathrm{d}x + a\omega_0^2 - f_{11}^{11}\omega_1 \end{aligned}$$

where $a = f_{10}^{21} - f_{01}^{21} (= 2a^{12})$, $b = f_{11}^{12}/f_{11}^{11}$ $(= f_{11}^{22}/f_{11}^{12}$ if $f_{11}^{12} \neq 0)$. It follows that $\operatorname{Adj} \mathrm{d}\breve{\alpha}$ is generated by forms $e\,\mathrm{d}x$, $e\omega_0^2$, ω_0, ξ. The $\mathscr{EL}$ conditions $e^1 = e^2 = 0$ are clearly equivalent to $e^1 = e = 0$ where *e is of the first order at most* (easy verification).

Let us deal with the function e in more detail.

Employing $\mathrm{d}\omega_s^i \equiv \mathrm{d}x \wedge \omega_{s+1}^i$, $\mathrm{d}\omega_s \equiv \mathrm{d}x \wedge \omega_{s+1} + \mathrm{d}b \wedge \omega_s^2$, one can obtain the congruence

$$0 = \mathrm{d}^2\breve{\alpha} \cong (\,\mathrm{d}e + a\omega_1 + e^1\,\mathrm{d}b) \wedge \omega_0^2 \wedge \mathrm{d}x - f_{11}^{11}\,\mathrm{d}b \wedge \omega_0^2 \wedge \omega_1 \pmod{\omega_0} \tag{5}$$

hence

$$\mathrm{d}b \wedge \omega_0^2 \wedge \omega_1 \cong 0 \ (\operatorname{mod} \mathrm{d}x, \omega_0), \ \mathrm{d}e \wedge \omega_0^2 \wedge \mathrm{d}x \cong 0 \ (\operatorname{mod} \mathrm{d}b, \omega_1)$$

which yields

$$\mathrm{d}b \cong 0,\ \mathrm{d}e \cong 0\ (\mathrm{mod}\,\mathrm{d}x, \omega_0, \omega_0^2, \omega_1). \tag{6}$$

Let us assume $\mathrm{d}e \neq 0$ for a moment. Then, in the domain where $e \neq 0$, obviously

$$\mathrm{Adj}\ \mathrm{d}\breve{\alpha} = \{\,\mathrm{d}x, \omega_0, \omega_0^2, \omega_1\} = \{\,\mathrm{d}x,\ \mathrm{d}w_0^1,\ \mathrm{d}w_0^2,\ \mathrm{d}w_1^1 + b\,\mathrm{d}w_1^2\} \tag{7}$$

and therefore $\mathrm{d}b, \mathrm{d}e \in \mathrm{Adj}\ \mathrm{d}\breve{\alpha}$, consequently $\mathrm{d}(w_1^1 + bw_1^2) \in \mathrm{Adj}\ \mathrm{d}\breve{\alpha}$. Note that congruences (6) are equivalent to the identities

$$b_1^2 = b\, b_1^1, \quad e_1^2 = be_1^1 \tag{8}$$

which will be of frequent use. They are valid in the total space (even at points where $e = 0$).

In principle, e need not depend on variables w_1^1, w_1^2. This is the case if and only if $e_1^1 = (e^2 - be^1)_1^1 = 0$ which (expressed in terms of coefficient f) is a rather clumsy result. However, a more explicit identity $e_1^1 = f_{11}^{11} Xb - b_1^1 e^1 - a$ easily follows from (5) by looking at the summands $\omega_1 \wedge \omega_0^2 \wedge \mathrm{d}x$ and yields a simplified but equivalent formula

$$e_1^1 = f_{11}^{11}(b_x + w_1^1 b_0^1 + w_1^2 b_0^2) - b_1^1(f_0^1 - f_{1x}^1 - w_1^1 f_{10}^{11} - w_1^2 f_{10}^{12}) - a.$$

It follows that e_1^1 is identically vanishing only for exceptional densities (to be still specified below).

