REACTION-DIFFUSION SYSTEMS: DESTABILIZING EFFECT OF CONDITIONS GIVEN BY INCLUSIONS II, EXAMPLES

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Abstract. The destabilizing effect of four different types of multivalued conditions describing the influence of semipermeable membranes or of unilateral inner sources to the reaction-diffusion system is investigated. The validity of the assumptions sufficient for the destabilization which were stated in the first part is verified for these cases. Thus the existence of points at which the spatial patterns bifurcate from trivial solutions is proved.

Keywords: bifurcation, spatial patterns, reaction-diffusion system, mollification, inclusions

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9. Auxiliary assertions

This paper is a continuation of [1]. We study the bifurcation points $s_I \in \mathbb{R}$ at which nontrivial solutions to

$$\begin{aligned} &\sigma_1(s)u - b_{11}Au - b_{12}Av - N_1(u,v) = 0\\ &\sigma_2(s)v - b_{21}Au - b_{22}Av - N_2(u,v) \in -M_2(v) \end{aligned} \tag{9.1}$$

bifurcate from the trivial solution. Recall that solutions to (9.1) for M_2 defined by (2.9) are weak solutions to

$$\left.\begin{aligned} &\sigma_1(s)u + b_{11}u + b_{12}v + n_1(u,v) = 0\\ &\sigma_2(s)v + b_{21}u + b_{22}v + n_2(u,v) = 0 \end{aligned}\right. \quad \text{in } \Omega \tag{9.2}$$

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with the boundary conditions

$$(9.3)\quad u=v=0 \ \text{ on } \Gamma_D,\ \frac{\partial u}{\partial n}=0,\ \frac{\partial v}{\partial n}\in -\frac{m(v)}{\sigma_2(s)} \ \text{ on } \Gamma_U,\ \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0 \ \text{ on } \Gamma_N$$

(see Section 12). In the main result of [1]—Theorem 4.1—the existence of such bifurcation points is proved under certain assumptions on the multivalued mapping M. Here we shall concentrate on a verification of these assumptions for some important particular examples of the mapping M (related to a multivalued function m) to show the existence of bifurcation points of spatial patterns to (9.2).

Let us note that the references to Sections 1–8 correspond to [1].

In the sequel, we need the following assertions and the corresponding

Notation 9.1.
$H^1_L(\Omega)=\{\varphi\in W^1_2(\Omega);\ \Delta\varphi\in L^2(\Omega)\}$,
$\mathbb{H}=H^1_L(\Omega)\cap\mathbb{V}$,
$H^{\frac12}(\partial\Omega)$—the space of traces of functions from $W^1_2(\Omega)$,
$H^{-\frac12}(\partial\Omega)$—the dual space of $H^{\frac12}(\partial\Omega)$,
$\mathcal{D}(\Omega)$—the space of C^∞—smooth functions with compact support in Ω.

Observation 9.1. There is a uniquely defined continuous mapping $\mathfrak{T}$: $H^1_L(\Omega)\to H^{-\frac12}(\partial\Omega)$ such that $\mathfrak{T}u=\frac{\partial u}{\partial n}$ if $u\in C^1(\mathrm{cl}\,\Omega)$—see [5].

Let $m\colon\mathbb{R}\to 2^{\overline{\mathbb{R}}}$ be a multivalued function. Let $\underline{m}(\xi):=\inf\{m(\xi)\}$, $\overline{m}(\xi):=\sup\{m(\xi)\}$ for $\xi\in\mathbb{R}$. If $u\in\mathbb{V}$ then u on $\partial\Omega$ is understood in the sense of traces and $\frac{\partial u}{\partial n}$ is understood as a functional $\mathfrak{T}$ from $H^{-\frac12}(\partial\Omega)$. This means that $\frac{\partial u}{\partial n}(x)\in -m(u(x))$ stands for

$$-\int_{\Gamma_U}\overline{m}(u(x))\psi(x)\,\mathrm{d}\Gamma\leqslant\int_{\Gamma_U}\frac{\partial u}{\partial n}\psi\,\mathrm{d}\Gamma\leqslant-\int_{\Gamma_U}\underline{m}(u(x))\psi(x)\,\mathrm{d}\Gamma$$
$$\text{for any } \psi\in H^{\frac12}(\Gamma_U),\ \psi\geqslant 0 \text{ a.e. on } \Gamma_U.$$

Lemma 9.1 (Cf. [6], Theorem 3.2.). *Let $G\subset\mathbb{R}^k$, $\mathrm{meas}_k\,G<+\infty$. Let $u_n,u\in L^2(G)$ be a sequence of functions, $u_n\to u$ in $L^2(G)$. Let g_n,g be continuous functions on $\mathbb{R}$. Let h_n, h be Nemytskii operators corresponding to the functions g_n,g. Let $C>0$ be a constant such that $|g_n(\xi)|\leqslant C(1+|\xi|)$, $|g(\xi)|\leqslant C(1+|\xi|)$ for all $\xi\in\mathbb{R}$. Let $g_n(u_n)\to g(u)$ everywhere on G. Then $h_n(u_n)\to h(u)$ in $L^2(G)$.*

Proof. We have $h_n(u_n)(x)=g_n(u_n(x))$, $h(u)(x)=g(u(x))$ for all $x\in G$.

We can choose a subsequence (let us denote it $\{u_n\}$ again) that is Cauchy. We can choose again a subsequence $\{u_{n_j}\}$ such that

$$(9.4)\qquad \|u_{n_j}-u_{n_{j+1}}\|_{L^2(G)}<\frac{1}{2^j}.$$

The convergence of g_n ensures $h_{n_j}(u_{n_j}) \to h(u)$ a.e. on G. The growth of g_n and g together with (9.4) gives

$$|h_{n_j}(u_{n_j})(x)| \leqslant C(1 + |u_{n_j}(x)|) \leqslant C\Big(1 + \sum_{j=1}^{\infty} |u_{n_{j+1}}(x) - u_{n_j}(x)| + |u_{n_1}(x)|\Big) =: f(x) \in L^2(G).$$

It follows from the Lebesgue theorem that

$$\|h_{n_j}(u_{n_j}) - h(u)\|_{L^2(G)} \to 0.$$

Therefore every subsequence of the original sequence $\{u_n\}$ contains a subsequence for which $h_{n_j}(u_{n_j})$ converges to $h(u)$ in $L^2(G)$. Our assertion is proved. $\square$

10. Examples

Example 10.1. We shall investigate the Model Example from [1]. The multivalued mapping M is given here by a function $m\colon \mathbb{R} \to 2^{\mathbb{R}}$ which is singlevalued, real and continuous on $\mathbb{R} \setminus \{0\}$ and multivalued at $\xi = 0$—see (2.9).

The set $\partial\Omega$ is Lipschitz. Thus there exist a system of positive constants a_i, b_i, a system of sets $U_i \subset \mathbb{R}^{\mathfrak{n}}$ covering Γ_U, a system of balls $B_i \subset \mathbb{R}^{\mathfrak{n}-1}$ centered at 0 and a system of bi-Lipschitzian homeomorphisms $Q_i\colon U_i \to B_i \times (-a_i, b_i)$ such that $Q_i(U_i \cap \Omega) = B_i \times (0, b_i)$ and $Q_i(U_i \cap (\mathbb{R}^{\mathfrak{n}} \setminus \operatorname{cl}\Omega)) = B_i \times (-a_i, 0)$, $i = 1, \dots, M$. Let

$$\alpha_1, \dots, \alpha_M \tag{10.1}$$

be a C^1-smooth partition of unity on Γ_U subordinated to the covering U_i and $\alpha_{M+1} := 1 - \sum\limits_{i=1}^{M} \alpha_i$. Then $\operatorname{supp}\alpha_{M+1} \cap \operatorname{cl}\Gamma_U = \emptyset$. For $G_0 := \Omega \cup \bigcup\limits_{i=1}^{M} U_i$ we have $d_0 := \operatorname{dist}(\Gamma_U, \mathbb{R}^{\mathfrak{n}} \setminus G_0) > 0$. For the Lipschitz cut-off function $\eta\colon x \mapsto [1 - \frac{2}{d_0}\operatorname{dist}(x, \Omega)]^+$ and for the continuous extension $\widetilde{E}\colon \mathbb{V} \to W^{1,2}(\mathbb{R}^{\mathfrak{n}})$ ensured e.g. by Theorem 2.3.10 from [5] we take $G = \mathbb{R}^{\mathfrak{n}}$ and define $E := \eta\widetilde{E}$ (cf. [1], Notation 4.1). Let us recall the mollification operator $\Phi^\delta\colon \mathbb{V} \to W^{1,2}(G) \cap C^0(\operatorname{cl}\Omega)$ introduced in Notation 4.1.

In the sequel we will need

Proposition 10.1.

(i) *For any $v \in \mathbb{V}$, $\Phi^\delta(v)$ is a continuous function on* $\operatorname{cl}\Omega$.
(ii) *If $v_n, v \in \mathbb{V}$, $v_n \to v$ in $\mathbb{V}$ and $\delta > 0$ is fixed then $\Phi^\delta(v_n) \to \Phi^\delta(v)$ in $C^0(\operatorname{cl}\Omega)$.*
(iii) *Let $v \in \mathbb{V}$, $\delta_n \to 0_+$. Then $\Phi^{\delta_n}(v) \to Ev$ in $W_0^{1,2}(G)$.*

(iv) *Let* $v_n \rightharpoonup v$ *weakly in* $\mathbb{V}$ *and* $\delta_n \to 0_+$. *Then* $\Phi^{\delta_n}(v_n) \rightharpoonup Ev$ *weakly in* $W_0^{1,2}(G)$.

(v) *If* $v_n \rightharpoonup v$ *weakly in* $\mathbb{V}$ *and* $\delta_n \to 0_+$ *then* $\Phi^{\delta_n}(v_n) \to v$ *in* $L^2(\partial\Omega)$.

(vi) *For any* $\psi \in \mathbb{V}$, $\psi \geqslant 0$ *a.e. on* Γ_U *there are* $w_n = w_n(\psi) \in \mathbb{V}$ *and* $\delta_n > 0$ *such that* $w_n \to \psi$ *strongly in* $\mathbb{V}$ *and* $\Phi^\delta(w_n) \geqslant 0$ *on* Γ_U *for any* $\delta \in (0, \delta_n)$.

Proof. (i) and (ii) are obvious.

(iii): It follows from the definition of Φ^δ and [5], Theorem 2.1.2 that $\Phi^{\delta_n}(v) \to Ev$ and $\frac{\partial}{\partial x_j}\Phi^{\delta_n}(v) = \Phi^{\delta_n}(\frac{\partial}{\partial x_j}v) \to \frac{\partial}{\partial x_j}Ev$ in $L^2(G)$ for any $j = 1, \ldots, \mathfrak{n}$. (For the proof of the identity $\frac{\partial}{\partial x_j}\Phi^\delta(f) = \Phi^\delta(\frac{\partial}{\partial x_j}f)$ see [5], Theorem 2.2.1.) Therefore, $\Phi^{\delta_n}(v) \to Ev$ in $W_0^{1,2}(G)$.

