ASYMPTOTIC PROPERTIES OF SOLUTIONS OF SECOND ORDER QUASILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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Abstract. This paper establishes existence of nonoscillatory solutions with specific asymptotic behaviors of second order quasilinear functional differential equations of neutral type. Then sufficient, sufficient and necessary conditions are proved under which every solution of the equation is either oscillatory or tends to zero as $t \to \infty$.

Keywords: quasilinear differential equations of neutral type, oscillatory, non-oscillatory solutions, Schauder-Tychonoff fixed point theorem

MSC 2000: 34K40, 34K25

1. Introduction

We consider quasilinear differential equations of neutral type in the form

$$(L_1^\alpha x(t))' + f(t, x(g(t))) = 0, \quad t \geqslant a > 0, \tag{E}$$

where $\alpha > 0$ is a constant and L_1^α is a differential operator defined by

$$L_0 x(t) = x(t) - p(t)x(h(t)), \tag{1.1}$$

$$L_1^\alpha x(t) = r(t)|L_0' x(t)|^{\alpha-1} L_0' x(t). \tag{1.2}$$

The conditions we always assume for (E) are listed below:

(C_1) $r\colon [a,\infty) \to (0,\infty)$ is continuous and

$$\int_a^\infty (r(t))^{\frac{-1}{\alpha}}\,\mathrm{d}t < \infty;$$

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(C_2) $p\colon [a,\infty) \to [0,\lambda]$ is continuous, $0 < \lambda < 1$;

(C_3) $h\colon [a,\infty) \to \mathbb{R}$ is continuous and strictly increasing, $h(t) < t$ for $t \geqslant a$ and $\lim\limits_{t\to\infty} h(t) = \infty$;

(C_4) $g\colon [a,\infty) \to \mathbb{R}$ is continuous and $\lim\limits_{t\to\infty} g(t) = \infty$;

(C_5) $f\colon [a,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(t,x)$ is nondecreasing in x and satisfies $xf(t,x) > 0$ for all $x \neq 0$ and $t \geqslant a$.

Let $t_1 \geqslant a$ be such that

$$t_0 = \min\{h(t_1), \inf_{t\geqslant t_1} g(t)\} \geqslant a. \tag{1.3}$$

By a proper solution of (E) we mean a continuous function $x\colon [t_0,\infty) \to \mathbb{R}$ which has the property that $L_0x(t)$ and $L_1^\alpha x(t)$ are continuously differentiable on $[t_1,\infty)$, and satisfies the equation (E) at every point of $[t_1,\infty)$. The solutions which vanish for all large t will be excluded from our consideration. A proper solution of (E) is said to be oscillatory if it has infinite sequences of zeros tending to infinity; otherwise a proper solution is said to be nonooscillatory.

In this paper we shall study the oscillatory and nonoscillatory behavior of proper solutions of the equation (E). More specifically we first classify the set of nonoscillatory solutions of (E) according to their asymptotic behavior as $t \to \infty$ and present conditions for the existence of three types of nonoscillatory solutions of (E) with specified asymptotic behavior. We then establish criteria for oscillation of all proper solutions of the equation (E).

Equations of the form (E) include as special cases the neutral equations of the type

$$(r(t)(x(t) - p(t)x(h(t))')' + f(t, x(g(t))) = 0, \quad t \geqslant a \tag{E$_1$}$$

and the non-neutral equations of the type

$$(r(t)|x'(t)|^{\alpha-1}x'(t))' + f(t, x(g(t))) = 0, \quad t \geqslant a, \tag{E$_2$}$$

both of which have been objects of intensive investigation in recent years. We refer to the papers [3–5, 7, 16] and to [1, 2, 8–15, 17, 19, 20] for typical oscillation and nonoscillation results regarding (E_1) and (E_2), respectively.

The oscillatory behavior of equations of the form (E) was first studied in the paper [6] under the hypothesis that the function $r(t)$ defining the operator L_1^α satisfies $\int_a^\infty (r(s))^{\frac{-1}{\alpha}}\,\mathrm{d}s = \infty$. The purpose of this paper is to turn our attention to the equation (E) with $r(t)$ satisfying the condition (C_1): $\int_a^\infty (r(s))^{\frac{-1}{\alpha}}\,\mathrm{d}s < \infty$ and develop an oscillation theory for it in the same spirit as in [6].

Extensive use will be made of the function $\varrho_\alpha(t)$ defined by

$$\varrho_\alpha(t) = \int_t^\infty (r(s))^{\frac{-1}{\alpha}}\,\mathrm{d}s, \quad t \geqslant a. \tag{1.4}$$

Note that $\varrho_\alpha(t) \to 0$ as $t \to \infty$ by (C_1).