Let us assume $e_1^1 \neq 0$ for a moment. Recall that the $\mathscr{C}$-curves satisfy the equations $e^1 = 0$ and $e = 0$, hence $Xe = 0$. *It follows that also the function* $f_{11}^{11} Xe/e_1^1 + e^1$ *briefly denoted by*

$$\tilde{e} = \frac{f_{11}^{11}}{e_1^1}(e_x + w_1^1 e_0^1 + w_1^2 e_0^2) + f_0^1 - f_{1x}^1 - w_1^1 f_{10}^{11} - w_1^2 f_{10}^{12}$$

of the order at most one (use (8_2)) *vanishes on all* $\mathscr{C}$*-curves.*

Let us deal with the function $\tilde{e}$ in more detail.

To this aim, using developments of the kind

$$\mathrm{d}g = Xg\,\mathrm{d}x + g_0^1\omega_0 + (g_0^2 - bg_0^1)\omega_0^2 + g_1^1\omega_1 + (g_1^2 - bg_1^1)\omega_1^2 + \dots, \tag{9}$$

formula (4) can be rewritten as

$$\mathrm{d}\breve{\alpha} = e\omega_0^2 \wedge \mathrm{d}x + \omega_0 \wedge \Big(\tilde{e}\,\mathrm{d}x + A\omega_0^2 - \frac{f_{11}^{11}}{e_1^1}\,\mathrm{d}e\Big) \tag{10}$$

where $A = a + f^{11}_{11}(e^2_0 - be^1_0)/e^1_1$. On the subspace where $e = 0$ is satisfied, clearly

$$0 = \mathrm{d}^2\breve{\alpha} = \mathrm{d}\omega_0 \wedge (\tilde{e}\,\mathrm{d}x + A\omega^2_0) - \omega_0 \wedge (\,\mathrm{d}\tilde{e} \wedge \mathrm{d}x + \mathrm{d}A \wedge \omega^2_0 + A\,\mathrm{d}x \wedge \omega^2_1).$$

By inserting $\mathrm{d}\omega_0 = \omega_1 \wedge \mathrm{d}x + \mathrm{d}b \wedge \omega^2_0$ and the developments (9) for the functions $g = \tilde{e}$, $g = A$, $g = b$, one can obtain the identities

$$\tilde{e}^2_1 - b\tilde{e}^1_1 = A, \quad A^2_1 - bA^1_1 = 0 \qquad (\text{when } e = 0). \tag{11}$$

In general $A \neq 0$ and then $(11_1, 8_2)$ imply $\mathrm{d}e \wedge \mathrm{d}\tilde{e} \neq 0\,(\mathrm{mod}\,\mathrm{d}x, \omega_0, \omega^2_0)$. It follows that *the system $e = \tilde{e} = 0$ is equivalent to the primary $\mathscr{E}\mathscr{L}$ equations $e^1 = e^2 = 0$ and can be uniquely brought into the shape*

$$w^1_1 = g^1(x, w^1_0, w^2_0), \quad w^2_1 = g^2(x, w^1_0, w^2_0) \tag{12}$$

with derivatives separated on the left. We have assumed $e^1_1 \neq 0$, $A \neq 0$ in this generic case.

3. The extremality in the generic case, see also I 5 (iv) and I 6. Since the $\mathscr{E}\mathscr{L}$ subspace $\mathbf{e}\colon \mathbf{E} \subset \mathbf{M}$ (2) is defined by the equations $X^k(w^i_1 - g^i) \equiv 0$ with the vector field I (8) where $m = 2$, the functions x, w^1_0, w^2_0 may be used for the coordinates on $\mathbf{E}$. Let us consider a $\mathscr{C}$-curve $\mathrm{P(t)} \in \mathbf{E}(0 \leqslant t \leqslant 1)$. Let moreover

