ON THE OSCILLATION OF CERTAIN NEUTRAL DIFFERENCE EQUATIONS

S. R. Grace, Giza, G. G. Hamedani, Milwaukee

(Received July 15, 1998)

Abstract. Various new criteria for the oscillation of nonlinear neutral difference equations of the form

$$\Delta^i\,(x_n - x_{n-h}) + q_n |x_{n-g}|^c \operatorname{sgn} x_{n-g} = 0, \quad i = 1, 2, 3 \text{ and } c > 0,$$

are established.

Keywords: nonlinear difference equations, oscillatory solutions, asymptotic behavior of solutions

MSC 2000: 34K25, 34K99

1. Introduction

Let $\mathbb{N}^*$ be the set of all non-negative intergers, and let Δ be the first order forward difference operator, $\Delta x_n = x_{n+1} - x_n,\ n \in \mathbb{N}^*$. For $i \geqslant 1$, let Δ^i be the i-th order forward operator, $\Delta^i x_n = \Delta(\Delta^{i-1} x_n)$.

Consider the neutral difference equations

$$\Delta^i(x_n - x_{n-h}) + q_n\, |x_{n-g}|^c \operatorname{sgn} x_{n-g} = 0, \quad i = 1, 2, 3, \tag{E_i}$$

and

$$\Delta^i(x_n - x_{n-h}) - q_n\, |x_{n-g}|^c \operatorname{sgn} x_{n-g} = 0, \quad i = 1, 2, 3, \tag{N_i}$$

where $\{q_n\}$ is a sequence of non-negative real numbers, c is a positive constant, and h and g are positive integers. A solution $\{x_n\},\ n \in \mathbb{N}^*$ of the equations (E_i) (or of (N_i)) is said to be oscillatory if for every $n_0 \geqslant 0$, there exists an $n \geqslant n_0$ such that

$x_n\, x_{n+1} \leqslant 0$. Otherwise the solution is called nonoscillatory. The equation (E_i) is called oscillatory if every solution of (E_i) is oscillatory.

The problem of obtaining sufficient conditions under which all the solutions or all the bounded solutions of certain classes of neutral delay difference equations are oscillatory has been studied by a number of authors. A large portion of the results reported have been for neutral difference equations of the form

$$(\mathrm{P}_i) \qquad \Delta^i(x_n + a x_{n-h}) + q_n\,|x_{n-g}|^c \operatorname{sgn} x_{n-g} = 0, \quad i \geqslant 1,\ c > 0,$$

where $a \neq -1$. Here, we refer to [1–11] and the references cited therein.

Much less is known regarding the oscillatory behavior of (E_1) when $c = 1$, though a number of authors have considered this problem. For recent works in this direction, we refer the reader to [1, 4, 8]. It seems that in these results the condition

$$\sum_{j=n_0 \geqslant 0}^{\infty} q_j = \infty, \tag{1.1}$$

is essential for the oscillation of the equation (E_1) for $c = 1$. In view of Theorem 1 of [12], for the continuous analogue of (E_1) with $c = 1$, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(x(t) - x(t-h)) + q(t)\,x(t-g) = 0.$$

where $q\colon\ [t_0, \infty) \longrightarrow (0, \infty)$ is continuous and g and h are positive real numbers, one can easily show that (E_1) with $c = 1$ is oscillatory if

$$\sum^{\infty} n q_n \sum_{j=n}^{\infty} q_j = \infty. \tag{1.2}$$

Very little is known, as far as we have gathered, regarding the oscillation of nonlinear equations (E_i) and (N_i), $i = 1, 2, 3$. The purpose of this paper is to establish some new criteria for the oscillation of all solutions (all bounded solutions) of (E_i) (of (N_i)), $i = 1, 2, 3$. The results of this paper can be applied to superlinear ($c > 1$), linear ($c = 1$) and sublinear ($0 < c < 1$) equations of type (E_i) and (N_i). We would also like to point out that the result obtained for (E_1) extends the two oscillation criteria mentioned above.

2. Oscillation of (E_i), $i = 1, 2, 3$

First we investigate the oscillation of (E_3) by considering two cases:

Case 1. For $n \geqslant n_0 \geqslant 0$, $Q_n = \sum\limits_{j=n}^{\infty} q_j < \infty$.