(iv): If $v_n \rightharpoonup v$ weakly in $\mathbb{V}$ then $v_n \to v$ strongly in $L^2(\Omega)$ and $\frac{\partial}{\partial x_j}v_n \rightharpoonup \frac{\partial}{\partial x_j}v$ weakly in $L^2(\Omega)$ for any $j = 1, \ldots, \mathfrak{n}$ by the embedding theorems. Let $T_n, T_0\colon L^2(G) \to L^2(G)$, $T_n f := \Phi^{\delta_n}(f)$, $T_0 f := Ef$ for any $f \in L^2(\Omega)$. It follows from [5], Theorem 2.1.2 that $T_n f \to T_0 f$ in $L^2(G)$ for any $f \in L^2(\Omega)$. T_n, T_0 are linear continuous operators, therefore they are uniformly bounded by the Banach-Steinhaus theorem. We obtain

$$\begin{aligned}\|T_n v_n - T_0 v\|_{L^2(G)} &\leqslant \|T_n v_n - T_n v\|_{L^2(G)} + \|T_n v - T_0 v\|_{L^2(G)}\\ &\leqslant \|T_n\|_{L(L^2(G),L^2(G))} \cdot \|v_n - v\|_{L^2(\Omega)} + \|T_n v - T_0 v\|_{L^2(G)} \to 0.\end{aligned}$$

Now, let $f_n \rightharpoonup f$ weakly in $L^2(\Omega)$ and let $g \in W^{1,2}(\Omega)$ be arbitrary. We have by using the Fubini theorem that

$$\begin{aligned}(T_n f_n - T_0 f, Eg)_{L^2(G)} &= (T_n f_n - T_n f, Eg)_{L^2(G)} + (T_n f - T_0 f, Eg)_{L^2(G)}\\ &= (Ef_n - Ef, T_n g)_{L^2(G)} + (T_n f - T_0 f, Eg)_{L^2(G)} \to 0.\end{aligned}$$

By the choice $f_n := \frac{\partial}{\partial x_j}v_n$, $f := \frac{\partial}{\partial x_j}v$ in the second part and due to the fact that $\frac{\partial}{\partial x_j}T_n f = T_n\frac{\partial}{\partial x_j}f$ for any $j = 1, \ldots, \mathfrak{n}$ the proof is completed.

(v) follows from the embedding theorems and (iv).

(vi): We decompose $\psi = \psi^+ - \psi^-$, where ψ^+, ψ^- denotes the positive and negative parts, respectively. We have $\psi^+, \psi^- \in \mathbb{V}$ by [3]. The "bad" term is ψ^-. Therefore, we can assume without loss of generality that $\psi \in \mathbb{V}$ is such that $\psi \leqslant 0$ in Ω and $\psi = 0$ a.e. on Γ_U. Let us denote $g_i = \alpha_i E\psi$, α_i from (10.1), $i = 1, \ldots, M$. Since the boundary of Γ_U is Lipschitz with respect to $\partial\Omega$, we can assume that $Q_i(U_i \cap \Gamma_U)$ is starshaped[1] in B_i with respect to $0 \in \mathbb{R}^{\mathfrak{n}-1}$. Let us denote $g_i^r(x) := g_i(Q_i^{-1}(rQ(x)))$ for any $r \in (0,1)$ and $x \in U_i$. Let $\widetilde{\Gamma}_U$ be the extension of Γ_U such that $g_i^r(x) = 0$ for any $x \in \widetilde{\Gamma}_U$. Thus we have constructed $\widetilde{\Gamma}_U$ such that $\operatorname{dist}(\Gamma_U, \partial\Omega \setminus \widetilde{\Gamma}_U) = \delta^{(0)} > 0$.

[1] A set $X \subset \mathbb{R}^k$ is called *starshaped* with respect to a set Y, $Y \subset X$, if any ray with its origin in Y has a unique common point with ∂X—see [4].

We have $g_i^r \to g_i$ in $W_0^{1,2}(U_i)$ for $r \to 1_-$ (this fact can be proved in a similar way as (iii) or [5], Theorem 2.1.1). For $i = 1, \ldots, M$ let O_j, $j = 1, \ldots, k_i$ be a covering of $V_i := \{x \in \mathbb{R}^n;\ \operatorname{dist}(x, \widetilde{\Gamma}_U \cap \operatorname{supp}\alpha_i) \leqslant \delta^{(i)}\}$ in U_i such that $\bigcup\limits_{j=1}^{k_i} O_j \cap (\partial\Omega \setminus \widetilde{\Gamma}_U) = \emptyset$ and $\delta^{(i)} \in (\min\{\frac{\delta^{(0)}}{2}, \operatorname{dist}(\operatorname{supp}\alpha_i, \mathbb{R}^n \setminus U_i)\})$, $i = 1, \ldots, M$. Let $\beta_{1,i}, \ldots, \beta_{k_i,i}$ be a C^1-smooth partition of unity on V_i subordinated to the covering $\{O_j;\ j = 1, \ldots, k_i\}$, $\beta_{k_i+1,i} := 1 - \sum\limits_{j=1}^{k_i} \beta_{j,i}$. Then $\operatorname{supp}\beta_{k_i+1,i} \cap V_i = \emptyset$.

We have $\beta_{j,i} g_i^r \in W_0^{1,2}(O_j \cap \Omega)$. Therefore there are $\varphi_n^{r,i,j}$ which are C^1-smooth with $\operatorname{supp}\varphi_n^{r,i,j} \subset (O_j \cap \Omega)$ and such that $\varphi_n^{r,i,j} \to \beta_{j,i} g_i^r$ in $W_0^{1,2}(O_j \cap \Omega)$ for $n \to +\infty$, $j = 1, \ldots, k_i$ and $i = 1, \ldots, M$—see [5]. We can choose $r_n \to 1_-$ and

$$w_n := \sum_{i=1}^{M} \Big(\sum_{j=1}^{k_i} \varphi_n^{r_n,i,j} + \beta_{k_i+1,i}\, g_i^r \Big) + \alpha_{M+1}\psi \to \psi \text{ in } \mathbb{V}$$

for $n \to +\infty$ and there are $\delta_n > 0$ such that $w_n = 0$ in a δ_n-neighbourhood of Γ_U. □

With help of Proposition 10.1 we verify all assumptions of Theorem 4.1 (the verification is contained in Section 12) and as a consequence of [1], Theorem 4.1 and Remark 4.2 for Example 10.1 we obtain

Theorem 10.1. *Let* (SIGN) *and* (1.1) *hold, let $\sigma(s)$ be a differentiable curve satisfying* (4.15), *let $d^0 \in C_p$ and let* (4.16) *hold. Let m be the multivalued function from Model Example and let us assume that there exists an eigenfunction e_p corresponding to the eigenvalue κ_p of the Laplacian with* (1.3) *such that* (4.13) *is fulfilled with $e = e_p$. Then stationary spatially nonconstant weak solutions (spatial patterns) of* (SRD), (1.2) *with diffusion parameters $d_1 = \sigma_1(s)$ and $d_2 = \sigma_2(s)$ (i.e. of* (9.2), (9.3)*) bifurcate at some $s_I \in (s_0, \tilde{s}]$.*

This is actually [1], Corollary 4.1 and the proof follows from [1], Theorem 4.1 and Remark 4.2 and the fact that no nontrivial constant function can satisfy (1.3).

E x a m p l e 10.2. Let us consider the same multivalued function m as in Model Example and define the corresponding mapping $M\colon \mathbb{V}^2 \to 2^{\mathbb{V}^2}$ by $M(U) = [\{0\}, M_2(v)]$,

$$M_2(v) = \Big\{ z \in \mathbb{V};\ \int_{\Omega_1} \underline{m}(v)\varphi \,\mathrm{d}x \leqslant \langle z, \varphi\rangle \leqslant \int_{\Omega_1} \overline{m}(v)\varphi \,\mathrm{d}x \text{ for all } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ a.e. in } \Omega_1 \Big\}$$

for any $v \in \mathbb{V}$, where $\Omega_1 \subset \Omega$ is a given domain such that a δ_0-neighbourhood of Ω_1 (with some $\delta_0 > 0$) belongs to Ω. Then a solution of (9.1) is a weak solution of the problem

$$\begin{aligned}
&d_1\Delta u + b_{11}u + b_{12}v + n_1(u,v) = 0 && \text{in } \Omega,\\
&d_2\Delta v + b_{21}u + b_{22}v + n_2(u,v) = 0 && \text{in } \Omega\setminus\Omega_1,\\
&d_2\Delta v + b_{21}u + b_{22}v + n_2(u,v) \in m(v) && \text{in } \Omega_1
\end{aligned} \tag{10.2}$$

with the boundary conditions (1.3). Such a model describes a similar situation as in Example 10.1 with a source in the interior of the domain Ω.

Let us define the corresponding homogeneous mapping M_0 by

$$M_0(U) = [\{0\}, M_{02}(v)]$$

with

$$\begin{aligned}
&M_{02}(v) = \{z \in \mathbb{V};\ \langle z, v\rangle = 0,\ \langle z,\varphi\rangle \leqslant 0 \text{ for all } \varphi\in\mathbb{V}, \varphi \geqslant 0 \text{ in } \Omega_1\} \text{ if } v \geqslant 0 \text{ in } \Omega_1,\\
&M_{02}(v) = \emptyset \text{ if } v < 0 \text{ in a subset of } \Omega_1 \text{ of a positive measure.}
\end{aligned}$$

Then a solution of (2.11) is a weak solution of

$$\begin{aligned}
&d_1\Delta u + b_{11}u + b_{12}v = 0 \quad \text{in } \Omega,\\
&d_2\Delta v + b_{21}u + b_{22}v = 0 \quad \text{in } \Omega\setminus\Omega_1,\\
&d_2\Delta v + b_{21}u + b_{22}v \leqslant 0,\ v \geqslant 0,\ (d_2\Delta v + b_{21}u + b_{22}v)\, v = 0 \quad \text{in } \Omega_1
\end{aligned} \tag{10.3}$$

with boundary conditions (1.3). Note that in this example, the problem (2.11) is equivalent to (2.13) with $K = \mathbb{V} \times \{\varphi \in \mathbb{V};\ \varphi \geqslant 0 \text{ in } \Omega_1\}$.

Remark 10.1. Let v be a solution to (2.11) with (10.3). Let us denote $\Omega_v^+ := \{x \in \Omega_1;\ v(x) > 0\}$ and $\Omega_v^0 := \{x\in\Omega_1;\ v(x) = 0\}$. If $\partial\Omega_v^0$ is a Lipschitzian manifold then we have

$$\frac{\partial v}{\partial n} = v = 0 \quad \text{in } \partial\Omega_v^+ \cap \Omega_1$$

(for the proof of this fact see Section 12).