$$Q(t) = (x(t), w^1_0(t), w^2_0(t), w^1_1(t), w^2_1(t), \ldots) \in \mathbf{M}(2),\ 0 \leqslant t \leqslant 1, \tag{13}$$

be a near $\mathscr{A}$-curve (hence $Q^*\omega^i_s \equiv 0$) with the same end points $Q(0) = P(0), Q(1) = P(1)$. Denoting by

$$R(t) = (x(t), w^1_0(t), w^2_0(t), r^1_1(t), r^2_1(t), \ldots) \in \mathbf{E},\ \ 0 \leqslant t \leqslant 1,$$

its projection into $\mathbf{E}$ (hence $r^i_s \equiv X^{s-1}g^i$ evaluated on $Q(t)$), then

$$\int_Q \alpha - \int_P \alpha = \Big(\int_Q \alpha - \int_R \breve{\alpha}\Big) + \Big(\int_R \breve{\alpha} - \int_P \breve{\alpha}\Big) \tag{14}$$

(since $\int_P \alpha = \int_P \breve{\alpha}$, cf. also I (25)) with the summands

$$\int_Q \alpha - \int_R \breve{\alpha} = \int \mathscr{E}\,\mathrm{d}x,\ \int_R \breve{\alpha} - \int_P \breve{\alpha} = \iint A\omega_0 \wedge \omega^2_0 \tag{15}$$

where

$$\mathscr{E} = f(\ldots, w^1_1, w^2_1) - f(\ldots, r^1_1, r^2_2) - \sum f^i_1(\ldots, r^1_1, r^2_1)\,(w^i_1 - r^i_1) \tag{16}$$

($\ldots = x, w_0^1, w_0^2$ and the variable t is omitted) is the common Weierstrass function, moreover the Green formula and (10) evaluated at $e = \tilde{e} = 0$ were employed in (15_2). One can observe that

$$\text{(17)} \qquad \mathrm{d}\breve{\alpha} = A\omega_0 \wedge \omega_0^2 = A(\,\mathrm{d}w_0^1 - g^1\,\mathrm{d}x) \wedge (\,\mathrm{d}w_0^2 - g^2\,\mathrm{d}x) = \,\mathrm{d}u \wedge \,\mathrm{d}v$$

in virtue of (10) with $e = \tilde{e} = 0$ substituted, where u, v are appropriate first integrals of the system (12).

Let us briefly look at the result. Concerning (15_1), the graph of the function $f = f(\ldots, w_1^1, w_1^2)$ with $\ldots = x, w_0^1, w_0^2$ kept fixed represents a surface in the space of the variables f, w_1^1, w_1^2, and the sense of $\mathscr{E}$ is well-known: it is the oriented vertical distance between the surface and its tangent planes. A certain difficulty lies in the fact that we have a developable surface (cf. (1_2)) therefore $\mathscr{E} = 0$ is vanishing along the generating lines. Otherwise the strong inequalities $\mathscr{E} > 0$ or $\mathscr{E} < 0$ can be clearly ensured in favourable cases and then the constant sign of (15_1) is guaranteed. Concerning the summand (15_2), it measures "the number of $\mathscr{C}$-curves" going through the loop which consists of the arc $R(t)$ and the reversely oriented arc $P(t)$; see (17) and especially (17_4). One can therefore see that the value of this summand can be made quite arbitrary by an appropriate choice of the curve $Q(t)$; see also I 2 (even with $c = 1$) for an analogous situation.

Altogether taken, the behaviour of each summand (15) is clear, alas, *the sign of the total sum* (14_2) *seems to be ambigous* and neither the negative nor the positive conjecture concerning the extremality can be stated at this place. We will return to this remarkable problem in Part III.

4. Non-generic densities (1) such that either $e_1^1 = 0$, or $e_1^1 \neq 0$ but $A = 0$ can occur, see the next Section for the realization.