Theorem 2.1. *If*

$$\sum_{n=n_0}^{\infty} (nQ_n)^c q_n = \infty, \tag{2.1}$$

then (E_3) *is oscillatory.*

Proof. Let $\{x_n\}$ be an eventually positive nonoscillatory solution of (E_3). Then there exists $n_1 \geqslant n_0$ such that $x_{n-a} > 0$ for $n \geqslant n_1$, where $a = \max\{g, h\}$. Let

$$y_n = x_n - x_{n-h}. \tag{2.2}$$

Then

$$\Delta^3 y_n = -q_n x_{n-g}^c \leqslant 0 \quad \text{for} \quad n \geqslant n_1, \tag{2.3}$$

which implies that $\Delta^i y_n, i = 0, 1, 2$ are eventually of one sign and that $\Delta^2 y_n$ is nonincreasing for $n \geqslant n_1$ and is eventually positive. There are four cases to consider:

(A) $y_n < 0$ and $\Delta y_n < 0$ eventually,
(B) $y_n < 0$ and $\Delta y_n > 0$ eventually,
(C) $y_n > 0$ and $\Delta y_n < 0$ eventually,
(D) $y_n > 0$ and $\Delta y_n > 0$ eventually.

Assume (A) holds. Since y_n is nonincreasing for $n \geqslant n_1$, there exist a constant $c_1 > 0$ and $N \geqslant n_1$ such that

$$y_n < -c_1 \quad \text{for} \quad n \geqslant N.$$

Thus,

$$x_N = y_N + x_{N-h} < -c_1 + x_{N-h},$$

or

$$x_{N+h} = y_{N+h} + x_N < -c_1 + x_N < -2c_1 + x_{N-h}.$$

Hence for any integer $m > 1$

$$x_{N+mh} < -(m+1)c_1 + x_{N-h} \longrightarrow -\infty \quad \text{as} \quad m \to \infty,$$

a contradiction.

Assume (B) holds. Since $\Delta^2 y_n > 0$ eventually, we must have $y_n > 0$ eventually, a contradiction.

Assume (C) holds. Here we have

$$x_n > x_{n-h} \quad \text{for} \quad n \geqslant n_1.$$

Hence, there exist a constant $b > 0$ and $N_1 \geqslant n_1 + g$ such that

$$x_{n-g} \geqslant b \quad \text{for} \quad n \geqslant N_1.$$

Then

$$\Delta^3 y_n \leqslant -b^c q_n \quad \text{for} \quad n \geqslant N_1, \tag{2.4}$$

and hence

$$\Delta^2 y_s - \Delta^2 y_n \leqslant -b^c \sum_{j=n}^{s-1} q_j, \quad n \geqslant N_1.$$

Now, letting $s \to \infty$ we have

$$\Delta^2 y_n \geqslant b^c Q_n \quad \text{for} \quad n \geqslant N_1. \tag{2.5}$$

In view of the monotonicity of Δy_n and $\Delta^2 y_n$ we obtain for every $m_2 \geqslant m_1 \geqslant k \geqslant N_1$

$$y_k \geqslant (m_1 - k + 1)(-\Delta y_{m_1}), \tag{2.6}$$

and

$$-\Delta y_{m_1} \geqslant (m_2 - m_1 + 1)\, \Delta^2 y_{m_2}. \tag{2.7}$$

Thus, for $n \geqslant N_2 \geqslant N_1 + 2h$, we have

$$y_{n-2h} \geqslant (h+1)^2\, \Delta^2 y_n. \tag{2.8}$$

Using (2.8) in (2.5), we obtain

$$y_n \geqslant C\ Q_{n+2h}, \quad n \geqslant N_2, \tag{2.9}$$

where $C = b^c (h+1)^2$.

Let $N_2 + (m-2)h \leqslant n \leqslant N_2 + (m-1)h$, then

$$x_n \geqslant C\left(Q_{n+2h} + Q_{n+h} + \dots Q_{n-(m-3)h}\right) + x_{n-mh} \geqslant C(m-2)Q_n. \tag{2.10}$$

From (2.3) and (2.10) we obtain

$$\Delta^3 y_n \leqslant -C^c (m-2)^c Q_n^c q_n = -M_n. \tag{2.11}$$

In view of the fact that $\frac{n}{m} \to h$ as $n \to \infty$, we have

$$\frac{M_n}{(nQ_n)^c q_n} = C^c \left(\frac{m-2}{n}\right)^c \longrightarrow \frac{C^c}{h^c} \quad \text{as} \quad n \to \infty. \tag{2.12}$$

Clearly (2.1) and (2.12) imply that

$$\sum_{n \geqslant N_2}^{\infty} M_n = \infty. \tag{2.13}$$

Then (2.11) and (2.13) yield

$$\Delta^2 y_n \longrightarrow -\infty \quad \text{as} \quad n \to \infty,$$

which contradicts the fact that $\Delta^2 y_n > 0$ eventually.