Proposition 10.2. *If $v_n \rightharpoonup v$ weakly in $\mathbb{V}$ and $\delta_n \to 0_+$ then $v_n^{\delta_n} \to v$ in $L^2(\Omega_1)$. Moreover, for any $\psi \in \mathbb{V}$, $\psi \geqslant 0$ a.e. in Ω_1 there are $w_n = w_n(\psi) \in \mathbb{V}$ and $\delta_n > 0$ small such that $w_n \to \psi$ strongly in $\mathbb{V}$ and $\Phi^\delta(w_n) \geqslant 0$ in Ω_1 for any $\delta \in (0, \delta_n)$.*

Proof. The first assertion follows from the embedding theorems and Proposition 10.1, (iv).

Similarly as in the proof of Proposition 10.1 we can assume without loss of generality that $\psi \in \mathbb{V}$ is such that $\psi \leqslant 0$ in Ω and $\psi = 0$ a.e. in Ω_1. Therefore, $E\psi \in W_0^{1,2}(\widetilde{G} \setminus \operatorname{cl}\Omega_1)$ with $\widetilde{G} := \operatorname{supp} E\psi \cup \Omega$ and there exist C^1-smooth functions φ_n with $\operatorname{supp}\varphi_n \subset \widetilde{G} \setminus \operatorname{cl}\Omega_1$ (this implies $\varphi_n \in \mathbb{V}$) such that $\varphi_n \to E\psi$ in $W_0^{1,2}(\widetilde{G})$—see [5]. We take $w_n := \varphi_n$ and there are $\delta_n > 0$ such that $w_n = 0$ in a δ_n-neighbourhood of Ω_1. $\square$

Now we define M^δ and M_0^δ in the same way as in Notation 4.1 with Γ_U replaced by Ω_1. Propositions 10.1 and 10.2 ensure the assumptions of Theorem 4.1 hold also for such M, M_0, M^δ, M_0^δ corresponding to the problem (10.2).

Theorem 10.2. *Let* (SIGN) *hold, let $\sigma(s)$ be a differentiable curve satisfying* (4.15)*, let $d^0 \in C_p$ and* (4.16) *hold. Let m be the multivalued function from Model Example and let us assume that there exists an eigenfunction e_p corresponding to an eigenvalue κ_p of the Laplacian with* (1.3) *such that $e_p \leqslant -\varepsilon$ in a δ_0-neighbourhood of Ω_1 with some $\varepsilon > 0$. Then stationary spatially nonconstant weak solutions of the problem* (10.2), (1.3) *with diffusion parameters $d_1 = \sigma_1(s)$ and $d_2 = \sigma_2(s)$ bifurcate at some $s_I \in (s_0, \tilde{s}]$.*

Again, this follows from Theorem 4.1 and Remark 4.2 from [1] and the fact that no nontrivial constant function can satisfy (1.3).

E x a m p l e 10.3. Let a be a positive constant and for any $\varphi \in \mathbb{V}$ let us denote

$$\overline{\varphi} := a \int_{\Gamma_U} \varphi(x)\,\mathrm{d}\Gamma.$$

Here e.g. $a := (\operatorname{meas}_{n-1}\Gamma_U)^{-1}$ can be taken. Let $m\colon \mathbb{R} \to 2^{\mathbb{R}}$ be the function from Model Example. Let us define the corresponding mapping $M\colon \mathbb{V}^2 \to 2^{\mathbb{V}^2}$ by $M(U) = [\{0\}, M_2(v)]$,

$$M_2(v) = \{z \in \mathbb{V};\ \underline{m}(\overline{v})\ \overline{\varphi}\ \leqslant \langle z, \varphi\rangle \leqslant \overline{m}(\overline{v})\ \overline{\varphi}\ \text{ for all } \varphi \in \mathbb{V}, \overline{\varphi} \geqslant 0\} \tag{10.4}$$

for any $v \in \mathbb{V}$. Then a solution of (9.1) is a weak solution of (9.2) with the boundary conditions

$$\begin{aligned} &u = v = 0 \quad \text{on } \Gamma_D,\\ &\frac{\partial u}{\partial n} = 0,\ \frac{\partial v}{\partial n} = \text{const} \in -\frac{m(\overline{v})}{\sigma_2(s)} \text{ on } \Gamma_U,\\ &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \Gamma_N. \end{aligned} \tag{10.5}$$

The corresponding homogeneous mapping is $M_0(U) = [\{0\}, M_{02}(v)]$ with

$$M_{02}(v) = \{z \in \mathbb{V};\ \langle z, v\rangle = 0,\ \langle z, \varphi\rangle \leqslant 0 \text{ for } \varphi \in \mathbb{V}, \overline{\varphi} \geqslant 0\} \text{ if } \overline{v} \geqslant 0,$$
$$M_{02}(v) = \emptyset \text{ if } \overline{v} < 0.$$

The associated convex cone is $K = \mathbb{V} \times K_2$, $K_2 = \{\varphi \in \mathbb{V};\ \overline{\varphi} \geqslant 0\}$ with $\emptyset \neq \operatorname{int} K_2 = \{\varphi \in \mathbb{V};\ \overline{\varphi} > 0\}$. Again, (2.11) is equivalent to (2.13).

Theorem 10.3. *Let* (SIGN) *hold, let* $\sigma(s)$ *be a differentiable curve satisfying* (4.15), *let* $d^0 \in C_p$ *and* (4.16) *hold. Let* m *be the multivalued function from Model Example and let us assume that there exists an eigenfunction* e_p *corresponding to an eigenvalue* κ_p *of the Laplacian with* (1.3) *satisfying* $\int_{\Gamma_U} e_p \, \mathrm{d}\Gamma < 0$. *Then stationary spatially nonconstant weak solutions of* (9.2), (10.5) *bifurcate at some* $s_I \in (s_0, \tilde{s}]$.

This follows from Theorem 4.1 and Remark 4.2 from [1] and the fact that no nontrivial constant function can satisfy (1.3).

R e m a r k 10.2. We have $\operatorname{int} K \neq \emptyset$, therefore no regularization is necessary; we can define $M^\delta := M$, $M_0^\delta := M_0$ and we have $K^\delta = K$, $P_\tau^\delta = P_\tau$.

11. Another example, where sensor and source are at different points

In the situation of Examples 10.1–10.3, the homogeneous problem (2.11) is equivalent to the variational inequality (2.13). In the next example we will consider a boundary condition such that the corresponding weak homogeneous problem (2.11) is not equivalent to (2.13).

E x a m p l e 11.1. (Cf. [2], Section 5.) Let $\Omega = (0,1)$, $\mathbb{V} = \{\varphi \in W_2^1(0,1);\ \varphi(0) = 0\}$. Let $x_0 \in (0,1)$ be fixed. Let us consider the multivalued function $m\colon \mathbb{R} \to 2^{\mathbb{R}}$ and the corresponding singlevalued functions $\underline{m}$ and $\overline{m}$ as in Model Example. Define the related mapping $M\colon \mathbb{V}^2 \to 2^{\mathbb{V}^2}$, $M(U) = [\{0\}, M_2(v)]$ for $U = [u, v]$ by

$$M_2(v) = \{z \in \mathbb{V};\ \underline{m}(v(x_0))\varphi(1) \leqslant \langle z, \varphi\rangle \leqslant \overline{m}(v(x_0))\varphi(1), \varphi \in \mathbb{V}, \varphi(1) \geqslant 0\} \tag{11.1}$$

for any $v \in \mathbb{V}$. Then a solution of (9.1) is a weak solution of the problem (9.2) with the boundary conditions

$$u(0) = v(0) = u_x(1) = 0,\ v_x(1) \in -\frac{m(v(x_0))}{\sigma_2(s)}. \tag{11.2}$$

The multivalued condition in (11.2) describes e.g. a semipermeable membrane on the boundary like in Model Example but with a sensor in the interior of the domain, i.e. the sensor is located at a different point than the source. In the situation of Example 10.1, we had $x_0 = 1$ (in the case $\mathfrak{n} = 1$ and $\Omega = (0,1)$), i.e. the sensor was at the same point as the source (membrane). From this point of view, the multivalued condition in Example 11.1 is more general.

Let us define convex cones $K_{x_0} = \{\varphi \in \mathbb{V};\ \varphi(x_0) \geqslant 0\}$ and $K_1 = \{\varphi \in \mathbb{V};\ \varphi(1) \geqslant 0\}$. The corresponding homogeneous mapping M_0 is $M_0(U) = [\{0\}, M_{02}(v)]$, $U = [u,v]$ with

$$\begin{aligned}
&M_{02}(v) = \{0\} \quad \text{if } v(x_0) > 0,\\
&M_{02}(v) = \{z \in \mathbb{V};\ \langle z, \varphi\rangle \leqslant 0 \text{ for all } \varphi \in K_1\} \ \text{ if } v(x_0) = 0,\\
&M_{02}(v) = \emptyset \quad \text{if } v(x_0) < 0.
\end{aligned}$$

Then the set K from (2.14) is $\mathbb{V} \times K_{x_0}$. A solution of (2.11) is a weak solution of (2.12), i.e. of

$$d_1 u_{xx} + b_{11}u + b_{12}v = 0, \quad d_2 v_{xx} + b_{21}u + b_{22}v = 0$$

with the boundary conditions

$$\begin{aligned}
&u(0) = v(0) = u_x(1) = 0,\\
&v_x(1) \geqslant 0,\ v(x_0) \geqslant 0,\ v_x(1)\cdot v(x_0) = 0.
\end{aligned} \tag{11.3}$$

A suitable penalty operator for M is $P_\tau(U) = [0, P_{\tau,2}(v)]$ with

$$\langle P_{\tau,2}(v), \varphi\rangle := p_\tau(v(x_0))\varphi(1)$$

for all v, $\varphi \in \mathbb{V}$, where p_τ are the same functions as in Model Example. Set $\mathcal{K} = \mathbb{V} \times (K_1 \cap K_{x_0})$ and consider the condition

$$-U_0 \in E_B(d^0) \cap \operatorname{int}\mathcal{K} \tag{11.4}$$

instead of (4.14). Here, $\operatorname{int}\mathcal{K} = \mathbb{V} \times \{\varphi \in \mathcal{K};\ \varphi(x_0) > 0, \varphi(1) > 0\} \neq \emptyset$. Therefore, we can define $M^\delta := M$, $M_0^\delta := M_0$, $P_\tau^\delta := P_\tau$ for any $\delta > 0$ small. It is easy to see by using Observation 3.3 from [1] that the condition (11.4) is fulfilled for $d^0 \in C_p$ and $U_0 = [\alpha(d^0)e_p, e_p]$, $\alpha(d^0) > 0$, provided the eigenfunction e_p corresponding to the eigenvalue κ_p of Laplacian with the boundary conditions

$$u(0) = u_x(1) = 0 \tag{11.5}$$

satisfies $e_p(x_0) < 0$ and $e_p(1) < 0$. Note that the eigenvalues κ_j are simple in the one-dimensional case. Therefore the operator L_δ from (5.9) plays a role only in the case when $d^0 = \sigma(s_0)$ is an intersection point of two different hyperbolas. Replacing K by $\mathcal{K}$ at the appropriate places and (4.14) by (11.4), we can go through the whole procedure used in [1], Sections 6 and 7 and prove the assertion of Theorem 11.1 below also in this situation.