In more detail, assuming $e_1^1 = 0$ but $e = e(x, w_0^1, w_0^2) \neq 0$ then the (possible) $\mathscr{C}$-curves explicitly given by $e = 0$ must also satisfy the equation $e^1 = 0$. It seems that they are very exceptional and rather mysterious. On the contrary, the case when $e = 0$ is identically vanishing will be discussed in more detail below: then the $\mathscr{E}\mathscr{L}$ system is underdetermined and consists of the *single second order equation* $e^1 = 0$ *for two unknown functions* w_0^1, w_0^2.

Assuming $e_1^1 \neq 0$ but $A = 0$, it is necessary to distinguish the subcases when either $\tilde{e} \neq 0$ or $\tilde{e} = 0$ on the subspace of **M** (2) given by the equation $e = 0$. Both subcases may actually occur. The first seems to be rather unpleasant and we are not able to state any reasonable result, however, the second leads to an underdetermined $\mathscr{E}\mathscr{L}$ system consisting of the *single first order equation* $e = 0$ which can be easily investigated.

5. The realization problems. If we wish to find examples of various kind of degenerate densities, it is not appropriate to start from the primary formula (1) since it gives rather complicated requirements for the coefficient f. Instead, it is better to deal with the relevant $\mathscr{PC}$ form $\breve{\alpha}$ and use the results of Section 2.

We shall suppose $\mathrm{d}e \neq 0$ this time. Then $\mathrm{d}\breve{\alpha}$ can be expressed by means of four adjoint functions x, w_0^1, w_0^2 and a certain $z = z(x, w_0^1, w_0^2, w_1^1, w_1^2)$ where $\mathrm{d}z$ is proportional to either of forms

$$\xi \cong \omega_1 \cong \mathrm{d}w_1^1 + b\,\mathrm{d}w_1^2 \qquad (\mathrm{mod}\,\mathrm{d}x,\ \mathrm{d}w_0^1,\ \mathrm{d}w_0^2)$$

as follows from (7). (It would be possible to choose either of the functions $b, e, w_1^1 + bw_1^2$ for this z. More precisely, $\mathrm{d}\breve{\alpha}$ can be expressed in this manner in the subdomain where $e \neq 0$, the behaviour at $e = 0$ easily follows by the continuity argument.) So we may assume the formula

$$\breve{\alpha} = P\,\mathrm{d}x + Q\,\mathrm{d}w_0^1 + R\,\mathrm{d}w_0^2 + S\,\mathrm{d}z - \mathrm{d}W, \tag{18}$$

where $P, \ldots, W$ are functions of x, w_0^1, w_0^2, z. Then a comparison with (1) implies $S = \partial W/\partial z$, so we may suppose $S = W = 0$ in (18) by a mere change of notation.

Let us find conditions for the remaining coefficients P, Q, R ensuring that the right hand side of (18) indeed is a $\mathscr{PC}$ form. By virtue of condition I (12) with $i = 1, 2$ and $\pi_i \equiv \omega_0^i$, it is necessary to ensure $\mathrm{d}\breve{\alpha} \cong 0\,(\mathrm{mod}\,\Omega \wedge \Omega, \omega_0^1, \omega_0^2)$. This is a possible approach but instead we shall follow an alternative and quite simple method.

Inserting $z = z(x, w_0^1, w_0^2, w_1^1, w_1^2)$ into formula (18) with $S = W = 0$, one can obtain the expression

$$\breve{\alpha} = f\,\mathrm{d}x + q\omega_0^1 + r\omega_0^2 \qquad (f = p + qw_1^1 + rw_1^2) \tag{19}$$

in terms of primary variables and contact forms. The change of the type of letter denotes here the results of substitution, e.g.,

$$p = P(x, w_0^1, w_0^2, z(x, w_0^1, w_1^1, w_1^2))$$

and analogously with q and r. Clearly (19) is a $\mathscr{PC}$ form if and only if $q = f_1^1$ and $r = f_1^2$. One can verify that these conditions are equivalent to the identity