Assume (D) holds. There exist a constant $k > 0$ and $n_2 \geqslant n_1$ such that

$$x_{n-g} \geqslant y_{n-g} \geqslant k \quad \text{for} \quad n \geqslant n_2. \tag{2.14}$$

By Lemma 4.1 of [5], there exists an $M^* \geqslant n_2$ such that

$$\Delta y_n \geqslant \tfrac{1}{2} n\, \Delta^2 y_n \quad \text{for} \quad n \geqslant M^*. \tag{2.15}$$

Replacing n with $j \geqslant M^*$ in (2.3), summing from $n \geqslant M^*$ to $s-1 (\geqslant n)$ and letting $s \to \infty$, we obtain

$$\Delta^2 y_n \geqslant k^c Q_n, \quad n \geqslant M^*. \tag{2.16}$$

Using (2.15) in (2.16) we have

$$\Delta y_n \geqslant \tfrac{1}{2} k^c\, n Q_n, \quad n \geqslant M^*. \tag{2.17}$$

Now, for $m-1 \geqslant M^*$ we have

$$x_m \geqslant y_m \geqslant y_m - y_{m-1} \geqslant \tfrac{1}{2}k^c\,(m-1)Q_m, \tag{2.18}$$

and hence

$$x_{n-g} \geqslant \tfrac{1}{2}k^c\,(n-g-1)\,Q_n \quad \text{for} \quad n \geqslant M^* + g + 1.$$

There exists $M_1^* \geqslant M^* + g + 1$ such that

$$x_{n-g} \geqslant \tfrac{1}{4}k^c\,nQ_n \quad \text{for} \quad n \geqslant M_1^*. \tag{2.19}$$

Using (2.19) in (2.3) and summing from M_1^* to $M - 1 \geqslant M_1^*$, we have

$$0 < \Delta^2 y_M \leqslant \Delta^2 y_{M_1^*} - (\tfrac{1}{4}k^c)^c \sum_{n=M_1^*}^{M-1} (nQ_n)^c\,q_n \longrightarrow -\infty \text{ as } M \to \infty,$$

a contradiction. This completes the proof. □

From the proof of Theorem 2.1, one can easily extract the following two oscillation criteria.

Corollary 2.1. *If condition* (2.1) *holds, then equation* (E_1) *is oscillatory.*

P r o o f. The proof is contained in the proof of Theorem 2.1 cases (A) and (C) and hence is omitted. □

Corollary 2.2. *If*

$$\sum_{k=n_1 \geqslant n_0+g+1}^{\infty} q_k \left(\sum_{n=n_0}^{k-g-1} nQ_n\right)^c = \infty, \tag{2.20}$$

then every unbounded solution of the difference equation

$$\Delta^3 y_n + q_n\,|y_{n-g}|^c \operatorname{sgn} y_{n-g} = 0,\ c > 0, \tag{E_3^*}$$

where q_n *and* g *are defined as in the equation* (E_3), *is oscillatory.*

P r o o f. The proof is similar to that of Theorem 2.1 (D) and hence is omitted. □

The following example is illustrative.

E x a m p l e 2.1. Consider the difference equations

$$(\mathrm{F}_i) \qquad \Delta^i(x_n - x_{n-h}) + (1/n^a)|x_{n-g}|^c \operatorname{sgn} x_{n-g} = 0, \ c > 0, \ i = 1,3 \text{ and } n \geqslant 1,$$

where h, g are nonnegative integers, $h > 0$ and $a > 1$. One can easily check that

$$Q_n = \sum_{j=n}^{\infty} (1/j^a) \geqslant 1/(a-1)n^{a-1},$$

and hence condition (2.1) is satisfied if $1 < a \leqslant \frac{2c+1}{c+1}$.

Thus we conclude that (F_i), $i = 1, 3$ are oscillatory for $h > 0, g \geqslant 0$ and all a and c such that $1 < a \leqslant \frac{2c+1}{c+1}$.