The proofs of all assertions from Sections 6 and 7 in [1] can be done analogously with the exception of the proof of the boundedness of the branch of triplets of solutions to penalty equation in s (see [1], (7.6) and Lemmas 6.5–6.7) where the conditions (4.5) and (4.10) are used. However, now (4.5) and (4.10) are not satisfied for all $U \in \mathbb{V}^2$. We have to strengthen the condition (4.15) by

$$\lim_{s\to+\infty} \sigma_1(s) = +\infty, \qquad \lim_{s\to+\infty} \sigma_2(s) = +\infty \tag{11.6}$$

and prove the fact $s < s_0 + \zeta_1 < +\infty$ with some $\zeta_1 > 0$ in the following way.

Let us assume by contradiction that there exist s_n and $U_n = [u_n, v_n]$ such that $s_n \to +\infty$, $\|U_n\| \to 0$, $W_n = [w_n, z_n] = \frac{U_n}{\|U_n\|} \rightharpoonup W = [w, z]$ and

$$D(\sigma(s_n))U_n - BAU_n - \frac{\tau_n}{1+\tau_n}N(U_n) + \frac{D(\sigma(s_n))}{1+\tau_n}L_\delta(s_n)U_n + P_{\tau_n}(U_n) = 0 \tag{11.7}$$

holds. The embedding theorem gives $w_n \to w$, $z_n \to z$ in $C^0([0,1])$. Writing (11.7) in the components and multiplying the first equation of (11.7) by $w_n\|U_n\|^{-1}$ we obtain

$$\begin{aligned}\sigma_1(s_n)\|w_n\|^2 - b_{11}\langle Aw_n, w_n\rangle - b_{12}\langle Az_n, w_n\rangle - \frac{\tau_n}{1+\tau_n}\Big\langle \frac{N_1(U_n)}{\|U_n\|}, w_n\Big\rangle \\ + \Big\langle \frac{\sigma_1(s_n)}{1+\tau_n} L_\delta(s_n)w_n, w_n\Big\rangle = 0.\end{aligned} \tag{11.8}$$

We have $L_\delta(s_n) \equiv 0$ for $s_n > s_0 + \eta$ directly from the definition of L_δ. If we had $w \neq 0$ then the left hand side of (11.8) would tend to infinity by the assumption (11.6).

Multiplying the second equation of (11.7) by $\varphi\|U_n\|^{-1}$ with $\varphi \in K_1$, $\varphi(1) = 0$ we obtain

$$\begin{aligned}\sigma_2(s_n)\langle z_n, \varphi\rangle - b_{21}\langle Aw_n, \varphi\rangle - b_{22}\langle Az_n, \varphi\rangle - \frac{\tau_n}{1+\tau_n}\Big\langle \frac{N_2(U_n)}{\|U_n\|}, \varphi\Big\rangle \\ + \Big\langle \frac{\sigma_2(s_n)}{1+\tau_n} L_\delta(s_n)z_n, \varphi\Big\rangle = 0\end{aligned} \tag{11.9}$$

(recall that $K_1 = \{\varphi \in \mathbb{V};\ \varphi(1) \geqslant 0\}$). We have $L_\delta(s_n) \equiv 0$ for $s_n > s_0 + \eta$ again. Dividing (11.9) by $\sigma_2(s_n)$ and letting $n \to +\infty$ we obtain $\langle z_n, \varphi\rangle \to 0$ by using

(11.6). We have $z_n \rightharpoonup z$ and it follows that $\langle z, \varphi\rangle = 0$ for arbitrary $\varphi \in \mathbb{V}$, $\varphi(1) = 0$. Thus $z_{xx} = 0$, which implies $z(x) = k \cdot x$ with some $k \in \mathbb{R}$. Note that the embedding theorem ensures $z \in C([0,1])$ and $z \in \mathbb{V}$ gives $z(0) = 0$.

Multiplying the second equation of (11.7) by $\sigma_2^{-1}(s_n)v_n\|U_n\|^{-2}$ and passing to the limit we obtain

$$(11.10)\quad 0 \leqslant \int_0^1 z_x^2 \,\mathrm{d}x = k^2 = -\lim_{n\to+\infty} \frac{p_{\tau_n}(v_n(x_0))z_n(1)}{\sigma_2(s_n)\|U_n\|} = -k \lim_{n\to+\infty} \frac{p_{\tau_n}(v_n(x_0))}{\sigma_2(s_n)\|U_n\|}.$$

This together with the sign of p_{τ_n} implies $k \geqslant 0$. If $k = 0$ then $z = 0$ in $[0,1]$. If $k > 0$ then $z(x_0) = kx_0 > 0$ and the C^0-convergence of z_n implies $v_n(x_0) > 0$ for n large enough. But $p_{\tau_n}(v_n(x_0)) = 0$ and we obtain from (11.10) that

$$k = -\lim_{n\to+\infty} \frac{p_{\tau_n}(v_n(x_0))}{\sigma_2(s_n)\|U_n\|} = 0.$$

This contradicts $k > 0$. Therefore $z = 0$, which is a contradiction with the fact that $\|W\| = \|[w,z]\| = 1$.

We obtain

Theorem 11.1. *Let* (SIGN) *hold, let $\sigma(s)$ be a differentiable curve satisfying* (4.15) *and* (11.6)*, let $d^0 \in C_p$ and* (4.16) *hold. Let m be the multivalued function from Model Example and let us assume that there exists an eigenfunction e_p corresponding to an eigenvalue κ_p of the Laplacian with* (11.5) *satisfying $e_p(x_0) < 0$ and $e_p(1) < 0$. Then stationary spatially nonconstant weak solutions (spatial patterns) of* (9.2), (11.2) *bifurcate at some $s_I \in (s_0, +\infty)$.*

This follows from [1], Theorem 4.1 and Remark 4.2 and from the considerations above. Let us note that in this situation we have no information like $s_I < \tilde{s}$.

12. Validity of Propositions 4.1–4.5 for the examples investigated above

Let us consider the mappings as in Model Example. Recall that $K_2 = \{\varphi \in \mathbb{V}; \varphi \geqslant 0$ a.e. on $\Gamma_U\}$. First, we need to show that a solution of (9.1) is a weak solution of the problem (9.2), (9.3). For the sake of simplicity we will write d_1, d_2 instead of

$\sigma_1(s), \sigma_2(s)$. The inclusion (9.1) is equivalent to the following couple of formulae:

$$\int_\Omega d_1 \sum_{j=1}^{n} u_{x_j}\varphi_{x_j} - (b_{11}u + b_{12}v + n_1(u,v))\varphi \,\mathrm{d}x = 0 \qquad \text{for any } \varphi \in \mathbb{V}, \tag{12.1}$$

$$\begin{aligned} -\int_{\Gamma_U} \overline{m}(v)\psi \,\mathrm{d}\Gamma \leqslant \int_\Omega d_2 \sum_{j=1}^{n} v_{x_j}\psi_{x_j} - (b_{21}u + b_{22}v + n_2(u,v))\psi \,\mathrm{d}x \\ \leqslant -\int_{\Gamma_U} \underline{m}(v)\psi \,\mathrm{d}\Gamma \qquad \text{for any } \psi \in K_2. \end{aligned} \tag{12.2}$$

Moreover,

$$\int_\Omega d_1 \sum_{j=1}^{n} u_{x_j}\varphi_{x_j} - \big(b_{11}u + b_{12}v + n_1(u,v)\big)\varphi \,\mathrm{d}x = 0 \qquad \text{for any } \varphi \in \mathcal{D}(\Omega),$$

$$\int_\Omega d_2 \sum_{j=1}^{n} v_{x_j}\psi_{x_j} - \big(b_{21}u + b_{22}v + n_2(u,v)\big)\psi \,\mathrm{d}x = 0 \qquad \text{for any } \psi \in \mathcal{D}(\Omega),$$

i.e. (9.2) is satisfied in the sense of distributions. Therefore the equations

$$d_1\Delta u + b_{11}u + b_{12}v + n_1(u,v) = 0, \tag{12.3}$$

$$d_2\Delta v + b_{21}u + b_{22}v + n_2(u,v) = 0 \tag{12.4}$$

are satisfied a.e. in Ω, where all terms are represented by functions from $L^2(\Omega)$. For any u, $v \in \mathbb{H}$ we can use Green's formula for (12.1), (12.2) to obtain

$$\int_\Omega \big(d_1\Delta u + b_{11}u + b_{12}v + n_1(u,v)\big)\varphi \,\mathrm{d}x - d_1 \int_{\partial\Omega} \mathfrak{T}u \;\varphi \,\mathrm{d}\Gamma = 0 \quad \forall\varphi \in \mathbb{V}, \tag{12.5}$$

$$\begin{aligned} \int_{\Gamma_U} \underline{m}(v)\psi \,\mathrm{d}\Gamma \leqslant \int_\Omega \big(d_2\Delta v + b_{21}u + b_{22}v + n_2(u,v)\big)\psi \,\mathrm{d}x \\ - d_2 \int_{\partial\Omega} \mathfrak{T}v \;\psi \,\mathrm{d}\Gamma \leqslant \int_{\Gamma_U} \overline{m}(v)\psi \,\mathrm{d}\Gamma, \quad \forall\psi \in K_2. \end{aligned} \tag{12.6}$$

With help of (12.3) and (12.4) in (12.5) and (12.6) we obtain $\mathfrak{T}u = 0$ on $\Gamma_U \cup \Gamma_N$ and

$$\int_{\Gamma_U} \underline{m}(v)\psi \,\mathrm{d}\Gamma \leqslant -d_2 \int_{\partial\Omega} \mathfrak{T}v \;\psi \,\mathrm{d}\Gamma \leqslant \int_{\Gamma_U} \overline{m}(v)\psi \,\mathrm{d}\Gamma \qquad \text{for all } \psi \in K_2. \tag{12.7}$$

It follows from the definition of $\mathbb{V}$ that $u = v = 0$ on Γ_D in the sense of traces. The choice $\psi = 0$ on Γ_U in (12.7) gives $\mathfrak{T}v = 0$ on Γ_N while $\psi \geqslant 0$ on Γ_U gives $-d_2\mathfrak{T}v \in m(v)$ on Γ_U.

In a similar way we can show that (9.1) is a weak formulation of (10.2) with (1.3) or of (9.2) with (10.5) in the situation of Example 10.2 or 10.3, respectively.

Now we shall prove Propositions 4.1–4.5, i.e. we shall verify the conditions (4.1)–(4.6) and (4.11)–(4.12) for M^δ, M_0^δ and P_τ^δ under the situation from Examples 10.1–10.3 and/or 11.1. For the sake of simplicity, we will write v^δ instead of $\Phi^\delta(v)$.

