# A NOTE ON THE PARABOLIC VARIATION

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*Abstract.* A condition for solvability of an integral equation which is connected with the first boundary value problem for the heat equation is investigated. It is shown that if this condition is fulfilled then the boundary considered is $\frac{1}{2}$-Hölder. Further, some simple concrete examples are examined.

*Keywords*: heat equation, boundary value problem, integral equations, parabolic variation

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Let $\varphi$ be a continuous function with bounded variation on a compact interval $\langle a, b\rangle$,

$$K = \{[\varphi(t), t];\ t \in \langle a, b\rangle\}.$$

For $[x, t] \in \mathbb{R}^2$, $\alpha, r > 0$, $\alpha < +\infty$, let $n_{x,t}(r, \alpha)$ denote the number (finite or $+\infty$) of points of the set

$$K \cap \left\{[\xi, \tau] \in \mathbb{R}^2;\ t - \tau = \left(\frac{\xi - x}{2\alpha}\right)^2, 0 < t - \tau < r\right\}.$$

For fixed $[x, t] \in \mathbb{R}^2$, $r > 0$, the function $n_{x,t}(r, \alpha)$ is a measurable function of the variable $\alpha \in (0, +\infty)$ and thus one can define

$$V_K(r; x, t) = \int_0^{+\infty} \mathrm{e}^{-\alpha^2} n_{x,t}(r, \alpha)\,\mathrm{d}\alpha. \tag{1}$$

Note that the function $V_K(+\infty; \cdot, \cdot)$ is called the parabolic variation of the set (curve) $K$. Parabolic variation was defined in [2] in connection with the study of a

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heat potential in $\mathbb{R}^2$. Note that boundedness of the parabolic variation is a condition for the existence of limits of this potential on $K$ and for the validity of an analogue of the jump formula.

For $[x,t] \in \mathbb{R}^2$, $t > a$, let $\alpha_{x,t}$ be the function defined on the interval $\langle a, \min\{t,b\})$ by

$$\alpha_{x,t}(\tau) = \frac{x - \varphi(\tau)}{2\sqrt{t-\tau}}. \tag{2}$$

The function $\alpha_{x,t}$ has locally finite variation on $\langle a, \min\{t,b\})$ (under the assumption that $\varphi$ is of finite variation on $\langle a,b\rangle$) and

$$V_K(r;x,t) = \int_{\max\{a,t-r\}}^{\min\{t,b\}} \mathrm{e}^{-\alpha_{x,t}^2(\tau)}\, \mathrm{d}\big(\operatorname{var} \alpha_{x,t}(\tau)\big) \tag{3}$$

whenever $\max\{a, t-r\} < \min\{t,b\}$ [otherwise $V_K(r;x,t) = 0$].

Let $[x,t] \in K$, $t > a$. It is known that if $V_K(r;x,t) < +\infty$ for some $r > 0$ then there is a limit (finite or infinite)

$$\alpha_{x,t}(t) = \lim_{\tau \to t-} \alpha_{x,t}(\tau) \tag{4}$$

[by this equality the value $\alpha_{x,t}(t)$ is defined]. Further, let $G$ be the function defined on $\mathbb{R}^1$ by

$$G(t) = \begin{cases} 0, & t = -\infty, \\ \displaystyle\int_{-\infty}^{t} \mathrm{e}^{-x^2}\, \mathrm{d}x, & t > -\infty. \end{cases} \tag{5}$$