C a s e 2. We consider (E_3) when

$$\sum_{j=n_0}^{\infty} q_j = \infty. \tag{2.21}$$

Theorem 2.2. *If condition* (2.21) *holds, then* (E_3) *is oscillatory.*

P r o o f. Let x_n be an eventually positive solution of (E_3), say $x_n > 0$ for $n \geqslant n_0 \geqslant 0$. There exists $n_1 \geqslant n_0$ such that $x_{n-a} > 0$ for $n \geqslant n_1$ where $a = \max\{g, h\}$. Define y_n by (2.2) and as in the proof of Theorem 2.1, we see that $\Delta^i y_n$, $i = 0, 1, 2$ are eventually of one sign and the four cases (A)–(D) hold. The proofs of cases (A) and (B) are similar to those of Theorem 2.1 (A) and (B) and hence are omitted. Next, we consider the cases (C) and (D). In both cases we see that $\Delta^2 y_n > 0$ and $y_n > 0$ eventually. From (2.2), we have $x_n > x_{n-h}$ for $n \geqslant n_1$. Hence, there exist $b > 0$ and $n_2 \geqslant n_1$ such that

$$x_{n-g} \geqslant b \quad \text{for} \quad n \geqslant n_2. \tag{2.22}$$

Then,

$$\Delta^3 y_n \leqslant -b^c q_n \quad \text{for} \quad n \geqslant n_2. \tag{2.23}$$

Summing both sides of (2.23) from n_2 to $m - 1 (\geqslant n_2)$, we obtain

$$0 < \Delta^2 y_m \leqslant \Delta^2 y_{n_2} - b^c \sum_{n=n_2}^{m-1} q_n \longrightarrow -\infty \quad \text{as} \quad m \to \infty,$$

a contradiction. This completes the proof. □

The following two criteria are immediate.

Corollary 2.3. *If condition* (2.21) *holds, then* (E_1) *is oscillatory.*

Corollary 2.4. *If* $q_n = q$, q *is a positive real number, then* (E_i), $i = 1, 3$ *are oscillatory.*

Now, we pose the following question: "Is condition (2.21) (alone) a sufficient condition for the oscillation of (E_2)?" The following example gives a negative answer to this question.

Example 2.2. The second order neutral difference equation

$$\Delta^2(x_n - x_{n-3}) + (\mathrm{e}^3 - 1)(1 - \mathrm{e}^{-1})^2\,\mathrm{e}^{-g} x_{n-g} = 0, \tag{F$_2$}$$

has a nonoscillatory solution $\{\mathrm{e}^{-n}\}$.

Therefore, our objective here is to present the following criteria for the oscillation of (E_2).

Theorem 2.3. *If* $g \geqslant h$, *condition* (2.21) *holds and every bounded solution of the difference equation*

$$\Delta^2 z_n - q_n |z_{n-(g-h)}|^c \operatorname{sgn} z_{n-(g-h)} = 0, \tag{E$_2^*$}$$

is oscillatory, then (E_2) *is oscillatory.*

Proof. Let $\{x_n\}$ be an eventually positive solution of (E_2), say $x_n > 0$ and $x_{n-g} > 0$ for $n \geqslant n_1 \geqslant n_0 \geqslant 0$. Defining y_n by (2.2) we have, from (E_2),

$$\Delta^2 y_n = -q_n\, x_{n-g}^c \leqslant 0 \quad \text{for} \quad n \geqslant n_1, \tag{2.24}$$

which implies that $\{\Delta y_n\}$ is nonincreasing for $n \geqslant n_1$.
As in the proof of Theorem 2.1, we consider the four cases (A)–(D).
Proof of case (A) is similar to that of Theorem 2.1 (A) and hence is omitted.
(B) Suppose $y_n < 0$ and $\Delta y_n > 0$, $n \geqslant n_1$. Note that

$$0 < v_n = -y_n = x_{n-h} - x_n < x_{n-h},$$

and hence

$$x_n > v_{n+h} \quad \text{for} \quad n \geqslant n_1.$$

From (2.24), we have

$$\Delta^2 v_n \geqslant q_n \left(v_{n-(g-h)}\right)^c \quad \text{for} \quad n \geqslant n_1.$$

Now, in view of Theorem 2 of [7] and its proof, we see that (E_2^*) has eventually positive solution, a contradiction.

(C) Suppose $y_n > 0$ and $\Delta y_n < 0,\ n \geqslant n_1$. Since $\Delta^2 y_n \leqslant 0, n \geqslant n_1$, one can easily see that $y_n \to -\infty$ as $n \to \infty$, a contradiction.