In the situation of Example 10.1 (i.e. Model Example in the sense of [1]) we define for $\tau \geqslant 0$ the functions p_τ by

$$p_\tau(\xi) := \begin{cases} 0 & \xi \geqslant 0, \\ \tau\xi & \xi \in [\xi_\tau, 0) \qquad \xi_\tau := \frac{m^0}{\sqrt{1+\tau^2}}, \\ k(\tau)m(\xi) & \xi < \xi_\tau \qquad k(\tau) := \frac{m^0}{m(\xi_\tau)}\frac{\tau}{\sqrt{1+\tau^2}}. \end{cases}$$

The definitions of $M^\delta, M_0^\delta, P_\tau^\delta$ include the triviality of their first components. Hence only the second components are essential. Definition of M_0^δ yields that $K^\delta = \{U = [u, v] \in \mathbb{V}^2;\ v^\delta \geqslant 0 \text{ on } \Gamma_U\}$ and (4.1)–(4.6) are satisfied. It is easy to see that $\operatorname{int} K^\delta = \{U = [u, v] \in \mathbb{V}^2;\ v^\delta > 0 \text{ on } \operatorname{cl}\Gamma_U\}$.

Proof of Proposition 4.1. Let $U_n \to 0$, $W_n = [\xi_n, w_n] = \frac{U_n}{\|U_n\|} \rightharpoonup W = [\xi, w]$, $Z_n = [\eta_n, z_n] \to Z = [\eta, z]$, $d_n \to d \in \mathbb{R}_+^2$, $D(d_n)W_n + Z_n \in -\frac{M^\delta(U_n)}{\|U_n\|}$. This yields for the first coordinate that $d_1^n\xi_n = -\eta_n$ and immediately $\xi_n \to \xi$ because $\eta_n \to \eta$. For the second coordinate the inclusion gives

$$\begin{aligned} -\int_{\Gamma_U}\frac{\underline{m}(v_n^\delta)}{\|U_n\|}\big[\varphi^\delta\big]^+\mathrm{d}\Gamma + \int_{\Gamma_U}\frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[\varphi^\delta\big]^-\mathrm{d}\Gamma &\geqslant \langle d_2^n w_n + z_n, \varphi\rangle \\ \geqslant -\int_{\Gamma_U}\frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[\varphi^\delta\big]^+\mathrm{d}\Gamma + \int_{\Gamma_U}\frac{\underline{m}(v_n^\delta)}{\|U_n\|}\big[\varphi^\delta\big]^-\,\mathrm{d}\Gamma &\quad \text{for all } \varphi \in \mathbb{V}. \end{aligned} \tag{12.8}$$

We obtain by using the appropriate part of (12.8) that

$$d_2^n\langle w_n, w_n\rangle \leqslant -\int_{\Gamma_U}\frac{\underline{m}(v_n^\delta)}{\|U_n\|}\big[w_n^\delta\big]^+\mathrm{d}\Gamma + \int_{\Gamma_U}\frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w_n^\delta\big]^-\mathrm{d}\Gamma - \langle z_n, w_n\rangle, \tag{12.9}$$

$$d_2^n\langle w_n, w\rangle \geqslant -\int_{\Gamma_U}\frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w^\delta\big]^+\,\mathrm{d}\Gamma + \int_{\Gamma_U}\frac{\underline{m}(v_n^\delta)}{\|U_n\|}\big[w^\delta\big]^-\,\mathrm{d}\Gamma - \langle z_n, w\rangle. \tag{12.10}$$

We have

$$\underline{m}(v_n^\delta)\big[w_n^\delta\big]^+ = 0,\ \overline{m}(v_n^\delta)\big[w_n^\delta\big]^- \leqslant 0,\ \overline{m}(v_n^\delta)\big[w^\delta\big]^+ \leqslant 0,\ \underline{m}(v_n^\delta)\big[w^\delta\big]^- \leqslant 0 \tag{12.11}$$

on Γ_U. The embedding theorem gives $v_n \to 0$, $w_n \to w$ and $v_n^\delta \to 0$ in $L^2(\partial\Omega)$ and consequently $v_n^\delta \to 0$ and $w_n^\delta \to w^\delta$ in $C^0(\operatorname{cl}\partial\Omega)$.

Now we will show that $w^\delta \geqslant 0$ on Γ_U. Let us assume by contradiction that there is an $\varepsilon_0 > 0$ and a set $\mathcal{E} \subset \Gamma_U$ with $\mathrm{meas}_{n-1}\,\mathcal{E} > 0$ such that $w^\delta < -\varepsilon_0$ on $\mathcal{E}$. We have $v_n^\delta < 0$ on $\mathcal{E}$ for n large enough, consequently $\overline{m}(v_n^\delta) \to m^0 < 0$ and $\frac{\overline{m}(v_n^\delta)}{v_n^\delta} \to +\infty$ on $\mathcal{E}$. Furthermore, there are n_0 and $c < 0$ such that for all $n \geqslant n_0$ we have $w_n^\delta < c$ on $\mathcal{E}$. Then the Fatou lemma yields

$$\begin{aligned}\limsup_{n\to+\infty} \int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w_n^\delta\big]^- \,\mathrm{d}\Gamma &\leqslant \limsup_{n\to+\infty} \int_{\mathcal{E}} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w_n^\delta\big]^- \,\mathrm{d}\Gamma \\ &\leqslant \limsup_{n\to+\infty} \int_{\mathcal{E}} \frac{\overline{m}(v_n^\delta)}{v_n^\delta}\frac{v_n^\delta}{\|U_n\|}\big[w_n^\delta\big]^- \,\mathrm{d}\Gamma \to -\infty,\end{aligned}$$

which contradicts (12.9) because $\langle z_n, w_n\rangle \to \langle z, w\rangle \in \mathbb{R}$. Therefore $w^\delta \geqslant 0$ on Γ_U and the second integral in (12.10) vanishes. This together with (12.9) and (12.11) gives

$$\begin{aligned} d_2^n\langle w_n + z_n, w\rangle &\geqslant -\int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w^\delta\big]^+ \,\mathrm{d}\Gamma \geqslant 0 \\ &\geqslant \int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w_n^\delta\big]^- \,\mathrm{d}\Gamma \geqslant d_2^n\langle w_n + z_n, w_n\rangle.\end{aligned} \tag{12.12}$$

The assumptions $d_2^n \to d_2 > 0$, $z_n \to z$ together with (12.12) imply $\|w\|^2 \geqslant \limsup\limits_{n\to+\infty}\|w_n\|^2$, hence $w_n \to w$ in $\mathbb{V}$.

Now, (12.8) implies

$$\langle d_2^n w_n + z_n, \varphi\rangle \geqslant -\int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\varphi^\delta \,\mathrm{d}\Gamma \geqslant 0 \text{ for all } \varphi \in \mathbb{V}, \varphi^\delta \geqslant 0 \text{ on } \Gamma_U$$

and it follows that $\langle d_2 w + z, \varphi\rangle \geqslant 0$ for all $\varphi \in \mathbb{V}, \varphi^\delta \geqslant 0$ on Γ_U. By choosing $\varphi := w$ we have

$$\langle d_2 w + z, w\rangle = \lim_{n\to+\infty}\langle d_2^n w_n + z_n, w\rangle \geqslant -\int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|} w^\delta \,\mathrm{d}\Gamma \geqslant 0$$

and on the other hand, the last two inequalities in (12.12) give

$$\langle d_2 w + z, w\rangle = \lim_{n\to+\infty}\langle d_2^n w_n + z_n, w_n\rangle \leqslant \int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w_n^\delta\big]^- \,\mathrm{d}\Gamma \leqslant 0.$$

It follows that $\langle d_2 w + z, w\rangle = 0$ and, by the definition of M_{02}^δ, $d_2 w + z \in -M_{02}^\delta(w)$. $\square$

Proof of Proposition 4.2. We shall prove $\lim_{n\to+\infty} |\langle D^{-1}(d_n)P^{\delta}_{\tau_n,2}(U_n), U_n - U\rangle| = 0$. If $v_n \rightharpoonup v$ in $W^{1,2}(\Omega)$ then $v_n \to v$ in $L^2(\partial\Omega)$ and also $v^{\delta}_n \to v^{\delta}$ in $L^2(\partial\Omega)$.

Let $\tau_n \to \tau$. Let us suppose first that $\tau < +\infty$. Lemma 9.1 gives $p_{\tau_n}(v^{\delta}_n) \to p_{\tau}(v^{\delta})$ in $L^2(\Gamma_U)$, which implies that $p_{\tau_n}(v^{\delta}_n)$ are bounded in $L^2(\Gamma_U)$. We have

$$\begin{aligned}\lim_{n\to+\infty} \bigl|\langle P^{\delta}_{\tau_n,2}(v_n), v_n - v\rangle\bigr| &\leqslant \lim_{n\to+\infty} \int_{\Gamma_U} |p_{\tau_n}(v^{\delta}_n)(v^{\delta}_n - v^{\delta})| \,\mathrm{d}\Gamma \\ &\leqslant \lim_{n\to+\infty} \|p^{\tau_n}(v^{\delta}_n)\|_{L^2(\Gamma_U)} \cdot \|v^{\delta}_n - v^{\delta}\|_{L^2(\Gamma_U)} = 0.\end{aligned}$$

Now, let $\tau = +\infty$. We take a certain sufficiently small $\varepsilon > 0$ and define a continuous function $\underline{m}_{\varepsilon}\colon \mathbb{R} \to \mathbb{R}$ such that $\underline{m}_{\varepsilon}(\xi) = \underline{m}(\xi)$ for any $\xi \in (-\infty, 0]\cup[\varepsilon, +\infty)$ and $\underline{m}_{\varepsilon}$ is continuous and negative on $(0, \varepsilon)$. The Nemytskii theorem implies $\underline{m}_{\varepsilon}(v^{\delta}_n) \to \underline{m}_{\varepsilon}(v^{\delta})$ in $L^2(\Gamma_U)$ for $n \to +\infty$ with ε fixed. We have

$$\begin{aligned}\lim_{n\to+\infty} |\langle P^{\delta}_{\tau_n,2}(v_n), v_n - v\rangle| &= \lim_{n\to+\infty} \Bigl|\int_{\Gamma_U} p_{\tau_n}(v^{\delta}_n)(v^{\delta}_n - v^{\delta}) \,\mathrm{d}\Gamma\Bigr| \\ &\leqslant \lim_{n\to+\infty} \int_{\Gamma_U} |\underline{m}_{\varepsilon}(v^{\delta}_n)(v^{\delta}_n - v^{\delta})| \,\mathrm{d}\Gamma \\ &\leqslant \lim_{n\to+\infty} \|\underline{m}_{\varepsilon}(v^{\delta}_n)\|_{L^2(\Gamma_U)} \cdot \|v^{\delta}_n - v^{\delta}\|_{L^2(\Gamma_U)} = 0.\end{aligned} \tag{12.13}$$

The second part of (4.12) can be proved by the same considerations by using w_n, w and $\frac{P^{\delta}_{\tau_n,2}(v_n)}{\|U_n\|}$ instead of v_n, v and $P^{\delta}_{\tau_n,2}(v_n)$ and with help of the assumption that $\frac{P^{\delta}_{\tau_n,2}(v_n)}{\|U_n\|}$ are bounded. □