The condition

$$\lim_{r \to 0+} \sup_{t \in (a,b\rangle} \Big[\frac{2}{\sqrt{\pi}} V_K\big(r;\varphi(t),t\big) + \Big|1 - \frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t),t}(t)\big)\Big|\Big] < 1 \tag{6}$$

plays a role in connection with the first boundary value problem for the heat equation on the set

$$\{[x,t] \in \mathbb{R}^2\,;\ t \in (a,b),\ x > \varphi(t)\}$$

or on the set

$$\{[x,t] \in \mathbb{R}^2\,;\ t \in (a,b),\ x < \varphi(t)\}.$$

Analogous conditions appear also in connection with the first boundary value problem for the heat equation on sets of the form

$$\{[x,t] \in \mathbb{R}^2\,;\ t \in (a,b),\ \varphi_1(t) < x < \varphi_2(t)\},$$

where $\varphi_1, \varphi_2$ are continuous functions of bounded variation on $\langle a, b\rangle$. Under the condition (6) the relevant integral equation has a solution and the Fourier problem can be represented by the heat potential. It was shown in [1] that under the condition (6) the just mentioned integral equation is solvable also in the space of all bounded Baire functions and that the corresponding Neumann series converges.

Let us prove the following two assertions.

**Proposition 1.** *Let $\varphi$ be a Lipschitz function on $\langle a, b\rangle$. Then*

$$\lim_{r\to 0+} \sup_{t\in\langle a,b\rangle} \Big[\frac{2}{\sqrt{\pi}} V_K\big(r;\varphi(t),t\big) + \Big|1 - \frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t),t}(t)\big)\Big|\Big] = 0. \tag{7}$$

*Let $\varphi$ be a continuous function of bounded variation on $\langle a, b\rangle$, $a < a' < b' \leqslant b$ and suppose that $\varphi$ is Lipschitz on $\langle a', b'\rangle$. Then*

$$\lim_{r\to 0+} \sup_{t\in\langle a'+\delta,b'\rangle} \Big[\frac{2}{\sqrt{\pi}} V_K\big(r;\varphi(t),t\big) + \Big|1 - \frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t),t}(t)\big)\Big|\Big] = 0 \tag{8}$$

*for any $\delta > 0$, $\delta < b' - a'$.*

P r o o f. If $\varphi$ is Lipschitz on $\langle a, b\rangle$ then clearly $\alpha_{\varphi(t),t}(t) = 0$ and thus

$$\Big|1 - \frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t),t}(t)\big)\Big| = 0$$

for each $t \in (a, b\rangle$.

Recall a well known fact that if $F$ is a non-negative continuous function on an interval $J \subset \mathbb{R}^1$, $f, g$ continuous functions of locally bounded variation on $J$ then

$$\int_J F \,\mathrm{d}\operatorname{var}(fg) \leqslant \int_J F|f| \,\mathrm{d}\operatorname{var} g + \int_J F|g| \,\mathrm{d}\operatorname{var} f.$$

Let $k \in \mathbb{R}^1$ be such that for any $t_1, t_2 \in \langle a, b\rangle$

$$\big|\varphi(t_1) - \varphi(t_2)\big| \leqslant k|t_1 - t_2|;$$

then $|\varphi'(\tau)| \leqslant k$ for almost all $\tau \in (a, b)$. Using the expression (3) for $V_K$ we get for $t \in (a, b\rangle$, $r > 0$,

$$\begin{aligned}
V_K\bigl(r; \varphi(t), t\bigr) &= \int_{\max\{a, t-r\}}^{t} \mathrm{e}^{-\alpha^2_{\varphi(t),t}(\tau)} \,\mathrm{d}_\tau \operatorname{var} \alpha_{\varphi(t),t}(\tau) \\
&\leqslant \int_{\max\{a, t-r\}}^{t} \mathrm{d}_\tau \operatorname{var} \frac{\varphi(t) - \varphi(\tau)}{2\sqrt{t-\tau}} \\
&\leqslant \int_{\max\{a, t-r\}}^{t} \frac{1}{2\sqrt{t-\tau}} \,\mathrm{d}_\tau \operatorname{var}\bigl(\varphi(t) - \varphi(\tau)\bigr) \\
&\qquad + \int_{\max\{a, t-r\}}^{t} \bigl|\varphi(t) - \varphi(\tau)\bigr| \,\mathrm{d}_\tau \operatorname{var} \frac{1}{2\sqrt{t-\tau}} \\
&\leqslant \int_{\max\{a, t-r\}}^{t} \frac{|\varphi'(\tau)|}{2\sqrt{t-\tau}} \,\mathrm{d}\tau + \int_{\max\{a, t-r\}}^{t} k(t-\tau) \frac{1}{4(t-\tau)^{3/2}} \,\mathrm{d}\tau \\
&\leqslant \frac{3}{2} k \int_{\max\{a, t-r\}}^{t} \frac{\mathrm{d}\tau}{2\sqrt{t-\tau}} \leqslant \frac{3}{2} k\sqrt{r}.
\end{aligned}$$