(D) Suppose $y_n > 0$ and $\Delta y_n > 0,\ n \geqslant n_1$. From (2.2), we see that $x_n > x_{n-h}$ for $n \geqslant n_1$ and hence there exists $b > 0$ and $n_2 \geqslant n_1$ such that (2.22) holds. Using (2.22) in (2.24) and summing from n_2 to $(m-1)(\geqslant n_2)$, we have

$$0 < \Delta y_m \leqslant \Delta y_{n_2} - b^c \sum_{n=n_2}^{m-1} q_n \longrightarrow -\infty \quad \text{as} \quad n \to \infty,$$

a contradiction. This completes the proof. □

The following corolloary is immediate.

Corollary 2.5. *Let $g \geqslant h,\ c = 1$ and*

(2.25)
$$q_n \geqslant q > 0 \quad \textit{for} \quad n \geqslant n_0 \geqslant 0.$$

Then (E_2) *is oscillatory if one of the following conditions is satisfied:*

(2.26)
$$q \geqslant 1 \quad \textit{and} \quad g = h.$$

(2.27)
$$q > \frac{4k^k}{(2+k)^{(2+k)}}, \quad \textit{where} \quad k = g - h \geqslant 1.$$

P r o o f. Follows from the proof of Theorem 2.3 above and Corollary 2.2 (ii) and (iii) of [7]. □

The following result deals with the oscillatory and asymptotic behavior of all solutions of (E_2).

Corollary 2.6. *If condition* (2.21) *or* (2.25) *holds, then every solution* $\{x_n\}$ *of* (E_2) *is either oscillatory or* $x_n \to 0$ *monotonically as* $n \to \infty$.

P r o o f. Let $\{x_n\}$ be an eventually positive solution of (E_2) and let y_n be defined as in (2.2). Proceeding as in the proof of Theorem 2.3, we see that the cases (A), (C), and (D) are impossible. Next, we consider the case (B) and suppose that $x_n \to c_1 \geqslant 0$ as $n \to \infty$. We claim that $c_1 = 0$. To show this, assume that $c_1 > 0$. Then there exists an $n_2 \geqslant n_1$ such that

$$x_n \geqslant \tfrac{1}{2}c_1 \quad \text{for} \quad n \geqslant n_2. \tag{2.28}$$

Using (2.28) in (2.24) and summing from n_2 to $m-1 (\geqslant n_2)$, we obtain

$$0 < \Delta y_m \leqslant \Delta y_{n_2} - (\tfrac{1}{2}c_1)^c \sum_{n=n_2}^{m-1} q_n \to -\infty \quad \text{as} \quad n \to \infty,$$

a contradition. □

R e m a r k 2.1. The hypotheses of Corollary 2.6 are satisfied for (F_2), and hence, we see that $x_n = \mathrm{e}^{-n} \to 0$ monotonically as $n \to \infty$.

R e m a r k 2.2. The characteristic equation associated with the linear difference equation

$$\Delta^i\,(x_n - x_{n-h}) + q\,x_{n-g} = 0, \quad i = 1,2,3, \tag{L_i}$$

which is a special case of (E_i), $i = 1,2,3$ has the form

$$(m-1)^i(1 - m^{-h}) + q\,m^{-g} = 0, \quad i = 1,2,3, \tag{C_i}$$

where q is a positive real constant and g and h are positive integers. By Corollary 2.1, one may conclude that (C_i), $i = 1$ and 3 have no positive roots, while, by Corollary 2.5, one may observe that (C_2) has no positive roots if either condition (2.26) or (2.27) is satisfied.

3. Bounded Oscillation of (N_i), $i = 1,2,3$

The results of this section are concerned with the oscillatory behavior of every bounded solution of (N_i), $i = 1,2,3$.

Theorem 3.1. *If $g \geqslant h$ and every bounded solution of each of the equations*

$$\Delta^2 z_n + \left(\frac{n-g}{2}\right)^c q_n\,|z_{n-g}|^c \operatorname{sgn} z_{n-g} = 0, \tag{H_1}$$

and

$$\Delta^3 w_n + q_n\,|w_{n-(g-h)}|^c \operatorname{sgn} w_{n-(g-h)} = 0, \tag{H_2}$$

is oscillatory, then every bounded solution of (N$_3$) *is oscillatory.*

Proof. Let $\{x_n\}$ be a bounded and eventually positive solution of (N$_3$), say $x_n > 0$ and $x_{n-g} > 0$ for $n \geqslant n_1 \geqslant n_0 \geqslant 0$. Define y_n as in (2.2). Then (N$_3$) takes the form

$$\Delta^3 y_n = q_n x^c_{n-g} \geqslant 0, \quad \text{for} \quad n \geqslant n_1, \tag{3.1}$$

and hence $\Delta^i\, y_n,\ i = 0, 1, 2$ are eventually of one sign. Since x_n is bounded, $\Delta^2 y_n < 0$ eventually. Therefore, the following two cases are considered:

(I) $\Delta y_n > 0$ and $y_n < 0$ eventually.