Proof of Proposition 4.3. **A:** If $v_n \rightharpoonup v$ in $\mathbb{V}$ and $\tau_n \to \tau \in [0, +\infty)$, then $v_n \to v$ in $L^2(\partial\Omega)$. Lemma 9.1 gives $p_{\tau_n}(v^{\delta}_n) \to p_{\tau}(v^{\delta})$ in $L^2(\Gamma_U)$. We have

$$\sup_{\|\varphi\|\leqslant 1} \langle P^{\delta}_{\tau_n,2}(v_n) - P^{\delta}_{\tau,2}(v), \varphi\rangle = \sup_{\|\varphi\|\leqslant 1} \int_{\Gamma_U} \bigl[p_{\tau_n}(v^{\delta}_n) - p_{\tau}(v^{\delta})\bigr]\varphi^{\delta} \,\mathrm{d}\Gamma \to 0.$$

Let $\tau \to +\infty$. For any sufficiently small $\varepsilon > 0$ we define a function $\underline{m}_{\varepsilon}\colon \mathbb{R} \to \mathbb{R}$ such that $\underline{m}_{\varepsilon}(\xi) = \underline{m}(\xi)$ for any $\xi \in (-\infty, 0]\cup[\varepsilon, +\infty)$, $\underline{m}_{\varepsilon}$ is continuous and negative on $(0, \varepsilon)$, $\underline{m}_{\varepsilon_1} \leqslant \underline{m}_{\varepsilon_2}$ for $\varepsilon_1 \geqslant \varepsilon_2$ and $\underline{m}_{\varepsilon} \to \underline{m}$ with $\varepsilon \to 0$. Similarly, let us define continuous functions $\overline{m}_{\varepsilon}\colon \mathbb{R} \to \mathbb{R}$ by $\overline{m}_{\varepsilon} := p_{1/\varepsilon}$. Then $\overline{m}_{\varepsilon}(\xi) = \overline{m}(\xi)$ for $\xi \geqslant 0$, $\overline{m}_{\varepsilon}(\xi) \geqslant \overline{m}(\xi)$ for $\xi < 0$, $\overline{m}_{\varepsilon_1} \geqslant \overline{m}_{\varepsilon_2}$ for $\varepsilon_1 \geqslant \varepsilon_2$ and $\overline{m}_{\varepsilon} \to \overline{m}$ with $\varepsilon \to 0$. Therefore, for all $\varepsilon > 0$ we have $p_{\tau_n} \leqslant \overline{m}_{\varepsilon}$ on $\mathbb{R}$ for n large enough. For such n we have

$$\begin{aligned}&\int_{\Gamma_U} \underline{m}_{\varepsilon}(v^{\delta}_n)\bigl[\varphi^{\delta}\bigr]^{+} \,\mathrm{d}\Gamma - \int_{\Gamma_U} \overline{m}_{\varepsilon}(v^{\delta}_n)\bigl[\varphi^{\delta}\bigr]^{-} \,\mathrm{d}\Gamma \\ &\qquad\leqslant \int_{\Gamma_U} p_{\tau_n}(v^{\delta}_n)\bigl[\varphi^{\delta}\bigr]^{+} \,\mathrm{d}\Gamma - \int_{\Gamma_U} p_{\tau_n}(v^{\delta}_n)\bigl[\varphi^{\delta}\bigr]^{-} \,\mathrm{d}\Gamma = \langle P^{\delta}_{\tau_n,2}(v_n), \varphi\rangle\end{aligned}$$

for any $\varphi \in \mathbb{V}$. The Nemytskii theorem implies $\underline{m}_\varepsilon(v_n^\delta) \to \underline{m}_\varepsilon(v^\delta)$, $\overline{m}_\varepsilon(v_n^\delta) \to \overline{m}_\varepsilon(v^\delta)$ in $L^2(\Gamma_U)$ for $n \to +\infty$ with ε fixed. If $P^\delta_{\tau_n,2}(v_n) \to \psi$ in $\mathbb{V}$ then the limiting process $n \to +\infty$ gives

$$\int_{\Gamma_U} \underline{m}_\varepsilon(v^\delta)\bigl[\varphi^\delta\bigr]^+ \,\mathrm{d}\Gamma - \int_{\Gamma_U} \overline{m}_\varepsilon(v^\delta)\bigl[\varphi^\delta\bigr]^- \,\mathrm{d}\Gamma \leqslant \langle z, \varphi\rangle$$

for any $\varphi \in \mathbb{V}$. The Levi theorem gives

(12.14)

$$\begin{aligned}\int_{\Gamma_U} &\underline{m}(v^\delta)\bigl[\varphi^\delta\bigr]^+ \,\mathrm{d}\Gamma - \int_{\Gamma_U} \overline{m}(v^\delta)\bigl[\varphi^\delta\bigr]^- \,\mathrm{d}\Gamma \\ &= \lim_{\varepsilon\to 0_+} \int_{\Gamma_U} \underline{m}_\varepsilon(v^\delta)\bigl[\varphi^\delta\bigr]^+ \,\mathrm{d}\Gamma - \int_{\Gamma_U} \overline{m}_\varepsilon(v^\delta)\bigl[\varphi^\delta\bigr]^- \,\mathrm{d}\Gamma \leqslant \langle z, \varphi\rangle\end{aligned}$$

for any $\varphi \in \mathbb{V}$.

In a similar way we can prove

$$\int_{\Gamma_U} \overline{m}(v^\delta)[\varphi^\delta]^+ \,\mathrm{d}\Gamma - \int_{\Gamma_U} \underline{m}(v^\delta)[\varphi^\delta]^- \,\mathrm{d}\Gamma \geqslant \langle z, \varphi\rangle \tag{12.15}$$

for any $\varphi \in \mathbb{V}$. The inequalities in (12.14) and (12.15) imply $z \in M_2^\delta(v)$.

B: The assumptions $v_n \to 0$ in $\mathbb{V}$, $w_n := \frac{v_n}{\|U_n\|} \rightharpoonup w$ in $\mathbb{V}$ and $\tau_n \to 0$ together with the embedding theorem give that $v_n \to 0$ and $w_n \to w$ in $L^2(\partial\Omega)$ and $w_n^\delta \to w^\delta$ in $C^0(\mathrm{cl}\,\Gamma_U)$. For a fixed τ_0 and n large enough we have $v_n^\delta > \xi_{\tau_0}$ on Γ_U and

$$\Bigl|\frac{p_{\tau_n}(v_n^\delta)}{v_n^\delta}\Bigr| \leqslant \Bigl|\frac{\tau_n v_n^\delta}{v_n^\delta}\Bigr| = \tau_n \to 0 \quad \text{ on } \Gamma_U.$$

We obtain

$$\begin{aligned}\Bigl|\sup_{\|\varphi\|\leqslant 1}\Bigl\langle \frac{P^\delta_{\tau_n,2}(v_n)}{\|U_n\|}, \varphi\Bigr\rangle\Bigr| = \Bigl|\sup_{\|\varphi\|\leqslant 1}\int_{\Gamma_U} \frac{p_{\tau_n}(v_n^\delta)}{\|U_n\|}\varphi^\delta \,\mathrm{d}\Gamma\Bigr| = \Bigl|\sup_{\|\varphi\|\leqslant 1}\int_{\Gamma_U} \frac{p_{\tau_n}(v_n^\delta)}{v_n^\delta} w_n^\delta\varphi^\delta \,\mathrm{d}\Gamma\Bigr| \\ \leqslant \sup_{\|\varphi\|\leqslant 1} \tau_n \cdot \|w_n^\delta\|_{L^2(\Gamma_U)} \cdot \|\varphi^\delta\|_{L^2(\Gamma_U)} \to 0.\end{aligned}$$

C: Let $U_n \to 0$, $W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W$, $\tau_n \to \tau \in [0, +\infty)$. The embedding theorem gives that $v_n \to 0$, $w_n := \frac{v_n}{\|U_n\|} \to w$ in $L^2(\partial\Omega)$ and $v_n^\delta \to 0$, $w_n^\delta \to w^\delta$ in $C^0(\mathrm{cl}\,\partial\Omega)$. Set $\mathcal{E} := \{x \in \Gamma_U;\ w^\delta < 0\}$ and for $\tau \geqslant 0$ introduce $p_{\tau,0}\colon \xi \mapsto \begin{cases} 0, & \xi \geqslant 0, \\ \tau\xi, & \xi \leqslant 0. \end{cases}$, i.e. $p_{\tau,0} = p_\tau$ for $\xi \geqslant \xi_\tau$. Hence from the C^0-convergence of v_n^δ we have

$$\begin{aligned}\Bigl|\sup_{\|\varphi\|\leqslant 1}\Bigl\langle \frac{P^\delta_{\tau_n,2}(v_n)}{\|U_n\|} - P^\delta_{\tau_n,0,2}(w), \varphi\Bigr\rangle\Bigr| \leqslant \sup_{\|\varphi\|\leqslant 1}\Bigl|\int_{\mathcal{E}} \Bigl[\frac{p_{\tau_n}(v_n^\delta)}{v_n^\delta} w_n^\delta - \tau w^\delta\Bigr]\varphi^\delta \,\mathrm{d}\Gamma\Bigr| \\ \leqslant \sup_{\|\varphi\|\leqslant 1}\Bigl\|\frac{p_{\tau_n}(v_n^\delta)}{v_n^\delta} w_n^\delta - \tau \cdot w^\delta\Bigr\|_{L^2(\Gamma_U)} \cdot \|\varphi^\delta\|_{L^2(\Gamma_U)} \to 0.\end{aligned}$$

Let now $\tau_n \to +\infty$ and $\frac{P^\delta_{\tau_n}(U_n)}{\|U_n\|} \to Z$. By the same considerations as in the proof of Proposition 4.1 (using $p_{\tau_n}(v^\delta_n)$ instead of $\overline{m}(v^\delta_n)$) we can show that $w^\delta \geqslant 0$ on Γ_U: Let us suppose that there is an $\varepsilon_0 > 0$ and a set $\mathcal{E} \subset \Gamma_U$ with $\mathrm{meas}_{\mathfrak{n}-1}\,\mathcal{E} > 0$ such that $w^\delta < -\varepsilon_0$ on $\mathcal{E}$. The C^0-convergence of v^δ_n ensures the existence of $\tau_0 \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $v^\delta_n \in (\xi_{\tau_0}, 0)$ on $\mathcal{E}$ for all $n \geqslant n_0$. Then $\frac{p_{\tau_n}(v^\delta_n)}{v^\delta_n} = \tau_n \to +\infty$ for $n \to +\infty$ on $\mathcal{E}$. The Fatou lemma gives

$$\limsup_{n\to+\infty} \int_{\Gamma_U} \frac{p_{\tau_n}(v^\delta_n)}{\|U_n\|} \varphi^\delta \,\mathrm{d}\Gamma \leqslant \limsup_{n\to+\infty} \int_{\mathcal{E}} \frac{p_{\tau_n}(v^\delta_n)}{v^\delta_n} \frac{v^\delta_n}{\|U_n\|} \varphi^\delta \,\mathrm{d}\Gamma = -\infty \tag{12.16}$$

for any $\varphi \in \mathbb{V}$ such that $\varphi^\delta \geqslant 0$ on Γ_U and $\varphi^\delta > 0$ on $\mathcal{E}$, which is the contradiction with

$$\int_{\Gamma_U} \frac{p_{\tau_n}(v^\delta_n)}{\|U_n\|} \varphi^\delta \,\mathrm{d}\Gamma \to \langle z, \varphi\rangle. \tag{12.17}$$