$$P' + Q'w_1^1 + R'w_1^2 = 0 \qquad ('= \partial/\partial z). \tag{20}$$

So we have a rather explicit view of all densities (1): *the functions P, Q, R can be arbitrarily chosen, and* (20) *may be regarded as the implicit definition of the* *function* z (assuming $P'' + Q''w_1^1 + R''w_1^2 \neq 0$, hence not all P'', Q'', R'' vanishing, for certainty). Since we are interested in densities (1) with $f_{11}^{11} = q_1^1 \neq 0$, *the condition* $Q' \neq 0$ *should be satisfied.*

6. Continuation: the genericity. Using the explicit representation (19_2) of the coefficient f of the density (1), one can then directly find the formulae

$$e^1 = P_0^1 + R_0^1 w_1^2 - Q_x - Q_0^2 w_1^2 - Q'Xz, \tag{21}$$

$$e^2 = P_0^2 + Q_0^2 w_1^1 - R_x - R_0^1 w_1^1 - R'Xz, \tag{22}$$

$$e = P_0^2 - R_x + b(Q_x - P_0^1) + c(R_0^1 - Q_0^2),\ b = R'/Q',\ c = P'/Q', \tag{23}$$

$$\tilde{e} = \frac{Q'}{E'}(E_x + E_0^1 w_1^1 + E_0^2 w_1^2) + P_0^1 - Q_x + (R_0^1 - Q_0^2)w_1^2, \tag{24}$$

$$A = R_0^1 - Q_0^2 + \frac{1}{E'}(Q'E_0^2 - R'E_0^1). \tag{25}$$

More precisely, $z = z(x, w_0^1, w_0^2, w_1^1, w_1^2)$ should be moreover inserted into the right hand sides of (21–25) to obtain full accordance of variables. In particular, the function $E = E(x, w_0^1, w_0^2, z)$ may be exactly identified with the right hand side of (23), and then $e = E(x, w_0^1, w_0^2, z(x, \dots, w_1^1))$ follows by the substitution. Recall that we have assumed $e_1^1 \neq 0$ (hence $E' \neq 0$) in (24, 25).

Owing to these results, the existence of various kinds of densities satisfying $A = 0$ (either identically, or along the subspace where $E = 0$) immediately follows. The reasoning can be a little simplified by an appropriate choice of the function z.

For instance, let us choose $z = Q$, hence $Q' = 1, Q_x = Q_1^0 = Q_2^0 = 0$. Then the condition $A = 0$ reads $R_0^1 E' = E_0^2 - R'E_0^1$ and admits a lot of solutions. For this case, one can moreover observe an interesting fact: assuming $e_1^1 = 0$ (hence $E' = 0$) then $E_0^2 = R'E_0^1$ which means $e_0^2 = be_0^1$ (use $R' = b$) and consequently $(Xe)_1^2 = b(Xe)_1^1$. This is like (8_2) and it follows that *the function*

$$\tilde{\tilde{e}} = \frac{f_{11}^{11}}{e_0^1} X^2 e + e^1 \tag{26}$$

(a substitute for $\tilde{e}$ which does not exist) *of the order at most one vanishes on all* $\mathscr{C}$*-curves.*

Recall that $e_1^1 \neq 0$ but $\tilde{e} = 0$ identically vanishing (possibly only along the subspace where $e = 0$) implies $A = 0$. In this a little peculiar but favourable case, using (14) with the last summand vanishing, the extremal properties become quite clear.