(II) $\Delta y_n > 0$ and $y_n > 0$ eventually.

I. Assume $\Delta y_n > 0$ and $y_n < 0$ for $n \geqslant n_2 \geqslant n_1$. Note that

$$0 < v_n = -y_n = x_{n-h} - x_n < x_{n-h}. \tag{3.2}$$

Using (3.2) in (3.1), we have

$$\Delta^3 v_n + q_n\, v^c_{n-(g-h)} \leqslant 0, \quad n \geqslant n_2. \tag{3.3}$$

Now, in view of Theorem 1 of [7] and its proof, (H$_2$) has a bounded and eventually positive solution, a contradiction.

II. Assume $\Delta y_n > 0$ and $y_n > 0$ for $n \geqslant n_2 \geqslant n_1$. By Lemma 4.1 (d) of [5], there exists $n_3 \geqslant n_2$ such that

$$y_{n-g} \geqslant \frac{n-g}{2}\, \Delta y_{n-g} \quad \text{for} \quad n \geqslant n_3.$$

From (2.2), we see that

$$x_{n-g} \geqslant \frac{n-g}{2}\, \Delta y_{n-g} \quad \text{for} \quad n \geqslant n_3. \tag{3.4}$$

Using (3.4) in (3.1), we have

$$\Delta^2 u_n \geqslant \left(\frac{n-g}{2}\right) q_n\, u^c_{n-g} \quad \text{for} \quad n \geqslant n_3, \tag{3.5}$$

where $u_n = \Delta y_n > 0,\ n \geqslant n_3$. The rest of the proof is similar to that of Theorem 2.3 (B) and hence is omitted. $\square$

Theorem 3.2. *If $g \geqslant h$, condition* (2.21) *(or* (2.25)*) holds and every bounded solution of* (H_2) *is oscillatory, then every bounded solution of* (N_3) *is oscillatory.*

Proof. Let $\{x_n\}$ be a bounded and eventually positive solution of (N_3) and let y_n be defined as in (2.2). As in the proof of Theorem 3.1, we see that case (I) is impossible, and so, we consider case (II). From (2.2) and the fact that $y_n > 0$ for $n \geqslant n_1$, there exist $n_2 \geqslant n_1$ and $b > 0$ such that (2.22) holds for $n \geqslant n_2$. In view of condition (2.21) (or (2.25)), using (2.22) in (3.1), and summing from n_2 to $m-1 (\geqslant n_2)$ we have

$$0 > \Delta^2 y_m \geqslant \Delta^2 y_{n_2} + b^c \sum_{n=n_2}^{m-1} q_n \to \infty \quad \text{as} \quad m \to \infty,$$

a contradiction. □

From the proof of Theorem 3.1, we have the following oscillation result for (N_1).

Corollary 3.1. *If $g \geqslant h$ and the equation*

$$(\mathrm{H}_3) \qquad \Delta v_n + q_n |v_{n-(g-h)}|^c \operatorname{sgn} v_{n-(g-h)} = 0,$$

is oscillatory, then every bounded solution of (N_1) *is oscillatory.*

The following result deals with the oscillatory and asymptotic behavior of every bounded solution of each of the equations (N_i), $i = 1, 3$.

Corollary 3.2. *If condition* (2.21) *(or* (2.25)*) holds, then every bounded solution $\{x_n\}$ of each of the equations* (N_i), $i = 1, 3$, *is either oscillatory or $x_n \to 0$ monotonically as $n \to \infty$.*

Proof. Let $\{x_n\}$ be a bounded and eventually positive solution of (N_3) and let y_n be defined as in (2.2). As in the proof of Theorem 3.2, we see that case (II) is impossible. Now, we consider (I), and as in the proof of Theorem 3.1 (I), we obtain (3.1). Suppose $x_n \to c_1 \geqslant 0$ as $n \to \infty$. We claim that $c_1 = 0$. If $c_1 > 0$, there exists $n_2 \geqslant n_1$ such that (2.28) holds for $n \geqslant n_2$. Using (2.28) in (3.1) and summing from n_2 to $m-1 (\geqslant n_2)$ we have

$$0 > \Delta^2 y_m \geqslant \Delta^2 y_{n_2} + (\tfrac{1}{2} c_1)^c \sum_{n=n_2}^{m-1} q_n \to \infty \quad \text{as} \quad m \to \infty,$$

a contradiction. □

The following example is illustrative.

E x a m p l e 3.1. The difference equations

$$\Delta^i\left(x_n - x_{n-h}\right) = \left(1-\mathrm{e}^h\right)\left(\mathrm{e}^{-1}-1\right)^i \mathrm{e}^{-g}\, x_{n-g},\ i = 1,3, \tag{F$_3$}$$

where h and g are nonnegative integers, $h > 0$, has a nonoscillatory solution $x_n = \mathrm{e}^{-n} \to 0$ monotonically as $n \to \infty$. All conditions of Corollary 3.2 are satisfied.

R e m a r k 3.1. Proof of (N_1) is similar to that of (N_3) and hence is omitted.

The following result is concerned with the oscillation of all bounded solutions of (N_2).

Theorem 3.3. *Every bounded solution of* (N_2) *is oscillatory if one of the following conditions is satisfied:*
(i) *Condition* (2.1).
(ii) *Condition* (2.21) *or* (2.25).
(iii) *Every bounded solution of the difference equation*

$$\Delta^2 z_n - q_n\,|z_{n-g}|^c \operatorname{sgn} z_{n-g} = 0, \tag{H$_4$}$$

is oscillatory.

P r o o f. Let $\{x_n\}$ be a bounded and eventually positive solution of (N_2), say $x_n > 0$ and $x_{n-a} > 0$ for $n \geqslant n_1 \geqslant n_0 \geqslant 0$ and $a = \max\{g,h\}$. Let y_n be defined as in (2.2). Then (N_2) takes the form

$$\Delta^2 y_n = q_n\, x_{n-g}^c \quad \text{for} \quad n \geqslant n_1. \tag{3.6}$$

Since x_n is bounded, we must have $\Delta y_n < 0$ eventually and so y_n must be eventually positive. Assume (2.1) holds. There exist $n_2 \geqslant n_1$ and $b > 0$ such that (2.22) holds for $n \geqslant n_2$. Replacing n with $j \geqslant n_2$ in (3.6) and summing from $n(\geqslant n_2)$ to $m-1(\geqslant n)$, we have

$$-\Delta y_n \geqslant \Delta y_m - \Delta y_n \geqslant b^c \sum_{j=n}^{m-1} q_j \to b^c Q_n \quad \text{as} \quad m \to \infty \tag{3.7}$$

or

$$y_n \geqslant y_n - y_{n+1} \geqslant b^c Q_n \quad \text{for} \quad n \geqslant n_2.$$

The rest of the proof is similar to that of Theorem 2.1 (C) and hence is omitted.

Next, assume (ii) holds. Using (2.22) in (3.6) and summing from $n(\geqslant n_2)$ to $m-1(\geqslant n)$, we have

$$0 > \Delta y_n \geqslant \Delta y_{n_2} + b^c \sum_{n=n_2}^{m-1} q_n \to \infty \quad \text{as} \quad m \to \infty,$$

a contradiction. Finally assume (iii) holds. From (2.2) and the fact that $y_n > 0,\ n \geqslant n_1$, we have $x_n \geqslant y_n$ for $n \geqslant n_1$. Thus

$$\Delta^2 y_n \geqslant q_n\, y_{n-g}^c \quad \text{for} \quad n \geqslant n_2 \geqslant n_1.$$

The rest of the proof is similar to that of Theorem 2.3 (B) and hence is omitted. □

From Theorems 3.2 and 3.3 above and Corollary 1 of [7], we have the following result:

Corollary 3.3. *For the linear difference equations*

$$(\mathrm{L}_i^*) \qquad \Delta^i\,(x_n - x_{n-h}) = q\; x_{n-g}, \quad i = 1, 2, 3,$$

where q is a positive real number, $h > 0$ and $g \geqslant 0$ are integers, we have:

(i) *Every bounded solution of (L_1^*) is oscillatory if $q > 1$ for $g = h$ and*

$$q > \frac{k^k}{(1+k)^{(1+k)}} \quad \textit{for} \quad k = g - h \geqslant 1.$$

(ii) *Every bounded solution of (L_2^*) is oscillatory.*

(iii) *Every bounded solution of (L_3^*) is oscillatory if $q > 1$ for $g = h$ and*

$$q > \frac{27\, k^k}{(3+k)^{(3+k)}} \quad \textit{for} \quad k = g - h \geqslant 1.$$

The following examples are illustrative.

Example 3.2. Consider the difference equations

$$(\mathrm{F}_i^*) \qquad \Delta^i\ (x_n - x_{n-h}) - (1 - \mathrm{e}^{-h})(\mathrm{e} - 1)^i\ \mathrm{e}^g\ x_{n-g} = 0, \quad i = 1, 2, 3,$$

where $h > 0$ and $g \geqslant 0$ are integers. All conditions of Corollary 3.3 are satisfied if $g \geqslant h \geqslant 1$ and hence bounded solutions of each of the equations (F_i^*), $i = 1, 2, 3$ are oscillatory. We note that each of the equations (F_i^*), $i = 1, 2, 3$, has an unbounded nonoscillatory solution $x_n = \mathrm{e}^n$.

Example 3.3. Consider the neutral difference equation

$$(\mathrm{F}_4) \qquad \Delta^2\ (x_n - x_{n-h}) = n^{-a}\, |x_{n-g}|^c\ \operatorname{sgn} x_{n-g}, \quad a > 1,\ c > 0,$$

where $h > 0$ and $g \geqslant 0$ are integers. As in Example 2.1, we see that all bounded solutions of (F_4) are oscillatory by Theorem 3.3 (i).

R e m a r k 3.2.

1. The results of this paper are presented in a form which is essentially new. These results are applicable to superlinear, linear and sublinear equations of type (E_i) and (N_i), $i = 1, 2, 3$.
2. The results obtained here are concerned with the delay neutral difference equations (i.e., $g, h > 0$). The results for advanced equations of type (E_i) and (N_i), $i = 1, 2, 3$ (i.e., $g, h < 0$) can be obtained similarly. Here, we omit the details.
3. It would be interesting to obtain results similar to those presented here for equations (E_i) and (N_i), $i > 3$, as well as those for the oscillation of all solutions of equations (N_i), $i \geqslant 1$.

References

[1] *D. A. Gedrgion, E. A. Grove, G. Ladas*: Oscillation of neutral difference equations. Appl. Anal. *33* (1989), 243–253.

[2] *S. R. Grace, B. S. Lalli*: Oscillation theorems for forced neutral difference equations. J. Math. Anal. Appl. *187* (1994), 91–106.

[3] *S. R. Grace, B. S. Lalli*: Oscillation theorems for second order delay and neutral difference equations. Utilitas Math. *45* (1994), 197–211.

[4] *I. Györi, G. Ladas*: Oscillation Theory of Delay Differential Equations with Applications. Oxford University Presss, Oxford, 1991.

[5] *J. W. Hooker, W. T. Patula*: A second order nonlinear difference equation: Oscillation and asymptotic behavior. J. Math. Anal. Appl. *91* (1983), 9–29.

[6] *J. W. Hooker, W. T. Patula*: Growth and oscillation properties of solutions of a fourth order linear difference equation. J. Austral. Math. Soc. Ser. B *26* (1985), 310–328.

[7] *G. Ladas, C. Qian*: Comparison results and linearized oscillations for higher order difference equations. Internat. J. Math. Math. Sci. *15* (1992), 129–142.

[8] *B. S. Lalli, B. G. Zhang, Li Jan Zhao*: On oscillation and existence of positive solutions of neutral difference equations. J. Math. Anal. Appl. *158* (1991), 213–233.

[9] *B. S. Lalli, S. R. Grace*: Oscillation theorems for second order neutral difference equations. Appl. Math. Comput. *62* (1994), 47–60.

[10] *Ch. G. Philos*: Oscillation in a class of difference equations. Appl. Math. Comput. *48* (1992), 45–57.

[11] *Ch. G. Philos, Y. G. Sficas*: Positive solutions of difference equations. Proc. Amer. Math Soc. *108* (1990), 107–115.

[12] *J. Yu, Z. Wang, C. Qian*: Oscillation of neutral delay differential equations. Bull. Austral. Math. Soc. *45* (1992), 195–200.

Authors' addresses: *S. R. Grace*, Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12211, A. R. Egypt, e-mail: `srgrace@alpha1-eng.cairo.eun.eg`; *G. G. Hamedani*, Department of Mathematics, Statistics, and Computer Science, Marquette University, P. O. Box 1881, Milwaukee, WI 53201-1881, USA, e-mail: `hosseinh@mscs.mu.edu`.