This implies $w^\delta \geqslant 0$ on Γ_U, i.e. $w \in K^\delta_2$. Moreover, (12.17) and the sign of p_τ give $\langle z, \varphi\rangle \leqslant 0$ for all $\varphi \in K^\delta_2$ and for $\varphi := w$ we obtain $\langle z, w\rangle \leqslant 0$. On the other hand, the choice $\varphi := w_n$ implies

$$\langle z, w\rangle = \lim_{n\to+\infty} \int_{\Gamma_U} \frac{p_{\tau_n}(v^\delta_n)}{\|U_n\|} w^\delta_n \,\mathrm{d}\Gamma \geqslant 0,$$

because the signs of $p_{\tau_n}(v^\delta_n)$ and w^δ_n are the same on Γ_U. We obtain $\langle z, w\rangle = 0$, therefore $z \in M^\delta_{02}(w)$ by definition. □

P r o o f o f P r o p o s i t i o n 4.4. **A:** Let $U_n \to 0$, $W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W \notin K^\delta$, $\tau_n \to \tau_0 > 0$ and $V \in \operatorname{int} K^\delta$. The embedding theorems give $v_n \to 0$ and $w_n \to w$ in $L^2(\partial\Omega)$ and $v^\delta_n \to v^\delta$, $w^\delta_n \to w^\delta$ in $C^0(\operatorname{cl}\Gamma_U)$. The assumption $W \notin K^\delta$ ensures the existence of an $\varepsilon_0 > 0$ and a set $\mathcal{E} \subset \Gamma_U$ with $\mathrm{meas}_{\mathfrak{n}-1}\,\mathcal{E} > 0$ such that $w^\delta < -\varepsilon_0$ on $\mathcal{E}$. Then there exist $\tau_0 \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $v^\delta_n \in (\xi_{\tau_0}, 0)$ on $\mathcal{E}$ for all $n \geqslant n_0$. The assumption $V = [y, z] \in \operatorname{int} K^\delta$ means $z^\delta > 0$ on Γ_U. For $\varepsilon_n \to 0_+$ we obtain

$$\begin{aligned}\limsup_{n\to+\infty} \Big\langle \frac{P^\delta_{\tau_n}(U_n)}{\|U_n\|}, V\Big\rangle &= \limsup_{n\to+\infty} \Big\langle \frac{P^\delta_{\tau_n,2}(v_n)}{\|U_n\|}, z\Big\rangle = \limsup_{n\to+\infty} \int_{\Gamma_U} \frac{p_{\tau_n}(v^\delta_n)}{\|U_n\|} z^\delta \,\mathrm{d}\Gamma \\ &\leqslant \limsup_{n\to+\infty} \int_{\mathcal{E}} \frac{p_{\tau_n}(v^\delta_n)}{v^\delta_n} w^\delta_n z^\delta \,\mathrm{d}\Gamma \leqslant \limsup_{n\to+\infty} \int_{\mathcal{E}} \frac{(\tau_0 - \varepsilon_n)\cdot v^\delta_n}{v^\delta_n} w^\delta_n z^\delta \,\mathrm{d}\Gamma < 0.\end{aligned} \tag{12.18}$$

B: The proof of the second part is similar—in the final line of (12.18) we use the fact that

$$\frac{p_{\tau_n}(v^\delta_n)}{\tau_n v^\delta_n} = \frac{\tau_n v^\delta_n}{\tau_n v^\delta_n} = 1$$

on $\mathcal{E}$ for n large enough. □

P r o o f o f P r o p o s i t i o n 4.5, (4.11). Let $\delta_n \to 0_+$, $U_n = [u_n, v_n] \rightharpoonup U$, $Z_n = [\eta_n, z_n] \to Z$, $d_n \to d \in \mathbb{R}^2_+$, $D(d_n)U_n + Z_n \in -M^{\delta_n}(U_n)$. The first part of the inclusion is the equation

$$d_1^n u_n + \eta_n = 0$$

and we have immediately $u_n \to u$ because $\eta_n \to \eta$. The second part of the inclusion gives

(12.19)
$$\begin{aligned} -\int_{\Gamma_U} \underline{m}(v_n^{\delta_n})\big[\varphi^{\delta_n}\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \overline{m}(v_n^{\delta_n})\big[\varphi^{\delta_n}\big]^- \mathrm{d}\Gamma \geqslant \langle d_2^n v_n + z_n, \varphi\rangle \\ \geqslant -\int_{\Gamma_U} \overline{m}(v_n^{\delta_n})\big[\varphi^{\delta_n}\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}(v_n^{\delta_n})\big[\varphi^{\delta_n}\big]^- \mathrm{d}\Gamma \quad \text{for all } \varphi \in \mathbb{V}. \end{aligned}$$

The embedding theorem together with Proposition 10.1, (iv) give $v_n \to v$ and $v_n^{\delta_n} \to v$ in $L^2(\partial\Omega)$. By using the appropriate part of (12.19) we obtain that

(12.20) $$\langle d_2^n v_n + z_n, v_n\rangle \leqslant -\int_{\Gamma_U} \underline{m}(v_n^{\delta_n})\big[v_n^{\delta_n}\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \overline{m}(v_n^{\delta_n})\big[v_n^{\delta_n}\big]^- \mathrm{d}\Gamma,$$

(12.21) $$d_2^n \langle v_n + z_n, v\rangle \geqslant -\int_{\Gamma_U} \overline{m}(v_n^{\delta_n})\big[v^{\delta_n}\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}(v_n^{\delta_n})\big[v^{\delta_n}\big]^- \mathrm{d}\Gamma.$$

The terms have the following signs on Γ_U:

(12.22) $$\underline{m}(v_n^{\delta_n})\big[v_n^{\delta_n}\big]^+ = 0,\ \overline{m}(v_n^{\delta_n})\big[v_n^{\delta_n}\big]^- \leqslant 0,\ \overline{m}(v_n^{\delta_n})\big[v^{\delta_n}\big]^+ \leqslant 0,\ \underline{m}(v_n^{\delta_n})\big[v^{\delta_n}\big]^- \leqslant 0.$$

For any fixed $\varepsilon > 0$ small let us define continuous functions $\underline{m}_\varepsilon, \overline{m}_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ such that $\underline{m}_\varepsilon(\xi) = \underline{m}(\xi)$ for any $\xi \in (-\infty, 0] \cup [\varepsilon, +\infty)$ and $\underline{m}_\varepsilon$ is negative on $(0, \varepsilon)$, and $\overline{m}_\varepsilon(\xi) = \overline{m}(\xi)$ for any $\xi \in (-\infty, -\varepsilon] \cup [0, +\infty)$ and $\overline{m}_\varepsilon$ is negative on $(-\varepsilon, 0)$ and such that they converge monotonously to $\underline{m}$ or $\overline{m}$, respectively, for $\varepsilon \to 0_+$. It follows from (12.20), (12.21), (12.22) and from the above definitions of $\underline{m}_\varepsilon$ and $\overline{m}_\varepsilon$ that

(12.23) $$\langle d_2^n v_n + z_n, v_n\rangle \leqslant \int_{\Gamma_U} \overline{m}_\varepsilon(v_n^{\delta_n})\big[v_n^{\delta_n}\big]^- \mathrm{d}\Gamma,$$

(12.24) $$d_2^n \langle v_n + z_n, v\rangle \geqslant -\int_{\Gamma_U} \overline{m}_\varepsilon(v_n^{\delta_n})\big[v^{\delta_n}\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}_\varepsilon(v_n^{\delta_n})\big[v^{\delta_n}\big]^- \mathrm{d}\Gamma.$$

The limiting process for $n \to +\infty$ in (12.23), (12.24) by using the Nemytskii theorem gives

$$\begin{aligned} \limsup_{n\to+\infty} \langle d_2^n v_n + z_n, v_n\rangle \leqslant \int_{\Gamma_U} \overline{m}_\varepsilon(v) v^- \mathrm{d}\Gamma, \\ d_2\|v\|^2 + \langle z, v\rangle \geqslant -\int_{\Gamma_U} \overline{m}_\varepsilon(v) v^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}_\varepsilon(v) v^- \mathrm{d}\Gamma \end{aligned}$$

and the limiting process for $\varepsilon \to 0_+$ by using the Levi theorem implies

$$(12.25)\qquad \begin{aligned} d_2 \limsup_{n\to+\infty} \|v_n\|^2 + \langle z, v\rangle &\leqslant \int_{\Gamma_U} \overline{m}(v) v^- \,\mathrm{d}\Gamma,\\ d_2\|v\|^2 + \langle z, v\rangle &\geqslant -\int_{\Gamma_U} \overline{m}(v) v^+ \,\mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}(v) v^- \,\mathrm{d}\Gamma. \end{aligned}$$

We have $\overline{m}(v)v^+ = 0$, $\overline{m}(v)v^- = \underline{m}(v)v^-$ and therefore (12.25) gives $\|v\|^2 \geqslant \limsup\limits_{n\to+\infty} \|v_n\|^2$, which implies $v_n \to v$ strongly in $\mathbb{V}$.