7. The realization once more. We shall suppose $e = 0$ identically vanishing from now on. Then Adj $\mathrm{d}\breve{\alpha} = \{\omega_0, \xi\} = \{\mathrm{d}u, \mathrm{d}v\}$ for appropriate adjoint functions u, v. Moreover,

$$\mathrm{d}\breve{\alpha} = \omega_0 \wedge \xi = \mathrm{d}u \wedge \mathrm{d}v, \;\; \breve{\alpha} = u\,\mathrm{d}v - \mathrm{d}W \tag{27}$$

in the space of independent variables $x, w_0^1, w_0^2, w_1^1, w_1^2$. Since (27_2) may be regarded as a particular case of (18), the results of the preceding Section 5 remain true. In particular, (23) gives the relevant condition

$$0 = Q'(P_0^2 - R_x) + R'(Q_x - P_0^1) + P'(R_0^1 - Q_0^2) \tag{28}$$

for the coefficients ensuring $e = 0$. Choosing $z = Q$, one can find a lot of solutions but we shall mention a more effective method below. On this occasion, let us note that (28) can be expressed in a very concise manner: denoting $\breve{\alpha}' = P'\,\mathrm{d}x + Q'\,\mathrm{d}w_0^1 + R'\,\mathrm{d}w_0^2$, clearly $\breve{\alpha}' \wedge \mathrm{d}\breve{\alpha} = Q'E\,\mathrm{d}x \wedge \mathrm{d}w_0^1 \wedge \mathrm{d}w_0^2$, hence (28) means that $\breve{\alpha}' \wedge \mathrm{d}\breve{\alpha} = 0$.

We are passing to a better alternative method. Employing (27), the requirement $\mathrm{d}\breve{\alpha} \cong 0 (\mathrm{mod}\, \Omega \wedge \Omega, \omega_0^1, \omega_0^2)$ ensuring that we deal with a $\mathscr{PC}$ form yields the conditions

$$u_1^1 Xv - v_1^1 Xu = u_1^2 Xv - v_1^2 Xu = 0 \tag{29}$$

for the sought functions u, v. Then the top order terms of (29) imply $u_1^1 v_1^2 = v_1^1 u_1^2$, so we may assume $v = V(x, w_0^1, w_0^2, u)$. With this assumption, (29) reduces to the single requirement

$$V_x + w_1^1 V_0^1 + w_1^2 V_0^2 = 0. \tag{30}$$

Choosing $V = V(x, w_0^1, w_0^2, u)$ quite arbitrary, then (30) may be regarded as the implicit equation determining u, and (27_2) with this function $v = V$ and a (little specialized) $W = W(x, w_0^1, w_0^2, u)$ gives the sought density $\alpha = f\,\mathrm{d}x$ where

$$f = uV_x - W_x + w_1^1(uV_0^1 - W_0^1) + w_1^2(uV_0^2 - W_0^2). \tag{31}$$

the function W must be chosen such that $u\partial V/\partial u = \partial W/\partial u$, hence

$$W = \int uV_u\,\mathrm{d}u = uV - \int V\,\mathrm{d}u \tag{32}$$

in quite explicit terms.

Altogether, *formulae* (30–32) *provide all densities* (1) *with* $e = 0$ *identically vanishing* (at the same time we have resolved the equation $\breve{\alpha}' \wedge \mathrm{d}\breve{\alpha} = 0$). Note that the result can be directly verified: both e^1 and e^2 are proportional to Xu, hence the $\mathscr{E}\mathscr{L}$ system is equivalent to the single equation $u =$const.

***8. The parametrical subcase.** Choosing in particular $V = V(w_0^1, w_0^2, u)$ independent of the variable x, then (30) clearly determines a function u homogeneous of zeroth order in variables w_1^1, w_1^2, and (31, 32) determine a function f homogeneous of the first order in w_1^1, w_1^2. (More explicitly: the well-known identities

$$\begin{gathered} f(w_0^1, w_0^2, \lambda w_1^1, \lambda w_1^2) = \lambda f(w_0^1, w_0^2, w_1^1, w_1^2), \\ w_1^1 f_1^1 + w_1^2 f_1^2 = f, \\ w_1^1 f_{11}^{11} + w_1^2 f_{11}^{12} = w_1^1 f_{11}^{12} + w_1^2 f_{11}^{22} = 0 \end{gathered} \tag{33}$$