Now we see that (7) is valid.

Denote for a while

$$K' = \bigl\{[\varphi(t), t];\ t \in \langle a', b'\rangle\bigr\}.$$

Fix $\delta > 0$, $\delta < b' - a'$. Then for $r > 0$, $r < \delta$ and $t \in \langle a' + \delta, b'\rangle$

$$V_K\bigl(r; \varphi(t), t\bigr) = V_{K'}\bigl(r; \varphi(t), t\bigr).$$

The second part of the assertion follows now from the first part applied to the interval $\langle a', b'\rangle$. $\square$

If the condition (6) is fulfilled then the absolute values of limits

$$\alpha_{\varphi(t),t}(t) = \lim_{\tau \to t-} \frac{\varphi(t) - \varphi(\tau)}{2\sqrt{t-\tau}}$$

$[t \in (a, b\rangle]$ are bounded by a finite constant. This fact alone does not imply that $\varphi$ is $\frac{1}{2}$-Hölder. Nevertheless, the following assertion is valid.

**Proposition 2.** *Let $\varphi$ be a continuous function of bounded variation on $\langle a, b\rangle$. If the condition* (6) *is fulfilled then $\varphi$ is $\frac{1}{2}$-Hölder on $\langle a, b\rangle$.*

P r o o f. Let $r > 0$ be such that

$$\sup_{t \in (a,b\rangle} \Bigl[\frac{2}{\sqrt{\pi}} V_K\bigl(r; \varphi(t), t\bigr) + \Bigl|1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\Bigr|\Bigr] = \lambda < 1. \tag{9}$$

Then, of course,

$$\Bigl|1-\frac{2}{\sqrt{\pi}}G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\Bigr|\leqslant\lambda$$

for each $t\in(a,b\rangle$ and hence there is a $c_0\in\mathbb{R}^1$ such that for each $t\in(a,b\rangle$

$$|\alpha_{\varphi(t),t}(t)|=\lim_{\tau\to t-}\Bigl|\frac{\varphi(t)-\varphi(\tau)}{2\sqrt{t-\tau}}\Bigr|\leqslant c_0.$$

Choose $k\in\mathbb{R}^1$, $k>c_0$ such that

$$\frac{2}{\sqrt{\pi}}\int_0^k \mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha>\lambda. \tag{10}$$

Suppose that $\varphi$ is not $\frac{1}{2}$-Hölder continuous on $\langle a,b\rangle$. Then there are $t_1,t_2\in\langle a,b\rangle$, $|t_1-t_2|<r$, such that

$$\bigl|\varphi(t_1)-\varphi(t_2)\bigr|>2k\sqrt{|t_1-t_2|}. \tag{11}$$

We can assume that $t_1>t_2$. Put

$$c=\bigl|\alpha_{\varphi(t_1),t_1}(t_1)\bigr|=\lim_{\tau\to t_1-}\Bigl|\frac{\varphi(t_1)-\varphi(\tau)}{2\sqrt{t_1-\tau}}\Bigr|.$$

We have $c<k$, of course. Let $\delta>0$ be such that

$$c+\delta<k.$$

Then there is an $\varepsilon>0$, $\varepsilon<t_1-t_2$ such that

$$\bigl|\varphi(t_1)-\varphi(\tau)\bigr|<2(c+\delta)\sqrt{t_1-\tau} \tag{12}$$

for any $\tau\in(t_1-\varepsilon,t_1)$. Let us show that whenever $\alpha\in\mathbb{R}^1$,

$$c+\delta<\alpha<k,$$

then

$$n_{\varphi(t_1),t_1}(r,\alpha)\geqslant 1; \tag{13}$$

recall that for $[x,t]\in\mathbb{R}^2$, $\alpha,r>0$ the symbol $n_{x,t}(r,\alpha)$ stands for the number of points of the intersection

$$K\cap\Bigl\{[\xi,\tau]\in\mathbb{R}^2;\ t-\tau=\Bigl(\frac{\xi-x}{2\alpha}\Bigr)^2,\ 0<t-\tau<r\Bigr\}.$$

Consider for example the case $\varphi(t_2) > \varphi(t_1)$. Given $\alpha > c + \delta$ put

$$p = \Big\{[\xi, \tau] \in \mathbb{R}^2\,;\ t_1 - \tau = \Big(\frac{\xi - \varphi(t_1)}{2\alpha}\Big)^2,\ 0 < t_1 - \tau < r,\ \xi > \varphi(t_1)\Big\}.$$

If $[\xi, \tau] \in p$, $\tau \in (t_1 - \varepsilon, t_1)$ then (12) yields

$$\xi - \varphi(t_1) = 2\alpha\sqrt{t_1 - \tau} > 2(c + \delta)\sqrt{t_1 - \tau} > \varphi(\tau) - \varphi(t_1),$$

that is

$$\xi > \varphi(\tau). \tag{14}$$

If $[\xi, \tau] \in p$, $\tau = t_2$, then (11) yields (if $\alpha < k$)

$$\xi - \varphi(t_1) = 2\alpha\sqrt{t_1 - t_2} < 2k\sqrt{t_1 - t_2} < \varphi(t_2) - \varphi(t_1),$$

that is

$$\xi < \varphi(t_2).$$

This, together with the fact that for $[\xi, \tau] \in p$, $\tau \in (t_1 - \varepsilon, t_1)$ the inequality (14) holds, implies that

$$K \cap p \neq \emptyset,$$

which means that $n_{\varphi(t_1),t_1}(r, \alpha) \geqslant 1$ for any $\alpha \in (c + \delta, k)$, indeed. Since $\delta > 0$ was arbitrary we see that (13) is valid for any $\alpha \in (c, k)$. Similarly in the case $\varphi(t_2) < \varphi(t_1)$.

Since

$$\Big|1 - \frac{2}{\sqrt{\pi}}G(q)\Big| = \frac{2}{\sqrt{\pi}}\int_0^{|q|} \mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha$$

for $q \in {}^*\mathbb{R}^1$ we now obtain [using (1) and (10)]

$$\begin{aligned}
\frac{2}{\sqrt{\pi}}V_K\big(r; \varphi(t_1), t_1\big) &+ \Big|1 - \frac{2}{\sqrt{\pi}}G\big(\alpha_{\varphi(t_1),t_1}(t_1)\big)\Big| \\
&\geqslant \frac{2}{\sqrt{\pi}}\int_c^k \mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha + \frac{2}{\sqrt{\pi}}\int_0^c \mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha \\
&= \frac{2}{\sqrt{\pi}}\int_0^k \mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha > \lambda,
\end{aligned}$$

but this contradicts (9). $\square$

Now let us evaluate the values of the term

$$\lim_{r\to 0+}\sup_{t\in\langle a,b\rangle}\Big[\frac{2}{\sqrt{\pi}}V_K\big(r;\varphi(t),t\big)+\Big|1-\frac{2}{\sqrt{\pi}}G\big(\alpha_{\varphi(t),t}(t)\big)\Big|\Big]$$

in the following three simple particular cases.

E x a m p l e 1. Let $\langle a,b\rangle=\langle 0,1\rangle$ and

$$\varphi(t)=k\sqrt{t}$$

on $\langle a,b\rangle$, where $k>0$.

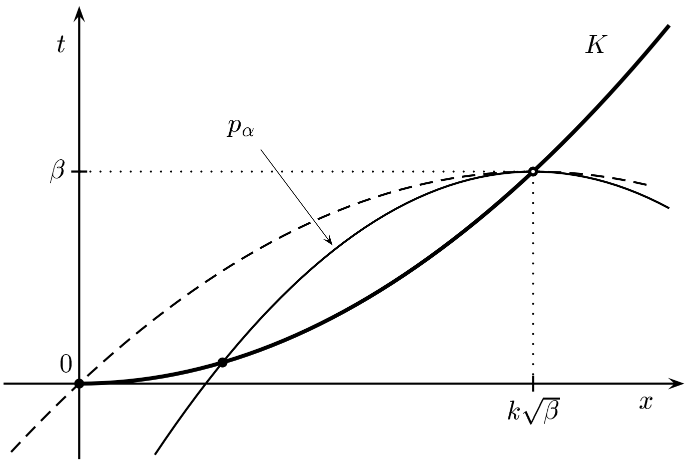


Figure 1

Denote

$$K=\big\{\,[\xi,\tau]\in\mathbb{R}^2\,;\ \tau\in\langle 0,1\rangle,\xi=k\sqrt{\tau}\,\big\}$$

and consider a point $[k\sqrt{\beta},\beta]\in K$, where $\beta\in(0,1\rangle$. For $\alpha>0$ denote further

$$p_\alpha=\Big\{[\xi,\tau]\in\mathbb{R}^2\,;\ \beta-\tau=\Big(\frac{k\sqrt{\beta}-\xi}{2\alpha}\Big)^2,\tau<\beta\Big\}.$$

It is easy to see that $K\cap p_\alpha\neq\emptyset$ if and only if $\alpha\leqslant\frac{k}{2}$. The dashed parabola in Figure 1 corresponds to the case $\alpha=\frac{k}{2}$ and the solid parabola $p_\alpha$ to the case $\alpha<\frac{k}{2}$. For $\alpha\leqslant\frac{k}{2}$ the intersection $K\cap p_\alpha$ contains just one point. Now we see that if $r>0$, $\beta<r$, then

$$n_{k\sqrt{\beta},\beta}(r,\alpha)=\begin{cases}1 & \text{for } \alpha\in(0,\frac{k}{2}\rangle,\\ 0 & \text{for } \alpha>\frac{k}{2},\end{cases}$$

and thus for such $r, \beta$ we always have

$$V_K\bigl(r;\varphi(\beta),\beta\bigr) = \int_0^{\frac{k}{2}} \mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha.$$

If $0 < r < \beta \leqslant 1$ then

$$V_K\bigl(r;\varphi(\beta),\beta\bigr) < \int_0^{\frac{k}{2}} \mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha.$$

As $\alpha_{\varphi(\beta),\beta}(\beta) = 0$ for any $\beta \in (0,1\rangle$ we see that for each $k > 0$

$$\lim_{r\to 0+}\ \sup_{\beta\in(0,1\rangle} \Bigl[\frac{2}{\sqrt{\pi}} V_K\bigl(r;\varphi(\beta),\beta\bigr) + \Bigl|1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(\beta),\beta}(\beta)\bigr)\Bigr|\Bigr] = \frac{2}{\sqrt{\pi}} \int_0^{\frac{k}{2}} \mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha < 1.$$

E x a m p l e 2. Let $\langle a,b\rangle = \langle -1,0\rangle$,

$$\varphi(t) = k\sqrt{-t}$$

on $\langle -1,0\rangle$, where $k > 0$.

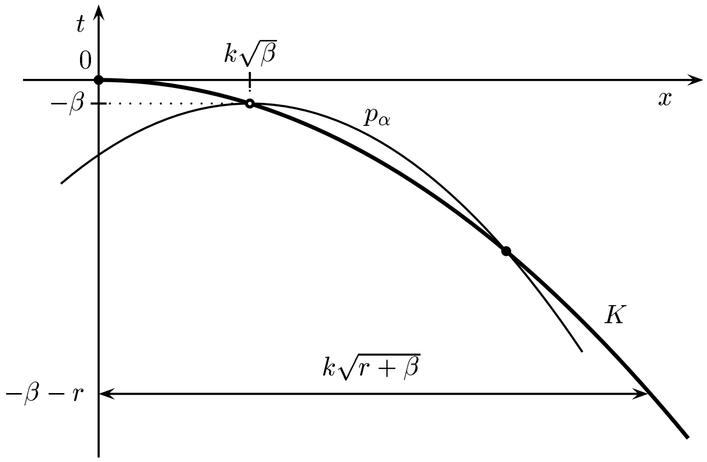


Figure 2

It is easy to see that for each $r > 0$

$$n_{0,0}(r,\alpha) = \begin{cases} +\infty & \text{for } \alpha = \frac{k}{2},\\ 0 & \text{for } \alpha \neq \frac{k}{2}, \end{cases}$$

hence

$$V_K(r;0,0) = 0.$$

On the other hand,

$$\alpha_{0,0}(0) = \lim_{\tau \to 0-} \frac{\varphi(0) - \varphi(\tau)}{2\sqrt{-\tau}} = -\frac{k}{2}$$

and thus

$$\Bigl|1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{0,0}(0)\bigr)\Bigr| = \frac{2}{\sqrt{\pi}} \int_0^{\frac{k}{2}} \mathrm{e}^{-\alpha^2}\, \mathrm{d}\alpha.$$

For $t \in (-1, 0)$ we have $\alpha_{\varphi(t),t}(t) = 0$, of course.

Now let $\beta \in (0,1)$, $r > 0$ and suppose that

$$r + \beta < 1.$$

For $\alpha > 0$ denote

$$p_\alpha = \Bigl\{[\xi, \tau] \in \mathbb{R}^2\,;\ -\beta - \tau = \Bigl(\frac{\xi - k\sqrt{\beta}}{2\alpha}\Bigr)^2, \tau < -\beta\Bigr\}.$$

Denote further, for a while,

$$K_\infty = \bigl\{\, [\xi, \tau] \in \mathbb{R}^2\,;\ \tau \in (-\infty, 0\rangle, \xi = k\sqrt{-\tau}\,\bigr\}.$$

It is easy to see that $K_\infty \cap p_\alpha \neq \emptyset$ if and only if $\alpha \in (0, \frac{k}{2})$. If $\alpha \in (0, \frac{k}{2})$ then $K_\infty \cap p_\alpha$ is a singleton and if $\{[\xi, \tau]\} = K_\infty \cap p_\alpha$ then

$$\xi = k\sqrt{\beta}\, \frac{k^2 + 4\alpha^2}{k^2 - 4\alpha^2}.$$

Now we see that $n_{k\sqrt{\beta}, -\beta}(r, \alpha) = 1$ if $\alpha \in (0, \frac{k}{2})$ is such that

$$(*) \qquad \sqrt{\beta}\, \frac{k^2 + 4\alpha^2}{k^2 - 4\alpha^2} < \sqrt{r + \beta},$$

otherwise $n_{k\sqrt{\beta}, -\beta}(r, \alpha) = 0$. Given $\alpha_0 \in (0, \frac{k}{2})$ then there is a $\beta > 0$ ($\beta < 1$) such that $(*)$ is fulfilled for each $\alpha \in (0, \alpha_0\rangle$. We obtain that for $\beta \in (0,1)$

$$V_K(r; k\sqrt{\beta}, -\beta) < \int_0^{\frac{k}{2}} \mathrm{e}^{-\alpha^2}\, \mathrm{d}\alpha$$

and

$$\sup_{\beta \in (0,1)} V_K(r; k\sqrt{\beta}, -\beta) = \int_0^{\frac{k}{2}} \mathrm{e}^{-\alpha^2}\, \mathrm{d}\alpha.$$

Altogether,

$$\lim_{r\to 0+}\sup_{t\in(-1,0\rangle}\Big[\frac{2}{\sqrt{\pi}}V_K\big(r;\varphi(t),t\big)+\Big|1-\frac{2}{\sqrt{\pi}}G\big(\alpha_{\varphi(t),t}(t)\big)\Big|\Big]=\frac{2}{\sqrt{\pi}}\int_0^{\frac{k}{2}}\mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha<1.$$

Example 3. Consider the case $\langle a,b\rangle=\langle -1,1\rangle$,

$$\varphi(\tau)=k\sqrt{|\tau|},$$

where $k>0$. By example 2 we have

$$\lim_{r\to 0+}\sup_{t\in(-1,0\rangle}\Big[\frac{2}{\sqrt{\pi}}V_K\big(r;\varphi(t),t\big)+\Big|1-\frac{2}{\sqrt{\pi}}G\big(\alpha_{\varphi(t),t}(t)\big)\Big|\Big]=\frac{2}{\sqrt{\pi}}\int_0^{\frac{k}{2}}\mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha.$$

It follows from Proposition 1 that for any $\varepsilon>0$, $\varepsilon<1$, we have

$$\lim_{r\to 0+}\sup_{t\in\langle\varepsilon,1\rangle}\Big[\frac{2}{\sqrt{\pi}}V_K\big(r;\varphi(t),t\big)+\Big|1-\frac{2}{\sqrt{\pi}}G\big(\alpha_{\varphi(t),t}(t)\big)\Big|\Big]=0.$$

Thus it suffices to examine points $[k\sqrt{\beta},\beta]$ for $\beta>0$ near zero.

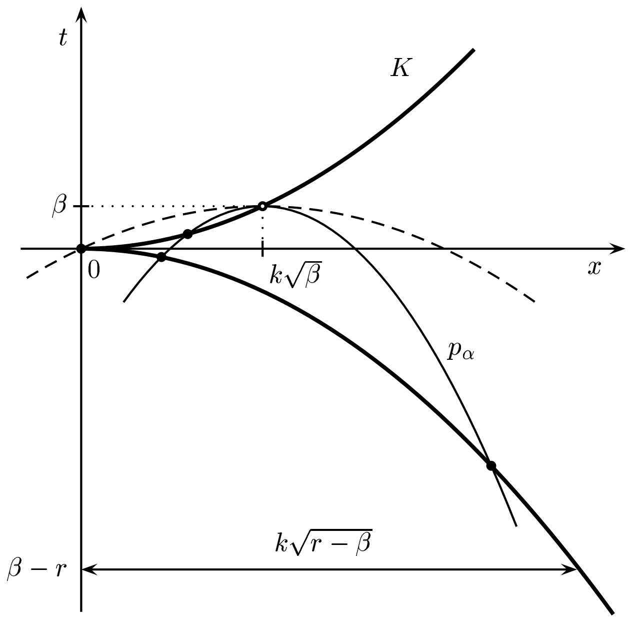


Figure 3

Let $r \in (0,1)$, $\beta > 0$, $\beta < \frac{r}{2}$. Denote

$$\begin{aligned}
p_\alpha &= \Big\{[\xi,\tau] \in \mathbb{R}^2\,;\ \beta - \tau = \Big(\frac{\xi - k\sqrt{\beta}}{2\alpha}\Big)^2, \tau < \beta\Big\},\\
p_\alpha^+ &= \big\{[\xi,\tau] \in p_\alpha\,;\ \xi > k\sqrt{\beta}\big\},\\
p_\alpha^- &= \big\{[\xi,\tau] \in p_\alpha\,;\ \xi < k\sqrt{\beta}\big\},\\
K_\infty &= K \cup \big\{[\xi,\tau] \in \mathbb{R}^2\,;\ \tau \in (-\infty, 0\rangle, \xi = k\sqrt{-\tau}\big\}.
\end{aligned}$$

It is easy to see that $K \cap p_\alpha^- = \emptyset$ for any $\alpha > \frac{k}{2}$, for $\alpha = \frac{k}{2}$ the intersection $K \cap p_\alpha^-$ is a singleton, and $K \cap p_\alpha^-$ consists of two points whenever $\alpha \in (0, \frac{k}{2})$.

Now consider $K_\infty \cap p_\alpha^+$. Elementary calculation yields that $K_\infty \cap p_\alpha^+ \neq \emptyset$ if and only if $\alpha \in (0, \frac{k}{2})$, and for $\alpha \in (0, \frac{k}{2})$ the intersection $K_\infty \cap p_\alpha^+$ is a singleton. If $\alpha \in (0, \frac{k}{2})$, $\{[\xi,\tau]\} = K_\infty \cap p_\alpha^+$ then

$$\xi = k\sqrt{\beta}\,\frac{k^2 + 2\sqrt{2}\alpha\sqrt{k^2 - 2\alpha^2}}{k^2 - 4\alpha^2}.$$

Thus we see that $n_{k\sqrt{\beta},\beta}(r,\alpha) = 0$ for $\alpha > \frac{k}{2}$, $n_{k\sqrt{\beta},\beta}(r, \frac{k}{2}) = 1$ and

$$2 \leqslant n_{k\sqrt{\beta},\beta}(r,\alpha) \leqslant 3$$

for $\alpha \in (0, \frac{k}{2})$. Here $n_{k\sqrt{\beta},\beta}(r,\alpha) = 3$ if and only if

$$(*) \qquad \sqrt{\beta}\,\frac{k^2 + 2\sqrt{2}\alpha\sqrt{k^2 - 2\alpha^2}}{k^2 - 4\alpha^2} < \sqrt{r - \beta}.$$

Given $\alpha_0 \in (0, \frac{k}{2})$ then there is a $\beta > 0$, $\beta < r/2$ such that for each $\alpha \in (0, \alpha_0)$ the condition $(*)$ is fulfilled. Now one can see that for $\varepsilon \in (0,1)$

$$\sup_{\beta \in (0,\varepsilon)} V_K\big(r; k\sqrt{\beta}, \beta\big) = 3\int_0^{\frac{k}{2}} \mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha.$$

Since $\alpha_{\varphi(t),t}(t) = 0$ for $t \in (0,1\rangle$ we obtain

$$\lim_{r\to 0+}\ \sup_{t \in (-1,1\rangle} \Big[\frac{2}{\sqrt{\pi}} V_K\big(r; \varphi(t), t\big) + \Big|1 - \frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t),t}(t)\big)\Big|\Big] = 3\frac{2}{\sqrt{\pi}}\int_0^{\frac{k}{2}} \mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha.$$

As

$$\frac{2}{\sqrt{\pi}}\int_0^{c} \mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha = \frac{1}{3}$$

for

$$c \approx 0.304570194\ldots$$

the condition

$$\lim_{r\to 0+}\sup_{t\in(-1,1\rangle}\Big[\frac{2}{\sqrt{\pi}}V_K\big(r;\varphi(t),t\big)+\Big|1-\frac{2}{\sqrt{\pi}}G\big(\alpha_{\varphi(t),t}(t)\big)\Big|\Big]<1$$

is not fulfilled for all $k>0$ but only if

$$k < 0.609140388\ldots$$

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