Similarly as above, we can estimate (12.19) both from below and above by using $\underline{m}_\varepsilon$ and $\overline{m}_\varepsilon$ to obtain

(12.26)
$$\begin{aligned} -\int_{\Gamma_U} \underline{m}_\varepsilon(v_n^{\delta_n})\big[\varphi^{\delta_n}\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \overline{m}_\varepsilon(v_n^{\delta_n})\big[\varphi^{\delta_n}\big]^- \mathrm{d}\Gamma &\geqslant \langle d_2^n v_n + z_n, \varphi\rangle\\ \geqslant -\int_{\Gamma_U} \overline{m}_\varepsilon(v_n^{\delta_n})\big[\varphi^{\delta_n}\big]^+ \,\mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}_\varepsilon(v_n^{\delta_n})\big[\varphi^{\delta_n}\big]^- \,\mathrm{d}\Gamma &\quad \text{for all } \varphi \in \mathbb{V}. \end{aligned}$$

The "double" limiting process in (12.26) (first for $n \to +\infty$, then for $\varepsilon \to 0_+$) gives

(12.27)
$$\begin{aligned} -\int_{\Gamma_U} \underline{m}(v)\varphi^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \overline{m}(v)\varphi^- \mathrm{d}\Gamma &\geqslant \langle d_2 v + z, \varphi\rangle\\ \geqslant -\int_{\Gamma_U} \overline{m}(v)\varphi^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}(v)\varphi^- \mathrm{d}\Gamma &\quad \text{for all } \varphi \in \mathbb{V}, \end{aligned}$$

which is equivalent to

$$-\int_{\Gamma_U} \underline{m}(v)\varphi \,\mathrm{d}\Gamma \geqslant \langle d_2 v + z, \varphi\rangle \geqslant -\int_{\Gamma_U} \overline{m}(v)\varphi \,\mathrm{d}\Gamma \qquad \text{for all } \varphi \in K_2$$

and we have $D(d)U + Z \in -M(U)$ by definition. □

P r o o f o f P r o p o s i t i o n 4.5, (4.12). Let $\delta_n \to 0_+$, $U_n \rightharpoonup U$, $Z_n \to Z$, $d_n \to d \in \mathbb{R}^2_+$, $D(d_n)U_n + Z_n \in -M_0^{\delta_n}(U_n)$. Again, as in the proof of (4.11), the first part of the inclusion gives $u_n \to u$ strongly. The second part of the inclusion gives

$$(12.28)\qquad \langle d_2^n v_n + z_n, v_n\rangle = 0,$$

$$(12.29)\qquad \langle d_2^n v_n + z_n, \varphi\rangle \geqslant 0 \quad \text{for all } \varphi \in \mathbb{V},\ \varphi^{\delta_n} \geqslant 0 \text{ on } \Gamma_U.$$

The embedding theorem gives $v_n \to v$ and $v_n^{\delta_n} \to v$ in $L^2(\partial\Omega)$. We have $v_n^{\delta_n} \geqslant 0$ on Γ_U and therefore also $v \geqslant 0$ on Γ_U. We can choose a subsequence (let us denote it v_n again) and Proposition 10.1, (vi) ensures the existence of $w_n = w_n(v) \in \mathbb{V}$ such that $w_n^{\delta_n} \geqslant 0$ on Γ_U and $w_n \to v$ strongly in $\mathbb{V}$. We can put $\varphi := w_n$ in (12.29) to obtain with help of (12.28) that

$$\langle d_2^n v_n + z_n, w_n\rangle \geqslant 0 = \langle d_2^n v_n + z_n, v_n\rangle.$$

The assumptions $d_2^n \to d_2 > 0$, $z_n \to z$ and the fact $w_n \to v$ imply $\limsup\limits_{n\to+\infty} \|v_n\|^2 \leqslant \|v\|^2$, therefore $v_n \to v$ strongly in $\mathbb{V}$.

Now, let $\psi \in \mathbb{V}$ be arbitrary such that $\psi \geqslant 0$ a.e. on Γ_U. Let $w_n = w_n(\psi)$ be the functions from Proposition 10.1, (vi) corresponding to ψ. Then the choice $\varphi := w_n(\psi)$ in (12.29) and the limiting process in (12.28) and (12.29) (we have $w_n(\psi) \to \psi$) gives $d_2 v + z \in -M_{02}(v)$. □

In the situation of Example 10.2, the verification of validity of Propositions 4.1–4.5 can be done analogously as in Model Example.

P r o o f o f R e m a r k 10.1. It follows from the last part of (10.3) that

$$d_2 \Delta v + b_{21} u + b_{22} v = 0 \quad \text{ in } \Omega_v^+.$$

Multiplying this equation by an arbitrary $\varphi \in \mathbb{V}$, $\varphi \geqslant 0$ in Ω_1, integrating over Ω_v^+ and using Green's formula we obtain

(12.30)
$$\begin{aligned} 0 &= \int_{\partial\Omega_v^+} d_2 \mathfrak{T} v\varphi \,\mathrm{d}\Gamma + \int_{\Omega_v^+} -d_2 \sum_{j=1}^{\mathfrak{n}} v_{x_j}\varphi_{x_j} + (b_{21}u + b_{22}v)\varphi \,\mathrm{d}x \\ &= \int_{\partial\Omega_v^+} d_2 \mathfrak{T} v\varphi \,\mathrm{d}\Gamma + \int_{\Omega_1} -d_2 \sum_{j=1}^{\mathfrak{n}} v_{x_j}\varphi_{x_j} + (b_{21}u + b_{22}v)\varphi \,\mathrm{d}x \\ &\qquad - \int_{\Omega_v^0} -d_2 \sum_{j=1}^{\mathfrak{n}} v_{x_j}\varphi_{x_j} + (b_{21}u + b_{22}v)\varphi \,\mathrm{d}x. \end{aligned}$$

It follows from (2.11) and the definition of $M_{02}(v)$ that

$$\int_{\Omega_1} -d_2 \sum_{j=1}^{\mathfrak{n}} v_{x_j}\varphi_{x_j} + (b_{21}u + b_{22}v)\varphi \,\mathrm{d}x \leqslant 0 \quad \text{for any } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ in } \Omega_1.$$

This fact together with (12.30) implies

$$\int_{\partial\Omega_v^+} d_2 \mathfrak{T} v\varphi \,\mathrm{d}\Gamma - \int_{\Omega_v^0} -d_2 \sum_{j=1}^{\mathfrak{n}} v_{x_j}\varphi_{x_j} + (b_{21}u + b_{22}v)\varphi \,\mathrm{d}x \geqslant 0 \tag{12.31}$$
$$\text{for any } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ in } \Omega_1.$$

Clearly, we have $\nabla v = 0$ in Ω_v^0 and the second term in (12.31) vanishes. As a test function in (12.31) we can choose $\varphi_n \in \mathbb{V}$, $\varphi_n \geqslant 0$ in Ω_1, $\varphi_n = 1$ in $\operatorname{cl}\Omega_v^+$, $\varphi_n = 0$ in $\Omega_v^n \subset \Omega_v^0$, $\operatorname{meas}(\Omega_v^0 \setminus \Omega_v^n) \to 0$ for $n \to +\infty$ to get

$$\int_{\partial\Omega_v^+} \mathfrak{T}v \, \mathrm{d}\Gamma \geqslant 0.$$

But $\mathfrak{T}v \leqslant 0$ on $\partial\Omega_v^+$ by the second condition in the last part of (10.3). Therefore $\int_{\partial\Omega_v^+} \mathfrak{T}v\varphi \, \mathrm{d}\Gamma \leqslant 0$ and, consequently, $\int_{\partial\Omega_v^+} \mathfrak{T}v\varphi \, \mathrm{d}\Gamma = 0$ for any $\varphi \geqslant 0$ in Ω_1. Thus we obtain $\mathfrak{T}v = 0$ on $\partial\Omega_v^+$. Finally, it is easy to see that $v = 0$ on $\partial\Omega_v^+$. $\square$

In the situation of Example 10.3 we have $\operatorname{int} K \neq \emptyset$ and there is no need to regularize via Φ^δ.

P r o o f of the fact $\frac{\partial v}{\partial n} = \text{const}$ on Γ_U from (10.5), i.e. of $\int_{\Gamma_U} \mathfrak{T}v\varphi \, \mathrm{d}\Gamma = C\,\overline{\varphi}$ for some $C \in \mathbb{R}$. In a similar way as at the beginning of Section 12 we can prove that (9.1) is a weak solution of (9.2) with (10.5) and we obtain

$$\underline{m}(\overline{v})\overline{\varphi} \leqslant -d_2 \int_{\Gamma_U} \mathfrak{T}v\varphi \, \mathrm{d}\Gamma \leqslant \overline{m}(\overline{v})\overline{\varphi} \qquad \text{for all } \varphi \in \mathbb{V}, \overline{\varphi} \geqslant 0 \tag{12.32}$$

(cf. (12.7)). The choice $\varphi \in \mathbb{V}$, $\overline{\varphi} = 0$ gives $\int_{\Gamma_U} \mathfrak{T}v\varphi \, \mathrm{d}\Gamma = 0$. If $\mathfrak{T}v$ were nonconstant on Γ_U then we would find $\varphi_1, \varphi_2 \in \mathbb{V}$, $\overline{\varphi}_1 = \overline{\varphi}_2$ and $C_1, C_2 \in \mathbb{R}$ such that

$$\int_{\Gamma_U} \mathfrak{T}v\varphi_j \, \mathrm{d}\Gamma = C_j\,\overline{\varphi}_j, \quad j = 1, 2.$$

Then $\int_{\Gamma_U} \mathfrak{T}v(\varphi_1 - \varphi_2) \, \mathrm{d}\Gamma = (C_1 - C_2)\overline{\varphi}_1 \neq 0$, which is a contradiction. $\square$

The verification of validity of Propositions 4.1–4.5 can be done analogously as in Model Example by using the functional $\overline{\varphi}$ instead of φ^δ.

In the situation of Example 11.1 (where $\Omega = (0,1)$) the embedding theorem guarantees nonempty interiors of the sets K, K_1 and $\mathcal{K}$, so we need not regularize (9.1).

P r o o f o f P r o p o s i t i o n 4.1. Let $U_n \to 0$, $W_n = [\xi_n, w_n] = \frac{U_n}{\|U_n\|} \rightharpoonup W = [\xi, w]$, $Z_n = [\eta_n, z_n] \to Z = [\eta, z]$, $d_n \to d \in \mathbb{R}^2_+$ and $D(d_n)W_n + Z_n \in -\frac{M(U_n)}{\|U_n\|}$. Analogously to the proof of Proposition 4.1 for Example 10.1 performed earlier we can prove $w(x_0) \geqslant 0$. Let us define a linear continuous functional $\mathcal{L}$ on $\mathbb{V}$ by $\mathcal{L}\varphi = \varphi(1)$ and $\mathbb{V} = \operatorname{Ker}\mathcal{L} \oplus \mathbb{V}_0$ with $\dim \mathbb{V}_0 = 1$. It follows from the definition of M that

$$-\frac{\underline{m}(v_n(x_0))}{\|U_n\|}\varphi(1) \geqslant \langle d_2^n w_n + z_n, \varphi\rangle \geqslant -\frac{\overline{m}(v_n(x_0))}{\|U_n\|}\varphi(1) \qquad \text{for all } \varphi \in K_1. \tag{12.33}$$

Therefore $\langle d_2^n w_n + z_n, \varphi\rangle = 0$ for any $\varphi \in \operatorname{Ker}\mathcal{L}$, i.e. $d_2^n w_n + z_n \in \mathbb{V}_0$. This together with the assumed convergences gives $w_n \to w$ in $\mathbb{V}$.

In the case $w(x_0) > 0$ the embedding theorem yields $w_n(x_0) > 0$ for n large enough. Then (12.33) is equivalent to $d_2^n w_n + z_n = 0$ and by the limiting process for $n \to +\infty$ we obtain $d_2 w + z = 0$, i.e. $d_2 w + z \in -M_{02}(w)$. In the case $w(x_0) = 0$ it follows from (12.33) and the sign of $\overline{m}$ that $\langle d_2 w + z, \varphi\rangle = \lim\limits_{n\to+\infty} \langle d_2^n w_n + z_n, \varphi\rangle \geqslant 0$ for any $\varphi \in K_1$, i.e. we have $d_2 w + z \in -M_{02}(w)$ again. $\square$

The verification of validity of Propositions 4.1–4.5 can be done analogously as in Model Example.

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