are satisfied.) So we have the familiar *parametrical integrals.* It may be interesting to mention the relevant $\mathscr{P}\mathscr{C}$ form:

$$\begin{aligned} \breve{\alpha} &= f_1^1 \,\mathrm{d}w_0^1 + f_1^2 \,\mathrm{d}w_0^2, \\ \mathrm{d}\breve{\alpha} &= (\,\mathrm{d}w_0^1 + b\,\mathrm{d}w_0^2) \wedge (a\,\mathrm{d}w_0^2 - f_{11}^{11}(\,\mathrm{d}w_1^1 + b\,\mathrm{d}w_1^2)) \end{aligned}$$

by easy calculation. It follows that $e^1 = aw_1^2 - f_{11}^{11}(w_2^2 + bw_1^2)$ and $e = 0$ by comparison with $(4, 5_3)$.

The results can be carried over to more general integrals (1) with the function $f = F(g, h, Xg, Xh)$, where $g = g(x, w_0^1, w_0^2), h = h(x, w_0^1, w_0^2)$ and F is homogeneous of the first order in the variables Xg, Xh.

9. On the Jacobi least action principle. Let us mention a Riemannian manifold with the first fundamental form g. Then (in rough terms) *the geodesics are* $\mathscr{C}$*-curves for the parametric* (hence degenerate) *variational integral* $\int (g)^{1/2}\,\mathrm{d}x$, and at the same time, *geodesics are* $\mathscr{C}$*-curves for the nondegenerate* (kinetic energy) *variational integral* $\int g\,\mathrm{d}x$. The parametrization is uncertain in the first approach, unlike the second where *the resulting parameter is proportional to the length.* The generalization in mechanics of conservative systems is also well-known as the *Maupertuis principle.* We shall however carry this result over to many other variational integrals (1) with $e = 0$ identically vanishing (which includes the parametrical case and much more).

Since we shall deal with several variational integrals at the same time, let us made our notation more precise: for a given density (1_1), we will write $e^i[f], e[f]$, and so like (instead of previous simpler e^i, e) to point out the dependence on the coefficient f. Then

$$\begin{aligned}
e^i[F(f)] &= F(f)^i_0 - XF(f)^i_1 \\
&= F'(f)F^i_0 - X(F'(f)f^i_1) \\
&= F'(f)(f^i_0 - Xf^i_1) - f^i_1 XF'(f) \\
&= F'(f)e^i[f] - f^i_1 XF'(f)
\end{aligned} \tag{34}$$

for the "composed" density $\beta = F(f)\,\mathrm{d}x$. Assuming degeneracy (1_2), the new density β need not be a degenerate one. In more detail

$$F(f)^{11}_{11}F(f)^{22}_{11} - \left(F(f)^{12}_{11}\right)^2 = \left(f^{11}_{11}f^2_1 - f^{12}_{11}f^1_1\right)^2 F''(f)F'(f)/f^{11}_{11}$$

after easy verification, hence β is degenerate if either $F'' = 0$ or $f^{11}_{11}f^2_1 = f^{12}_{11}f^1_1$ is satisfied. One can find that the second condition implies that the primary density α is of a rather particular kind: then $f = f(\ldots, w^1_1, w^2_1)$ with $\ldots = x, w^1_0, w^2_0$ kept fixed is a *cylindrical surface* with the axis parallel to the w^1_1, w^2_1 plane.

Passing to our intention, let us take a degenerate but "non-cylindrical" density $\alpha = f\,\mathrm{d}x$ with $e[f] = 0$ vanishing, and put $\beta = F(f)\,\mathrm{d}x$ with nonlinear F.