The following notation will be needed in the sequel:

$$h^{[0]}(t) \equiv t, \quad h^{[k]}(t) = h(h^{[k-1]}(t)), \qquad k = 1, 2, \ldots, \tag{1.5}$$

$$P_0(t) \equiv 1, \quad P_k(t) = \prod_{i=0}^{k-1} p(h^{[i]}(t)), \quad k = 1, 2, \ldots, \tag{1.6}$$

$$\gamma(t) = \sup\{s \geqslant a; g(s) \leqslant t\}, \quad \gamma_h(t) = \sup\{s \geqslant a; h(s) \leqslant t\}. \tag{1.7}$$

2. Classification of proper nonoscillatory solutions

We begin by classifying the set of possible nonoscillatory solutions of the equation (E) according to their asymptotic behavior as $t \to \infty$.

Let N denote the set of all nonoscillatory solutions of (E). If $x \in N$ then it follows from (E) and the assumptions (C_1)–(C_5) that the function

$$L_0x(t) = x(t) - p(t)x(h(t)) \tag{2.1}$$

has to be eventually of constant sign, so that either

$$x(t)L_0x(t) > 0 \tag{2.2}$$

or

$$x(t)L_0x(t) < 0 \tag{2.3}$$

for all sufficiently large t.

We use the notation

$$N^+ = \{x(t) \in N\colon\ x(t)L_0x(t) > 0 \quad \text{for all large } t\},$$
$$N^- = \{x(t) \in N\colon\ x(t)L_0x(t) < 0 \quad \text{for all large } t\}.$$

If $x \in N^-$ then by Remark 2.1 in [18] $\lim\limits_{t\to\infty} x(t) = 0$. Now in view of ($C_2$), ($C_3$), $\lim\limits_{t\to\infty} L_0x(t) = 0$. From this we obtain

R e m a r k 2.1. If $x(t) \in N^-$, then $\lim\limits_{t\to\infty} x(t) = 0$, $\lim\limits_{t\to\infty} L_0x(t) = 0$.

Let $x(t) \in N^+$ for $t \geqslant t_1$. Then from (2.1) we have

$$x(t) = L_0x(t) + p(t)L_0x(h(t)) + P_2(t)x(h^{[2]}(t)), \quad t \geqslant t_2 \geqslant \gamma_h(t_1). \tag{2.4}$$

From (2.4) in view of (C_2) we get

$$|x(t)| \geqslant |L_0x(t)|, \quad t \geqslant t_1. \tag{2.5}$$

Repeating the application of (2.1) and (2.4) we obtain

$$x(t) = \sum_{k=0}^{n(t)-1} P_k(t)L_0x(h^{[k]}(t)) + P_{n(t)}x(h^{[n(t)]}(t)), \quad t \geqslant t_{n(t)} \geqslant \gamma_h(t_{n(t)-1})), \tag{2.6}$$

where $n(t)$ denotes the least positive integer such that $h(t_1) < h^{[n(t)]} \leqslant t_1$.

Let K_x be a constant such that $|x(t)| \leqslant K_x$ for $t \in [h(t_1), t_1]$. If $L_0x(t)$ is nondecreasing on $[t_1, \infty)$, then (2.6) in view of (C_2) and (1.6) yields

$$|x(t)| \leqslant \frac{|L_0x(t)|}{1-\lambda} + K_x, \quad t \geqslant t_2 \geqslant t_1. \tag{2.7}$$

Lemma 2.1. *Let $x(t)$ be a nonoscillatory solution of* (E) *on $[t_0, \infty)$. If $x(t) \in N^+$, then there exist positive constants c_1, c_2 and $T \geqslant t_0$ such that*

$$c_1\varrho_\alpha(t) \leqslant |L_0x(t)| \leqslant c_2 \quad \text{for} \quad t \geqslant T. \tag{2.8}$$

P r o o f. Let $x \in N^+$. Without loss of generality we may suppose that $x(t) > 0$ and $L_0x(t) > 0$ for $t \geqslant t_0$. In view of the assumptions (C_1)–(C_5) the equation (E) implies that

$$L_1^\alpha x(t) = r(t)|L_0'x(t)|^{\alpha-1}L_0'x(t) \tag{2.9}$$

is decreasing for $t \geqslant t_1 \geqslant \gamma(t_0)$. Hence in view of ($C_1$) either $L_0'x(t) > 0$ for $t \geqslant t_1$ or there exists $t_2 \geqslant t_1$ such that $L_0'x(t) < 0$ for $t \geqslant t_2$.

i) Suppose that $L_0'x(t) > 0$ on $[t_1, \infty)$. Then with regard to (2.9) there exists a constant $K_1^\alpha > 0$ such that $L_1^\alpha x(t) = r(t)(L_0'x(t))^\alpha \leqslant K_1^\alpha$ for $t \geqslant t_1$. From the last inequality we obtain $L_0x(t) - L_0x(t_1) \leqslant K_1\varrho_\alpha(t_1)$, which implies that

$$L_0x(t) \leqslant c_2, \quad t \geqslant t_1, \tag{2.10}$$

where $c_2 = L_0x(t_1) + K_1\varrho_\alpha(t_1)$.

ii) Suppose that $L_0'x(t) < 0$ on $[t_2, \infty)$. Since $L_1^\alpha x(t) = -r(t)(-L_0'x(t))^\alpha$ is decreasing for $t \geqslant t_2$ we have

$$-L_0'x(t) \geqslant (r(t_2))^{\frac{1}{\alpha}}|L_0'x(t_2)|(r(t))^{\frac{-1}{\alpha}}, \quad t \geqslant t_2, \tag{2.11}$$

from which via integration over $[t, \infty)$, $t \geqslant t_2$, it follows that

$$L_0x(t) \geqslant c_1\varrho_\alpha(t), \quad t \geqslant t_2, \tag{2.12}$$

where $c_1 = (r(t_2))^{\frac{1}{\alpha}}|L_0'x(t_2)|$. Let $T = \max\{t_1, t_2\}$. The desired inequality (2.8) follows from (2.12) and (2.10).

Using Lemma 2.1, (2.5) and (2.7) we obtain

$$0 \leqslant \liminf_{t\to\infty} |x(t)|, \ \limsup_{t\to\infty} |x(t)| < \infty.$$

Then in view of the monotonicity of $L_0x(t)$ there exists a limit $\lim\limits_{t\to\infty} |L_0x(t)| = b_0 < \infty$. Let $\liminf\limits_{t\to\infty} |x(t)| = 0$. Then by Lemma 1 and Lemma 2 [16] we have

$$\lim_{t\to\infty} |L_0x(t)| = 0 \ \text{ and } \ \lim_{t\to\infty} |x(t)| = 0.$$

Combining Lemma 2.1 with (2.6), (2.7), we conclude that the following three types of asymptotic behavior are possible for nonoscillatory solutions $x(t) \in N^+$ of (E):

$$0 < \liminf_{t\to\infty} |x(t)|, \quad \limsup_{t\to\infty} |x(t)| < \infty, \tag{I}$$

$$\lim_{t\to\infty} x(t) = 0, \quad \limsup_{t\to\infty} \frac{|x(t)}{\varrho_\alpha(t)} = \infty, \tag{II}$$

$$0 < \liminf_{t\to\infty} \frac{|x(t)|}{\varrho_\alpha(t)}, \quad \limsup_{t\to\infty} \frac{|x(t)|}{\varrho_\alpha(t)} < \infty. \tag{III}$$

$\square$

In this section we establish criteria for the existence of nonoscillatory proper solutions of the equation (E) of type (I), (II) or (III) mentioned above.

Theorem 3.1. *The equation* (E) *has nonoscillatory solutions of type* (I) *if an only if*

$$\int_a^\infty \left(\frac{1}{r(t)}\int_a^t |f(s,c)|\,\mathrm{d}s\right)^{\frac{1}{\alpha}} \mathrm{d}t < \infty, \quad T \geqslant a \tag{3.1}$$

for some constant $c \neq 0$.

P r o o f. (The "only if" part.) Let $x(t)$ be a nonoscillatory solution of (E) of type (I) on $[t_0,\infty)$, $t_0 \geqslant a$. We may suppose that $x(t)$ is eventually positive. Then there exist positive constants c, c_1 and $t_1 \geqslant t_0$ such that

$$c \leqslant x(g(t)) \leqslant c_1 \quad \text{for} \quad t \geqslant t_1. \tag{3.2}$$

In view of (C_4), (C_5) and (3.2) we see from (E) that

$$(L_1^\alpha x(t))' \leqslant -f(t,c), \quad t \geqslant t_1. \tag{3.3}$$

The last inequality implies that $L_1^\alpha x(t) = r(t)|L_0'x(t)|^{\alpha-1}L_0'x(t)$ is decreasing on $[t_1,\infty)$. Then in view of (C_1), there exists a $t_2 \geqslant t_1$ such that $L_0'x(t)$ is either positive or negative for $t \geqslant t_2$.

i) Suppose that $L_0'x(t) > 0$ on $[t_2,\infty)$. Then, integrating (3.3) over $[t_2,t]$ we have

$$\int_{t_2}^t f(s,c)\,\mathrm{d}s \leqslant L_1^\alpha x(t_2), \quad t \geqslant t_2,$$

which implies because of (C_1) that

$$\int_{t_2}^\infty \left(\frac{1}{r(t)}\int_{t_2}^t f(s,c)\,\mathrm{d}s\right)^{\frac{1}{\alpha}} \mathrm{d}t < \infty.$$

This shows that (3.1) is valid.

ii) Suppose that $L_0'x(t) < 0$ on $[t_2,\infty)$. Integration of (3.3) over $[t_2,t]$ gives

$$r(t)|L_0'x(t)|^\alpha \geqslant \int_{t_2}^t f(s,c)\,\mathrm{d}s$$

or

$$-L_0' x(t) \geqslant \left(\frac{1}{r(t)} \int_{t_2}^{t} |f(s,c)| \,\mathrm{d}s\right)^{\frac{1}{\alpha}}, \ t \geqslant t_2.$$

Integrating the above inequality over $[t_2, \infty)$ and noting that $x \in N^+$ we see that (3.1) holds.

(The "if" part.) Suppose that (3.1) holds for some constant $c > 0$. The case of a negative c can be treated similarly. Let b and d be positive constants such that $0 < d < b\frac{1-\lambda}{1+\lambda}$ and $\frac{b+d}{1-\lambda} \leqslant c$, where λ is as in (C_2). Take $T \geqslant a$ such that

$$T_0 = \min\{h(T), \inf_{t \geqslant T} g(t)\} > a \tag{3.4}$$

and

$$\int_T^{\infty} \left(\frac{1}{r(t)} \int_T^t f(s,c) \,\mathrm{d}s\right)^{\frac{1}{\alpha}} \mathrm{d}t < \frac{d}{2}. \tag{3.5}$$

Let $C[T_0, \infty)$ be the locally convex space of all continuous functions defined on $[T_0, \infty)$ which are constant on $[T_0, T]$ with the topology of uniform convergence on any compact subinterval of $[T_0, \infty)$.

Define a closed convex subset Y of $C[T_0, \infty)$ by

$$\begin{aligned} Y = \{y \in C[T_0, \infty);\ b - d \leqslant y(t) \leqslant b + d \ \text{ on } \ [T, \infty) \\ \text{and } y(t) = y(T) \ \text{ on } \ [T_0, T]\}. \end{aligned} \tag{3.6}$$

Using (2.5) we can associate to each $y \in Y$ the function $\tilde{y}\colon [T_0, \infty) \to \mathbb{R}$ defined by

$$\begin{aligned} \tilde{y}(t) &= \sum_{k=0}^{n(t)-1} P_k(t) y(h^{[k]}(t)) + P_{n(t)} \frac{y(T)}{1 - p(T)}, \quad t \geqslant T, \\ \tilde{y}(t) &= \frac{y(T)}{1 - p(T)}, \quad t \in [T_0, Y], \end{aligned} \tag{3.7}$$

where $n(t)$ denotes the least positive integer such that $T_0 \leqslant h^{[n(t)]}(t) \leqslant T$.

It is easy to verify that

$$y(t) = \tilde{y}(t) - p(t)\tilde{y}(h(t)), \quad t \geqslant T_0, \tag{3.8}$$

and

$$b - d \leqslant y(t) \leqslant \tilde{y}(t) \leqslant \frac{b+d}{1-\lambda}, \quad t \geqslant T. \tag{3.9}$$

We now define an operator $\mathcal{F}\colon\ Y \to C[T_0,\infty)$ by

$$(\mathcal{F}y)(t) = b + \int_t^\infty \left(\frac{1}{r(\tau)} \int_T^\tau f(s, \tilde{y}(g(s)))\,\mathrm{d}s\right)^{\frac{1}{\alpha}} \mathrm{d}\tau, \quad t \geqslant T,$$
$$(\mathcal{F}y)(t) = (\mathcal{F}y)(T), \qquad T_0 \leqslant t \leqslant T.$$

If $y \in Y$, then using (3.9), (3.5) and (C_5) we obtain

$$|(\mathcal{F}y)(t) - b| \leqslant \int_T^\infty \left(\frac{1}{r(\tau)} \int_T^\tau f\Big(s, \frac{b+d}{1-\lambda}\Big)\,\mathrm{d}s\right)^{\frac{1}{\alpha}} \mathrm{d}\tau < d,$$

which shows that the operator $\mathcal{F}$ maps Y into Y. It is a matter of routine calculation to verify that $\mathcal{F}$ is a continuous mapping and that $\mathcal{F}(Y)$ is relatively compact in the topology of $C[T_0,\infty)$. Therefore, the Schauder-Tychonoff fixed point theorem ensures the existence of an element $y_0 \in Y$ such that $\mathcal{F}y_0 = y_0$ and y_0 satisfies the integral equation

$$y_0(t) = b + \int_t^\infty \left(\frac{1}{r(\tau)} \int_T^\tau f(s, \tilde{y}_0(g(s)))\,\mathrm{d}s\right)^{\frac{1}{\alpha}} \mathrm{d}\tau, \quad t \geqslant T, \tag{3.10}$$

where $y_0(t) = \tilde{y}_0(t) - \tilde{y}_0(h(t)), t \geqslant T$.

Differentiating (3.10) we obtain that $\tilde{y}_0(t)$ is a nonoscillatory solution of (E) of type (I).

This completes the proof. □

Theorem 3.2. *The equation* (E) *has nonoscillatory solutions of type* (III) *if and only if*

$$\int_T^\infty |f(t, c\varrho_\alpha(t))|\,\mathrm{d}t < \infty, \quad T \geqslant a. \tag{3.11}$$

for some constant $c \neq 0$.

P r o o f. (The "only if" part.) Let $x(t)$ be a type (III)-solution of (E) on $[t_0,\infty)$, $t_0 \geqslant a$. We may suppose that $x(t)$ is eventually positive. Then there exist positive constants c, c_1 and $t_1 \geqslant t_0$ such that

$$c\varrho_\alpha(t) \leqslant x(g(t)) \leqslant c_1\varrho_\alpha(t) \ \text{ for } \ t \geqslant t_1. \tag{3.12}$$

In view of (3.12), (C_4) and (C_5), the equation (E) yields

$$(L_1^\alpha x(t))' \leqslant -f(t, c\varrho_\alpha(t)), \quad t \geqslant t_1. \tag{3.13}$$

The last inequality implies that $L_1^\alpha x(t) = r(t)|L_0'x(t)|^{\alpha-1}L_0'x(t)$ is decreasing on $[t_1,\infty)$. Then in view of (C_1) there exists a $t_2 \geqslant t_1$ such that $L_0'x(t)$ is either positive or negative for $t \geqslant t_2$.

i) If $L_0'x(t) > 0$ on $[t_2,\infty)$, then integrating (3.13) over $[t_2,\infty)$ we have

$$\int_{t_2}^{\infty} f(t, c\varrho_\alpha(t))\,\mathrm{d}t \leqslant L_1^\alpha x(t_2) < \infty.$$

ii) If $L_0'x(t) < 0$ on $[t_2,\infty)$, then, in view of the monotonicity of $L_1^\alpha x(t) = -r(t)|L_0'x(t)|^\alpha$, we have

$$-L_0'x(s) \geqslant \Big(\frac{r(t)}{r(s)}\Big)^{\frac{1}{\alpha}}|L_0'x(t)|, \quad s \geqslant t \geqslant t_2.$$

Integration of the last inequality over $[t,\infty)$ gives

$$L_0x(t) \geqslant (r(t))^{\frac{1}{\alpha}}|L_0'x(t)|\varrho_\alpha(t), \quad t \geqslant t_2 \tag{3.14}$$

which, combined with the inequality following from the integration of (3.13), yields

$$\Big(\frac{L_0x(t)}{\varrho_\alpha(t)}\Big)^\alpha \geqslant r(t)|L_0'x(t)|^\alpha \geqslant \int_{t_2}^{t} f(s, c\varrho_\alpha(s))\,\mathrm{d}s. \tag{3.15}$$

Combining (3.15) with (2.6) and (3.12) shows that (3.11) holds as desired.

(The "if" part.) Suppose that (3.11) holds for some nonzero constant c. We may suppose that c is positive. Let b and d be such that $0 < d < b\frac{1-\lambda^\alpha}{1+\lambda^\alpha}$, $\frac{(b+d)^{\frac{1}{\alpha}}}{(1-\lambda} \leqslant c$, where λ is as in (C_2). Take $T \geqslant a$ such that (3.4) holds and

$$\int_{T}^{\infty} f(s, c\varrho_\alpha x(t))\,\mathrm{d}t < d. \tag{3.16}$$

We define Y to be the closed convex subset of $C[T_0,\infty)$ as follows:

$$\begin{aligned} Y = \{y \in C[T,\infty)\colon\ &(b-d)^{\frac{1}{\alpha}}\varrho_\alpha(t) \leqslant y(t) \leqslant (b+d)^{\frac{1}{\alpha}}\varrho_\alpha(t) \quad \text{on } [T,\infty)\\ &\text{and } y(t) = c\varrho_\alpha(T) \quad \text{on } [T_0,T]\}. \end{aligned} \tag{3.17}$$

With each $y \in Y$ we associate the function ỹ defined by (3.7). Then it can be shown that the operator $\mathcal{F}\colon Y \to C[T_0,\infty)$ defined by

$$(\mathcal{F}y)(t) = \int_t^\infty \bigg(\frac{1}{r(\tau)}\bigg(b + \int_T^\tau f(s, \tilde{y}(g(s)))\,\mathrm{d}s\bigg)^{\frac{1}{\alpha}}\bigg)\,\mathrm{d}\tau, \quad t \geqslant T,$$

and

$$(\mathcal{F}y)(t) = (\mathcal{F}y)(T), \qquad T_0 \leqslant t \leqslant T$$

is a continuous mapping which sends Y into a relatively compact subset of Y. By the Schauder-Tychonoff fixed point theorem there exists an element $y_0 \in Y$ such that $\mathcal{F}y_0 = y_0$. This function $y_0 = y_0(t)$ satisfies the integral equation

$$y_0(t) = \int_t^\infty \left(\frac{1}{r(\tau)} \left(b + \int_T^\tau f(s, \tilde{y}_0(g(s)) \,\mathrm{d}s \right)^{\frac{1}{\alpha}} \right) \mathrm{d}\tau, \quad t \geqslant T, \tag{3.18}$$

where $y_0(t) = \tilde{y}_0(t) - \tilde{y}_0(h(t)), t \geqslant T$.

Differentiating (3.18) we conclude that $\tilde{y}_0(t)$ is a nonoscillatory solution of (E) of type (III). □

Let us turn to the solutions of type (II) of (E). Unlike the solutions of types (I) and (III) we have been unable to characterize the existence of this type of solutions.

Theorem 3.3. *The equation* (E) *has nonoscillatory solutions of type* (II) *if*

$$\int_a^\infty \left(\frac{1}{r(t)} \int_t^\infty |f(s, c)| \,\mathrm{d}s \right)^{\frac{1}{\alpha}} \mathrm{d}t < \infty, \tag{3.19}$$

for some constant $c \neq 0$ *and*

$$\int_a^\infty |f(t, k\varrho_\alpha(t))| \,\mathrm{d}t = \infty \tag{3.20}$$

for any $k \neq 0$.

P r o o f. Suppose that (3.19) holds for some constant $c > 0$. A parallel argument holds for the case of negative c. Let T be so large and d be such that $0 < d\varrho_\alpha(T) < c$ and

$$\int_T^\infty \left(\frac{1}{r(t)} \left(d\varrho_\alpha(T) + \int_T^\infty f(s, c) \,\mathrm{d}s \right)^{\frac{1}{\alpha}} \right) \mathrm{d}\tau < d\varrho_\alpha(T). \tag{3.21}$$

We define a closed convex subset Y of $C[T_0, \infty)$ and a mapping $\mathcal{F}\colon\ y \to [T_0, \infty)$ as follows:

$$\begin{aligned} Y = \{y \in C[T_0, \infty);\ & d\varrho_\alpha(t) \leqslant y(t) \leqslant c \text{ on } [T, \infty) \\ & \text{and } y(t) = y(T) \text{ on } [T_0, T]\}, \end{aligned}$$

$$\begin{aligned} (\mathcal{F}y)(t) &= \int_t^\infty \left(\frac{1}{r(\tau)} \left(d\varrho_\alpha(T) + \int_T^\tau f(s, \tilde{y}(g(s))) \,\mathrm{d}s \right)^{\frac{1}{\alpha}} \right) \mathrm{d}\tau, \quad t \geqslant T, \\ (\mathcal{F}y)(t) &= (\mathcal{F}y)(T), \qquad T_0 \leqslant t \leqslant T, \end{aligned}$$

where $\tilde{y}(t)$ denotes the function associated with $y(t)$ via (3.7). Observe that

$$d\varrho_\alpha(t) \leqslant y(t) \leqslant \tilde{y}(t) \leqslant \frac{c}{1-\lambda}$$

for $t \geqslant T$. It is a matter of routine calculation to verify that $\mathcal{F}$ is a continuous mapping and $\mathcal{F}(Y)$ is relatively compact in the topology of $C[T_0, \infty)$. Therefore by the Schauder-Tychonoff fixed point theorem there exists a fixed element $y_0 \in Y$ such that $\mathcal{F}y_0 = y_0$ and y_0 satisfies the integral equation

$$y_0(t) = \int_t^\infty \left(\frac{1}{r(\tau)}\left(d\varrho_\alpha(T) + \int_T^\tau f(s, \tilde{y}_0(g(s)))\,\mathrm{d}s\right)^{\frac{1}{\alpha}}\right)\mathrm{d}\tau, \quad t \geqslant T, \tag{3.22}$$

where $y_0(t) = \tilde{y}_0(t) - \tilde{y}_0(h(t)), t \geqslant T$. From (3.22) and (3.20) it follows that $\tilde{y}_0(t)$ is a nonoscillatory solution of (E) of type (II). □

4. Oscillation of proper solutions

In this section we give criteria for (E) to be almost oscillatory in the sense that $N = N^-$ or equivalently every solution of (E) is either oscillatory or tends to zero as $t \to \infty$. In order to obtain such criteria we need stronger hypotheses on the nonlinearity of the function $f(t, x)$ in (E) with respect to x.

Definition 4.1.
(i) The equation (E) is said to be strongly superlinear if there exists a constant $\beta > \alpha$ such that $|x|^{-\beta}|f(t, x)|$ is nondecreasing in $|x|$ for each fixed $t \geqslant a$.
(ii) The equation (E) is said to be strongly sublinear if there exists a constant $0 < \gamma < \alpha$ such that $|x|^{-\gamma}|f(t, x)|$ is nonincreasing in $|x|$ for each fixed $t \geqslant a$.

Theorem 4.1. *Let the equation* (E) *be strongly superlinear. Suppose that*

$$g(t) \leqslant t \quad \text{for} \quad t \geqslant a. \tag{4.1}$$

If

$$\int_a^\infty |f(t, c\varrho_\alpha(t))|\,\mathrm{d}t = \infty \tag{4.2}$$

for all constants $c \neq 0$ then every proper solution of (E) *is either oscillatory or tends to zero as $t \to \infty$.*

P r o o f. Let $x(t)$ be a nonoscillatory solution of (E). Without loss of generality we suppose that $x(g(t)) > 0$ for $t \geqslant t_0$. Then the equation (E) in view of (C_1)–(C_5) implies that the function $y(t) = L_0(t)$ is eventually of constant sign, i.e. either $x \in N^+$ or $x \in N^-$.

I. Let $x \in N^+$. Then (2.5) and (2.8) hold and so the function $y(t) = L_0 x(t)$ satisfies

$$x(t) \geqslant y(t), \quad t \geqslant t_1 \tag{4.3}$$

and

$$c_1 \varrho_\alpha(t) \leqslant y(t) \leqslant c_2, \quad t \geqslant t_1 \tag{4.4}$$

for some positive constants c_1, c_2 and $t_1 \geqslant t_0$.

Using the assumption (C_5) and (4.3), we obtain from (E)

$$(L_1^\alpha x(t))' \leqslant -f(t, y(g(t))), \quad t \geqslant t_2 = \gamma(t_1). \tag{4.5}$$

The function $L_1^\alpha x(t) = r(t)|y'(t)|^{\alpha-1} y'(t)$ is decreasing on $[t_2, \infty)$. Therefore there exists a $T \geqslant t_2$ such that $y'(t)$ is either positive or negative on $[T, \infty)$.

i) Suppose that $y'(t) > 0$ on $[T, \infty)$. Integrating (4.5) from T to ∞ and using (4.4) we have

$$\infty > r(T)(y'(T))^\alpha \geqslant \int_T^\infty f(t, c_1 \varrho_\alpha(g(t)))\,\mathrm{d}t \geqslant \int_T^\infty f(t, c_1 \varrho_\alpha(t))\,\mathrm{d}t,$$

which contradicts (4.2).

ii) Suppose that $y'(t) < 0$ on $[T, \infty)$. If $\beta > \alpha$ is the exponent of superlinearity of (E), then in view of (4.4) and the monotonicity of $y(t)$ we have

$$(c_1 \varrho_\alpha(t))^{-\beta} f(t, c_1 \varrho_\alpha(t)) \geqslant (y(g(t))^{-\beta} f(t, y(g(t))), \ t \geqslant T$$

or

$$f(t, y(g(t))) \geqslant \Big(\frac{y(t)}{c_1 \varrho_\alpha(t)}\Big)^\beta f(t, c_1 \varrho_\alpha(t)), \ t \geqslant T. \tag{4.6}$$

On the other hand, since $y'(t) < 0$ on $[T, \infty)$ we have (3.14), i.e.

$$\Big(\frac{y(t)}{c_1 \varrho_\alpha(t)}\Big)^\alpha \geqslant c_1^{-\alpha} r(t)|y'(t)|^\alpha, \ t \geqslant T_1 \geqslant T. \tag{4.7}$$

Integrating (4.5) from T_1 to t, we get

$$-L_1^\alpha x(t) > -L_1^\alpha x(t) + L_1^\alpha x(T_1) \geqslant \int_{T_1}^t f(s, y(g(s)))\,\mathrm{d}s, \ t \geqslant T_1. \tag{4.8}$$

Noting that $L_1^\alpha(t) = -r(t)|y'(t)|^\alpha < 0$ and using (4.6)–(4.8) we obtain

$$\begin{aligned}\Big(\frac{y(t)}{c_1\varrho_\alpha(t)}\Big)^\alpha &\geqslant c_1^{-\alpha}\int_{T_1}^t f(s,y(g(s)))\,\mathrm{d}s\\ &\geqslant c_1^{-\alpha}\int_{T_1}^t \Big(\frac{y(s)}{c_1\varrho_\alpha(s)}\Big)^\beta f(s,c_1\varrho_\alpha(s))\,\mathrm{d}s,\quad t\geqslant T_1.\end{aligned}$$

Denote by $z(t)$ the last integral in the above inequalities. We then have

$$z'(t) = c_1^{-\alpha}\Big(\frac{y(t)}{c_1\varrho_\alpha(t)}\Big)^\beta f(t,c_1\varrho_\alpha(t)) \geqslant c_1^{-\alpha}(z(t))^{\frac{\beta}{\alpha}} f(t,c_1\varrho_\alpha(t)),\ t\geqslant T_1.$$

We divide the above inequality by $z(t)^{\frac{\beta}{\alpha}}$ and integrate it from T_1 to ∞, obtaining

$$c_1^{-\alpha}\int_{T_1}^\infty f(t,c_1\varrho_\alpha(t))\,\mathrm{d}t \leqslant \frac{\alpha}{\beta-\alpha} z(T_1)^{1-\frac{\beta}{\alpha}} < \infty,$$

which contradicts (4.2).

II. Let $x\in N^-$. Then $\lim\limits_{t\to\infty} x(t) = 0$ by Remark 2.1.

The proof of Theorem 4.1 is complete. □

Theorem 4.2. *Let the equation* (E) *be strongly sublinear. Suppose that* (4.1) *holds. Every proper solution of* (E) *is either oscillatory or tends to* 0 *as* $t\to\infty$ *if and only if*

$$\int_a^\infty \Big(\frac{1}{r(t)}\int_T^t |f(s,c)|\,\mathrm{d}s\Big)^{\frac{1}{\alpha}}\mathrm{d}t = \infty \tag{4.9}$$

for all constants $c\neq 0$.

Proof. The "only if" part follows from Theorem 3.1.

To prove the "if" part we assume for a contradiction that (E) has a nonoscillatory solution $x(t)$ such that $\lim\inf\limits_{t\to\infty}|x(t)| > 0$. Without loss of generality we may suppose that $x(g(t)) > 0$ for $t\geqslant t_0$. Then the equation (E) in view of (C_1)–(C_5) implies that the function $L_0(t)$ is eventually of constant sign, i.e. either $x\in N^+$ or $x\in N^-$.

I. Let $x\in N^+$. Then the function $y(t) = L_0x(t)$ satisfies (4.3) and (4.4).

i) Suppose that $y'(t) > 0$ on $[t_1,\infty)$. Then there exist $K>0$ and $t_2\geqslant t_1$ such that $y(g(t))\geqslant K$ for $t\geqslant t_2$. It follows from (4.5) in view of (C_5) that

$$(L_1^\alpha x(t))' \leqslant -f(t,K),\quad t\geqslant t_2.$$

Integrating this inequality from t_2 to t yields

$$\int_{t_2}^{t} f(s,K)\,\mathrm{d}s \leqslant L_1^\alpha x(t_2) - L_1^\alpha x(t) \leqslant L_1^\alpha x(t_2) < \infty, \tag{4.10}$$

which, in view of the assumption (C_1), implies

$$\int_{t_2}^{\infty} \left(\frac{1}{r(t)} \int_{t_2}^{t} f(s,K)\,\mathrm{d}s\right)^{\frac{1}{\alpha}} \mathrm{d}t < \infty.$$

This contradicts (4.9).

ii) Suppose that $y'(t) < 0$ on $[t_1, \infty)$. Using the sublinearity of (E) and (4.4) we find

$$(y(g(t)))^{-\gamma} f(t, y(g(t))) \geqslant c_2^{-\gamma} f(t, c_2), \quad t \geqslant \gamma(t_1) = t_2,$$

where $\gamma \in (0, \alpha)$ is the exponent of sublinearity. Combining (4.5) with (4.11) shows that

$$-(L_1^\alpha x(t))' \geqslant c_2^{-\gamma}(y(g(t)))^{-\gamma} f(t, c_2), \quad t \geqslant T_2. \tag{4.12}$$

Integrating (4.12) from T_2 to t and using the decreasing nature of y and (4.1), we obtain

$$r(t)|y'(t)|^\alpha \geqslant c_2^{-\gamma}(y(t))^{-\gamma} \int_{T_2}^{t} f(s, c_2)\,\mathrm{d}s,$$

which is equivalent to

$$|y'(t)|(y(t))^{-\frac{\gamma}{\alpha}} \geqslant c_2^{-\frac{\gamma}{\alpha}} \left(\frac{1}{r(t)} \int_{T_2}^{t} f(s, c_2)\,\mathrm{d}s\right)^{\frac{1}{\alpha}}, \ t \geqslant T_2. \tag{4.13}$$

Integrating (4.13) from T_2 to ∞ we conclude that

$$c_2^{-\frac{\gamma}{\alpha}} \int_{T_2}^{\infty} \left(\frac{1}{r(t)} \int_{T_2}^{t} f(s,K)\,\mathrm{d}s\right)^{\frac{1}{\alpha}} \mathrm{d}t \leqslant \frac{\alpha}{\alpha - \gamma} (y(T_2))^{\frac{\alpha-\gamma}{\alpha}},$$

which contradicts (4.9).

II. Let $x \in N^-$. Then $\lim\limits_{t\to\infty} x(t) = 0$ by Remark 2.1.

This completes the proof of Theorem 4.2. $\square$

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