Then the $\mathscr{C}$-curves to the density α satisfy $e^1[f] = e^2[f] = 0$, however, the single equation $e^1[f] = 0$ is enough. Moreover the $\mathscr{C}$-curves to the density β satisfy $e^1[F(f)] = e^2[F(f)] = 0$ but using (34), this system is equivalent to $e^1[f] = XF'(f) = 0$, hence equivalent to

$$e^1[f] = 0,\ \ F'(f) = \text{const.} \tag{35}$$

The first equation means that we deal with $\mathscr{C}$-curves to the primary density α, the second can be interpreted as a specification of the independent variable x.

S u m m a r y: the $\mathscr{C}$-curves to the density $F(f)\,\mathrm{d}z$ are just the $\mathscr{C}$-curves to the density $f\,\mathrm{d}x$ with the independent variable satisfying (35_2).$_*$

10. The extremality for the case $e = 0$ does not make any difficulties. The $\mathscr{E}\mathscr{L}$ subspace $\mathbf{e}\colon\ \mathbf{E} \subset \mathbf{M}(2)$ is defined by equations

$$X^k e^1 = X^k(f^1_0 - Xf^1_1) = \ldots + (w^1_{k+2} + bw^2_{k+2})f^{11}_{11} \equiv 0,$$

hence the functions x, w^1_0, w^1_1, w^2_s $(s = 0, 1, \ldots)$ can be taken for the coordinates for $\mathbf{E}$. Since the form $\mathrm{d}\breve{\alpha}$ can be expressed by two variables, the Lagrange subspace $\mathbf{l}\colon\ \mathbf{L} \subset \mathbf{E}$ is of codimension one and we shall assume x, w^1_0, w^2_s $(s = 0, 1, \ldots)$ for coordinates on $\mathbf{L}$. Closely simulating I 6, we consider an embedded $\mathscr{C}$-curve

$P(t) \in \mathbf{L}(0 \leqslant t \leqslant 1)$, an arbitrary $\mathscr{A}$-curve (14) with the same end points, and its projection

$$R(t) = (x(t), w_0^1(t), w_0^2(t), r_1^1(t), w_1^2(t), \ldots) \in \mathbf{L},\ 0 \leqslant t \leqslant 1,$$

into $\mathbf{L}$. Then the decomposition (14) can be employed with the second summand on the right vanishing (since $\mathrm{d}\breve{\alpha} = 0 \,\mathrm{on}\, \mathbf{L}$), and the first summand (15_1) with $\mathscr{E} = f(w_1^1) - f(e_1^1) - f_1^1(r_1^1)(w_1^1 - r_1^1)$ where the variables x, w_0^1, w_2^2 are omitted for brevity. The inequalities $\mathscr{E} \geqslant 0$ or $\mathscr{E} \leqslant 0$ permit a quite reasonable geometric interpretation and resolve the problem analogously as in the nondegenerate case.

Comments. We cannot refer to any literature except for the parametrical subcase of Section 8. Then the function f does not depend on variable x, the reasonings of Section 10 can be repeated with alternative coordinates w_0^1, w_0^2 on $\mathbf{L}$, and the resulting achievement is the only one which is (rather thoroughly) discussed in all textbooks. Main contribution of this article consists in explicit realization of various kinds of degenerate problems and in transparent clarification of difficulties concerning the extremality. It should be noted that already the arrival at $\mathscr{E}\mathscr{L}$ systems causes serious difficulties in the optimal control theory, see [3]. Many interesting results are referred in [1], alas, they are rather general and of quite different kind.

References

[1] *José F. Cariñena*: Theory of singular Lagrangians. International centre for theoretical physics, Miramare-Trieste, 1989.

[2] *J. Chrastina*: Examples from the calculus of variations I. Nondegenerate problems. Math. Bohem. *125* (2000), 55–76.

[3] *R. Gabasov, F. M. Kirillova*: Singular Optimal Control. Moskva, 1973. (In Russian.)

Author's address: *Jan Chrastina*, Masaryk University, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic.